Progress on stabilisation of complex Langevin for real-time simulations of non-abelian gauge theories

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Doktoratskolleg Particles and Interactions

1 Introduction

Complex Langevin (CL) equation:

$$\mathrm{Re}\dot{z}(\theta) = \mathrm{Re}K(z(\theta)) + \eta(\theta), \qquad \textit{Drift term: } K(z) = i\frac{dS}{dz},\ z \in \mathcal{M}_C = \mathbb{C}^N,$$

$$\mathrm{Im}\dot{z}(\theta) = \mathrm{Im}K(z(\theta)), \qquad \textit{Noise term: } \langle\eta(\theta)\rangle = 0,\ \langle\eta(\theta)\eta(\theta')\rangle = 2\delta(\theta-\theta')$$

- Complex Fokker-Planck equation, complexification of degrees of freedom
- This allows us to compute oscillatory integrals:

$$\langle\mathcal{O}\rangle = \frac{1}{Z}\int dx\, \mathcal{O}(x)\exp\left[iS(x)\right] \approx \lim_{\theta_0\to\infty}\frac{1}{T}\int_{\theta_0}^{\theta_0+T} d\theta\, \mathcal{O}[z(\theta)]$$

- CL suffers from instabilities:
  - Runaway instabilities
    → avoided by an adaptive stepsize
  - Wrong convergence
    → mitigated by stabilization techniques
    → improvements investigated here
- We introduce a CL step for anisotropic lattices

2 Complex Langevin for real-time Yang-Mills simulations

- Complexification of the Lie algebra of the gauge group: $\mathrm{SU}(N) \to \mathrm{SL}(N,\mathbb{C})$
- Complex Langevin equation for Yang-Mills theories (continuum S_{YM}):

$$\frac{\partial A^a_\mu(\theta,x)}{\partial\theta} = i\frac{\delta S_{\mathrm{YM}}}{\delta A^a_\mu(\theta,x)} + \eta^a_\mu(\theta,x), \qquad S_{\mathrm{YM}} = -\frac{1}{4}\int_{\mathscr{C}} d^4x F^{\mu\nu}_a F^a_{\mu\nu}$$

$$\langle\eta^a_\mu(\theta,x)\rangle = 0, \quad \langle\eta^a_\mu(\theta,x)\eta^b_\nu(\theta',y)\rangle = 2\delta(\theta-\theta')\delta^{(d)}(x-y)\delta^{ab}\delta_{\mu\nu}.$$

- CL step using anisotropic a_μ (discrete S_W on $N_t \times N_s^3$ lattice):

$$U'_{x\mu} = \exp\left[-i\lambda_a(\epsilon_\mu D^a_{x,\mu}S_W[U] - \sqrt{\epsilon_\mu}\eta^a_{x,\mu})\right]U_{x\mu}, \qquad \epsilon_0 = \epsilon\frac{|a_t|^2}{a_s^2},\ \epsilon_i = \epsilon$$

- **Our observation:** Partial continuum limit with $|a_t| \ll a_s$ and $N_t|a_t| = const.$ improves convergence for real-time CL-simulations

3 Methods

- Path integral is regularized by tilting the time contour
- **Our focus:** Stabilizing the tilted part of the contour

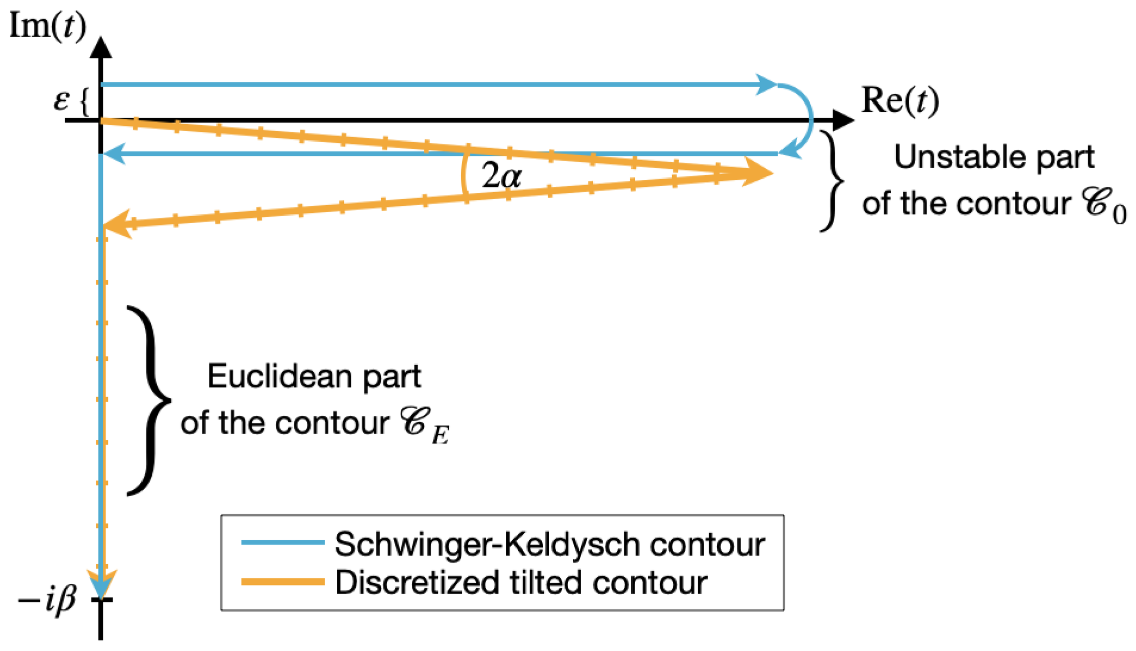

Figure 1: Cont. and discr. Schwinger-Keldysch contour.

Modern stabilization techniques:

- **Adaptive stepsize (AS) [2]** counter-acts runaways:

$$\epsilon \mapsto \tilde{\epsilon} = \epsilon\frac{B}{\max_{x,\mu,a}|K^a_{x,\mu}|}$$

- **Gauge cooling (GC) [3]** reduces "distance" $F[U]$ to the $\mathrm{SU}(N)$ group:

$$U_{x,\mu} \mapsto U^V_{x,\mu} = V_{x,\mu}U_{x,\mu}V^{-1}_{x+\mu,\mu},$$

$$F[U] = \sum_{x,\mu}\mathrm{Tr}\left[(U_{x,\mu}U^\dagger_{x,\mu} - 1)^2\right] \to \min$$

- **Dynamical stabilization (DS) [4]** reduces excursions to $\mathrm{SL}(N)$:

$$K^a_{x,\mu} \mapsto \tilde{K}^a_{x,\mu} = K^a_{x,\mu} + i\alpha M^a_x,$$

$$M^a_x \propto F[U], \quad \frac{d\mathcal{O}}{d\alpha} \approx 0$$

Sampling uncorrelated measurements: autocorrelation time

- **Autocorrelation function:**

$$R_{\mathcal{O}}(\tau) = \frac{\langle(\mathcal{O}_\theta - \langle\mathcal{O}_\theta\rangle)(\mathcal{O}_{\theta+\tau} - \langle\mathcal{O}_{\theta+\tau}\rangle)\rangle}{\sigma_\theta\sigma_{\theta+\tau}}$$

- **Autocorrelation time** $T_{\mathcal{O}}$:

$$R_{\mathcal{O}}(\tau) \approx \exp(-\tau/T_{\mathcal{O}})$$

4 Progress on the stabilization of real-time YM simulations

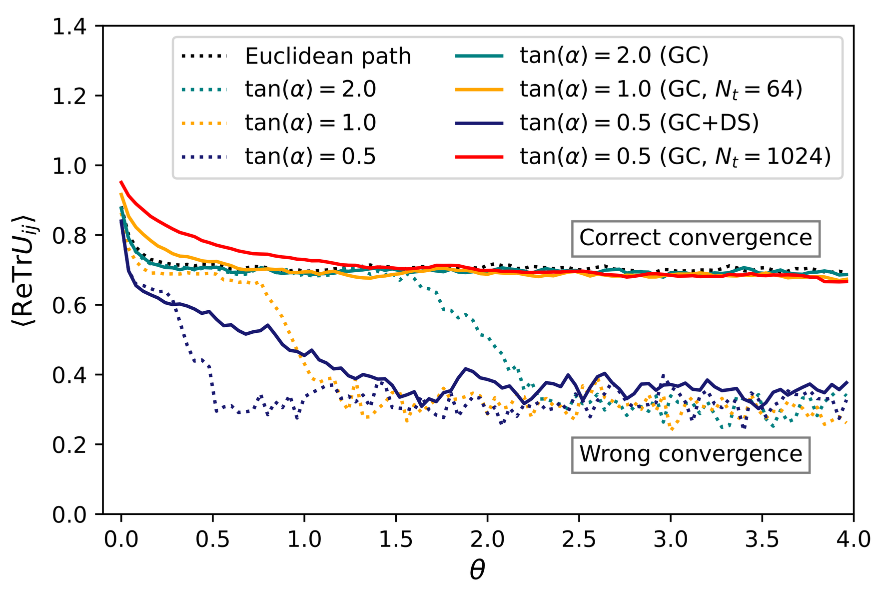

Figure 2: $\mathcal{O}$ for different contour tilt angles of $\mathscr{C}_0$ and stabilization techniques. If not mentioned otherwise $N_t = 16$.

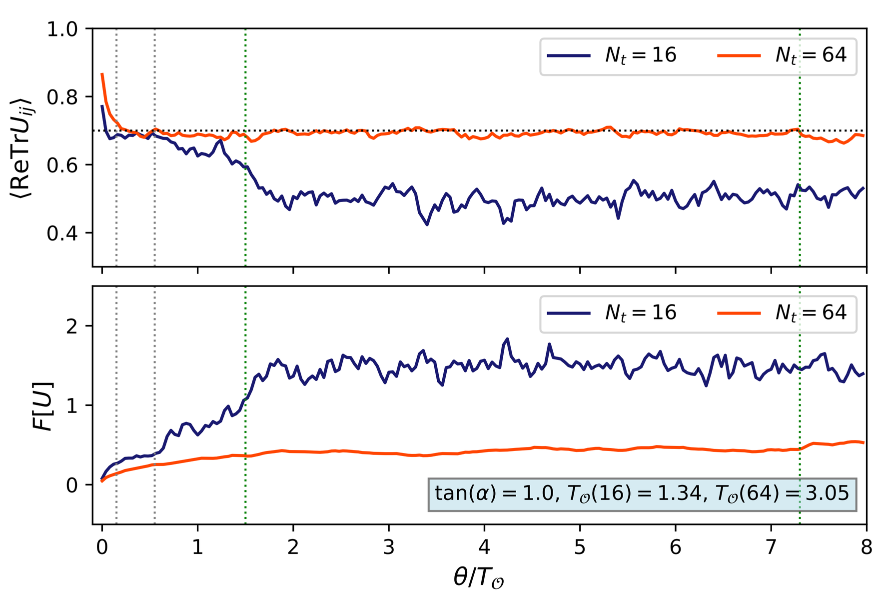

Figure 3: Observable (top) and unitarity norm (bottom) as functions of rescaled Langevin time (AS + GC, no DS).

Fig. 2 reproduces results of [1] for the average spatial plaquette ($\mathcal{O} = \mathrm{ReTr}U_{ij}$). We use AS (always), GC and DS to overcome runaways and wrong convergence.

- $\mathcal{O}$ without stabilization: short stability region or wrong convergence
- $\langle\mathcal{O}\rangle$ should be ind. of contour (time transl. inv.)
- Stabilization techniques and partial continuum limit improve convergence
- DS can improve stability but introduces bias
- Autocorrelation time grows with increasing N_t

Fig. 3 shows CL evolution scaled by the autocorr. time.

- Stable θ-region grows faster with N_t than $T_{\mathcal{O}}$
- No dynamical stabilization is needed
  ⇒ biased results are avoided
- *Novel* ϵ_μ *prescription* in CL step effectively enlarges stable θ-region

5 Conclusion

- Stabilization techniques extend the applicability of CL by mitigating known instabilities
- Anisotropic lattice discretization can enlarge the stable θ-regions
- Stability region increases faster with respect to N_t than the autocorrelation time $T_{\mathcal{O}}$

⇒ Extrapolation to Schwinger-Keldysh contour might be possible

⇒ Application to real-time observables

References:

[1] Berges, J. et al. (2007). PRD, 75, 045007. [arXiv:0609058]

[2] Aarts, G. et al. (2010). PLB, 687(2–3), 154–159. [arXiv:0912.3360]

[3] Seiler, E., Sexty, D., Stamatescu, I.-O. (2013). PLB, 723(1–3), 213–216. [arXiv:1211.3709]

[4] Attanasio, F., Jäger, B. (2019). EPJC, 79(1), 16. [arXiv:1808.04400]