# Critical point method and $\beta$-function at large $N_f$

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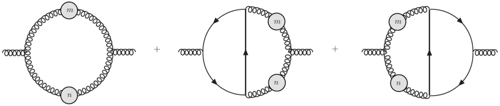
# Outlook

# Introduction

- Large-$N_f$ methods target theories with "large" flavor symmetries as $O(N_f)$ and $SU(N_f)$
- in case of $N_f \gg 1$, the series is better organized in powers of $1/N_f$
- at each order in $1/N_f$, infinite Feynman diagrams contribute and resummation techniques – or others – are needed




- all-order results are available in a closed form for 4d QFTs (nice)

# Large-$N_f$ expansion

Consider a gauge theory with coupling constant $e$

- define a 't Hooft coupling $K$ as

$$K = \frac{e^2}{4\pi} N_f , \quad K \to \# \text{ as } N_f \to \infty$$

- any amplitude $\mathcal{A}$ can be expanded as

$$\mathcal{A}(K; p_i) = \mathcal{A}_0(K; p_i) + \frac{1}{N_f}\mathcal{A}_1(K; p_i) + \frac{1}{N_f^2}\mathcal{A}_2(K; p_i) + \dots$$

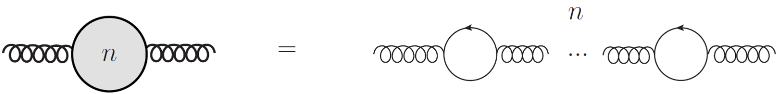
# Large-$N_f$ expansion

Counting the order in $1/N_f$:

I. diagram with $n_g$ gauge lines and $n_L$ fermion loops:

$$\mathcal{D}(n_g, n_L) \sim \left(\frac{1}{N_f}\right)^{n_g/2 - n_L}$$

II. for $n_g = 2$ and $n_L = 1$ we have $\mathcal{D}(2,1) \sim (1/N_f)^0$



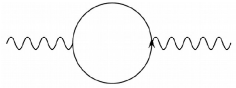

III. net effect:

$$\frac{1}{q^2} \to \frac{K^n \Pi_0^n}{(q^2)^{1+n\epsilon/2}}$$

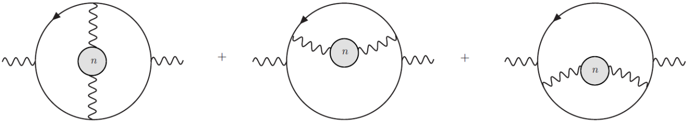
# Large-$N_f$ expansion

Consider the two-point function for the QED photon

I $\Pi_0(p)$: one-loop



II $\Pi_1(p)$: two basic topologies, all-order




III induced $1/N_f$ expansion for the $\beta$-function

$$\beta_K = \frac{2}{3}K^2 + \frac{1}{N_f}F_1(K) + \left\{ \frac{1}{N_f^2}F_2(K) + \ldots \right\}$$

# Resummation

Task: compute $F_1(K)$

$$Z_S = 1 - \frac{2}{3}\frac{K}{\epsilon} + \sum_{n=0}^{\infty} \mathrm{div}\left\{ \frac{K^{n+2}}{N_f}\left(1 - \frac{2}{3}\frac{K}{\epsilon}\right)^{-n} \Pi_1^{(n)}(p^2, \epsilon) \right\} + \mathcal{O}\left(\frac{1}{N_f^2}\right)$$

the $n$-bubble pieces $\Pi_1^{(n)}$ behave

$$\Pi_1^{(n)} \sim \frac{1}{n\epsilon^{n-1}} \sum_{j=0}^{\infty} \pi_j(p^2, \epsilon)(n\epsilon)^j$$

$\pi_j$ well-behaved in $\epsilon$; eventually, we have to resum

$$\sum_{n=0}^{\infty} K^{n+2} \mathrm{div}\left\{ \sum_{j=0}^{\infty} \frac{\pi_j(p^2, \epsilon)}{\epsilon^{n-j-1}} \sum_{k=0}^{n-2} \binom{n-2}{k} (n-k)^{j-1}(-1)^k \right\}$$

# Resummation

Euler's finite difference theorem (Planques-Mestre,Pascual 1984)

$$\sum_{k=0}^{n-2} \begin{pmatrix} n-2 \\ k \end{pmatrix} (n-k)^{j-1}(-1)^k = \begin{cases} \frac{(-1)^n}{n(n-1)} & j=0 \\ 0 & j \in (1, n-2) \\ a_{n,j}\, n! & j > n-2 \end{cases}$$

$j > n-2$ are finite terms in $\epsilon$ ($\Rightarrow$ *renormalons*)

$$Z_S = 1 - \frac{2}{3}K + \frac{1}{N_f} \sum_{n=2}^{\infty} \left(-\frac{K}{3}\right)^n \mathrm{div} \left\{ \frac{1}{\epsilon^{n-1}(n-1)n} \pi_0(\epsilon) \right\} + \mathcal{O}\left(\frac{1}{N_f^2}\right)$$

indeed, $\pi_0(p^2, \epsilon) \equiv \pi_0(\epsilon)$ independent of $p^2$

# Resummation

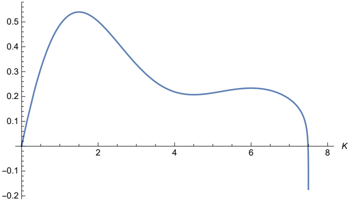
Only the $1/\epsilon$ part contributes to the $\beta$-function

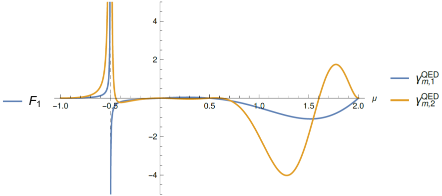
$$\sum_{n=1}^{\infty} \frac{K^{n-1}}{\epsilon^n} \pi_0(\epsilon)\Big|_{1/\epsilon} = \frac{1}{\epsilon} \sum_{n=0}^{\infty} K^n \pi_0^{(n)} = \frac{1}{\epsilon} \pi_0(K)$$

$$\sum_{n=1}^{\infty} \frac{K^n}{n\epsilon^n} \pi_0(\epsilon)\Big|_{1/\epsilon} = \frac{1}{\epsilon} \sum_{n=0}^{\infty} \frac{K^{n+1}}{n+1} \pi_0^{(n)} = \frac{1}{\epsilon} \int_0^K \pi_0(\epsilon) d\epsilon$$

the coupling $K$ and the dimension $d = 4 - \epsilon$ are somehow exchanged as final outcome of the large-$N_f$ resummation!

# Resumation




Result (QED):

$$\beta_K = \frac{2}{3}K^2 + \frac{1}{2N_f}\int_0^K I_1(t)\mathrm{d}t, \quad I_1(t) = \frac{(1-t)(1-t/3)(1+t/2)\Gamma(4-t)}{3\Gamma^2(2-t/2)\Gamma(3-t/2)\Gamma(1+t/2)}$$

and similarly for $\gamma_m$, known up to $\mathcal{O}(1/N_f^2)$

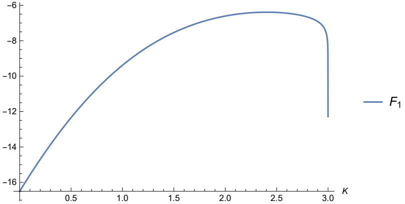
$\Rightarrow$ both $\beta$ and $\gamma_m$ singular at $K = 15/2$ (and more other points)

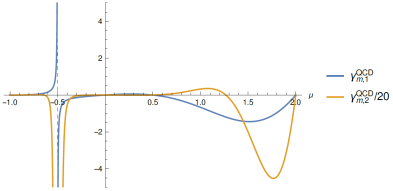
# Resummation

by means of bubble resummation, similar results have been obtained in a closed form for more and more theories:

- semi-simple gauge groups [1803.09770]
- gauge-Yukawa [1712.06859,1808.03252]
- Gross-Neveu-Yukawa [1806.06954]
- ...

# what about QCD?




gauge propagator dressed by fermion bubbles as in QED, but more basic topologies due to non-abelian vertices (*double chains*)

direct resummation impossible, results from critical point method by J. Gracey & B. Holdom

$\Rightarrow \beta$ has an early pole at $K = 3$, $\gamma_m$ at $K = 15/2$ as QED

# Critical point method

Bubble resummation: net effect dimension $\epsilon \rightarrow K$;
$\epsilon$ and $K$ are indeed related from the start in the context of critical point method.

This formalism, developed by A.N. Vasil'ev and J. Gracey, exploits conformal properties of the theory in arbitrary dimension close to the Wilson-Fisher fixed point.

Universality is used to connect theories in the same class (e.g., QCD and non-abelian Thirring Model at large $N_f$)

# Critical point method

In arbitrary dimension $d = d_c - \epsilon$, the $\beta$-function for a one-coupling theory is:

$$\beta(g) = -\epsilon g + b g^2 + \dots$$

the critical coupling $g_c$ at the WF fixed point satisfies

$$\beta(g_c) = 0 \Leftrightarrow g_c = \frac{\epsilon}{b} + \dots$$

which signals a phase transition whose properties are encoded in the critical exponents, e.g.

$$\omega = \beta'(g_c), \quad \eta = \gamma_\phi(g_c)$$

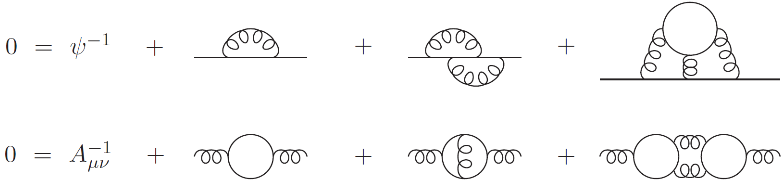
# Critical point method

The exponents $\omega, \eta$ are computed by:

I making a scaling ansatz for the propagators at the WF fixed point

$$\psi \sim A \frac{\not{p}}{(p^2)^{d/2-\alpha+1}}, \quad A_{\nu\sigma} \sim \frac{B}{(p^2)^{\mu-\beta}}$$

II solving the Schwinger-Dyson equation at large $N_f$, which yelds algebraic equations for the critical exponents ($d$ only variable)




III using the relations among the different exponents

# Hvar 2018

# Some questions

- How can we systematically translate critical exponents to particle physics (BSM) language? Critical point method more powerful but need to go through Gracey's literature...
- Is the knowledge of the critical exponents always enough to reconstruct the $\beta$-function?
- Large-$N_f$ functions – as $F_1$ – show poles; are higher orders singularities appearing at the same point?
- If yes, is it possible to resum them?

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# Bubble-resummation and critical-point methods for $\beta$-functions at large $N$

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# Critical Look at $\beta$-Function Singularities at Large $N$

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# One coupling

One-coupling theory:

$$\beta'(g_c) = \omega(d) \equiv -(d - d_c) + \sum_{n=1}^{\infty} \frac{1}{N^n} \omega_n(d)$$

ansatz for the $\beta$-function ($K = gN$):

$$\beta(g) = (d - d_c)g + g^2 \left( b\, N + c + \sum_{n=1}^{\infty} \frac{F_n(gN)}{N^{n-1}} \right)$$

the relation at the WF fixed point $\beta(g_c) = 0$ implies

$$d = d_c - g_c \left( b\, N + c + \sum_{n=1}^{\infty} \frac{F_n(g_c N)}{N^{n-1}} \right)$$

# One coupling

The relation $\beta'(g_c) = \omega\,[d]$ then implies

$$t^2 \sum_{n=1}^{\infty} \frac{F_n'(t)}{N^n} = \sum_{n=1}^{\infty} \frac{1}{N^n}\omega_n \left[ d_c - t \left( b + \frac{c}{N} + \sum_{n=1}^{\infty} \frac{F_n(t)}{N^n} \right) \right]$$

By comparing LH and RH sides, one finds:

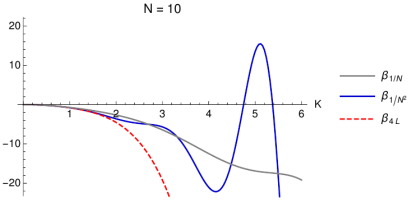
$$F_1(K) = \int_0^K \mathrm{d}t \frac{\omega_1 \left( d_c - bt \right)}{t^2},$$

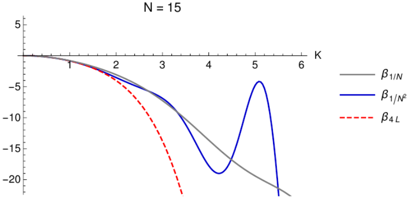
$$F_2(K) = \int_0^K \mathrm{d}t \left( \frac{c + F_1(t)}{b} \left( tF_1''(t) + 2F_1'(t) \right) + \frac{\omega_2 \left( d_c - bt \right)}{t^2} \right)$$

# One coupling

application: Gross-Neveu model

- interest in a possible IR fixed point
- no hint at LO, $\mathcal{O}(1/N_f)$ and w/ Pade' approximants
- by exploiting $\omega^{(2)}$, we find no hint also at $\mathcal{O}(1/N_f^2)$



# One coupling

To summarize,

- the slope of the $\beta$-function at criticality can be expanded and matched with $\omega$ order by order in $1/N_f$
- this procedure results in simple differential equations that can be solved iteratively
- the set of $F_n$'s is fully determined by the $\omega_n$'s
- $F_1$ contributes to all $F_n$'s: implication for the pole structure in higher order terms

# Two couplings

Gross-Neveu-Yukawa (GNY):

- GNY is the bosonised Gross-Neveu

$$\mathcal{L}_{\mathrm{GNY}} = \bar{\psi} \imath \partial\!\!\!/ \psi - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + g_1 \phi \bar{\psi} \psi + g_2 \phi^4 .$$

- rescaled couplings

$$y \equiv \frac{g_1^2 \mu^\epsilon}{8\pi^2}, \quad K \equiv 2yN, \quad \lambda \equiv \frac{g_2 \mu^\epsilon}{8\pi^2}.$$

- $\omega \to \omega^\pm(d)$, the eigenvalues of Jacobian $\beta(y_c, \lambda_c)$ at the two-dimensional WF fixed point
- known up to $\mathcal{O}(1/N_f^2)$, $\omega^\pm$ suggest a shrinking in the radius of convergence moving from $1/N_f \to 1/N_f^2$

# Two couplings

our ansatz at $\mathcal{O}(1/N_f)$ contains four unknown functions $F_{1,2,3,4}$:

$$\begin{aligned}\beta_y =&(d-d_c)y + y^2(2N+3+F_1(yN))\\ \beta_\lambda =&(d-d_c)\lambda + y^2(-N+F_2(yN))\\ &+\lambda^2(36+F_3(yN)) + y\lambda(4N+F_4(yN))\end{aligned}$$

$\omega^\pm$ provide two constraints:

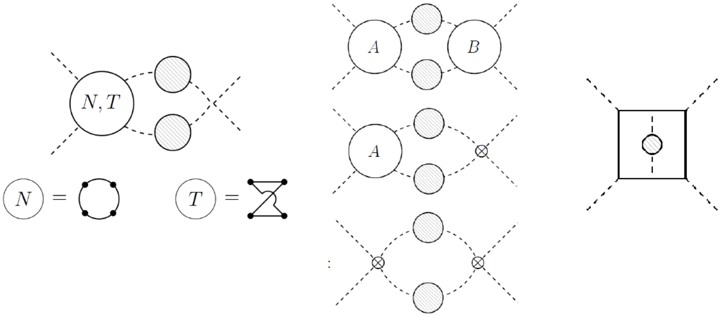
$$F_1(t) = \int_0^t \frac{\tilde{\omega}_-^{(1)}(2\epsilon)}{\epsilon^2}\mathrm{d}\epsilon,$$

$$30 - 2F_1(\epsilon/2) + F_3(\epsilon/2) + F_4(\epsilon/2) = 2\frac{\tilde{\omega}_+^{(1)}(\epsilon)}{\epsilon}.$$

$\Rightarrow$ critical exponents not enough to compute $\beta_\lambda$

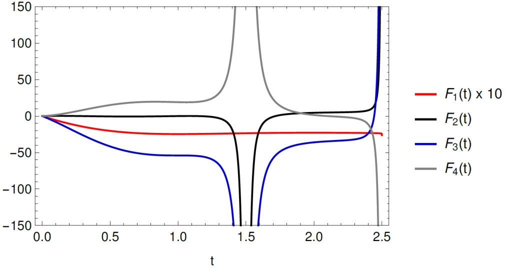
# Two couplings

To compute $\beta_\lambda$, we have to rely on bubble resummation:




- first time resummation with three-loop basic topology
- double chains can be effectively reduced to single chain

# Two couplings

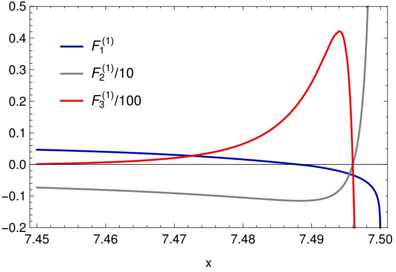



- we have computed the full system of $\beta$-functions at $\mathcal{O}(1/N_f)$
- radius of convergence does not shrink from $1/N_f \to 1/N_f^2$: the "new" pole is already there at $1/N_f$, it just cancels in the LHS of

$$30 - 2F_1(\epsilon/2) + F_3(\epsilon/2) + F_4(\epsilon/2) = 2\frac{\tilde{\omega}_+^{(1)}(\epsilon)}{\epsilon}$$

We find the following structure:

| | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\dots$ |
|---|---|---|---|---|
| $F_1$ | $F_1^{(1)}$ | | | |
| $F_2$ | $F_2^{(1)}$ | $F_2^{(2)}$ | | |
| $F_3$ | $F_3^{(1)}$ | $F_3^{(2)}$ | $F_3^{(3)}$ | |
| $\vdots$ | | | | $\ddots$ |




$$F_{n>1}^{(1)}(x) = \int_0^x \frac{dt}{t^2} \sum_{\ell=1}^{n-1} \frac{1}{\ell!} c_{n-\ell-1}^{(\ell)} \left(\frac{t}{b}\right)^{\ell} \frac{d^{\ell}}{dt^{\ell}} \left[t^2 F_1'(t)\right]$$

alternating poles!

# A critical look

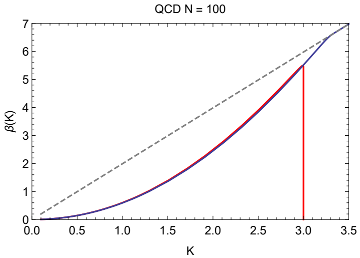
Is it possible to resum all $F_n^{(1)}$'s (first column)?

$$\mathcal{F}^{(1)} = \sum_{n=1}^{\infty} \frac{F_n^{(1)}(x)}{N^{n-1}}$$

$$\partial_x \mathcal{F}^{(1)} = \frac{1}{x^2}\omega_1 \left( d_c - x \left( b + \frac{c + \mathcal{F}^{(1)}}{N} \right) \right)$$

$$\mathcal{F}^{(1)} = N\left(\frac{a}{x} - b\right) - c, \quad x \gtrsim x_s$$

$$aN = -\omega_1(d_c - a)$$




- naked singularity in $F_1$ is removed and appears as a large-$N$ behaviour
- cancellation happens across different orders in $1/N$, how come? because $\omega(d)$ is truly a function of *one* variable

# A critical look

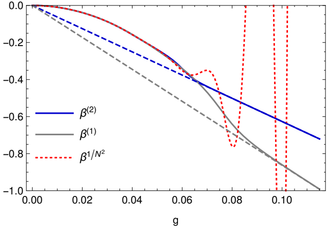
What happens if we add $\omega_2$ (second column)?

- closest singularity comes from $\omega_2$ and is positive: Landau pole, the scalar $O(N)$ model is an example of such
- otherwise: $a'N = -\omega_1(d_c - a') - \frac{1}{N}\omega_2(d_c - a')$

one does not need to know the exact form of $\omega_2$!

Gross-Neveu – we do know $\omega_2$:

- no singularities but wild oscillations
- our resummation gets rid of unphysical fixed points



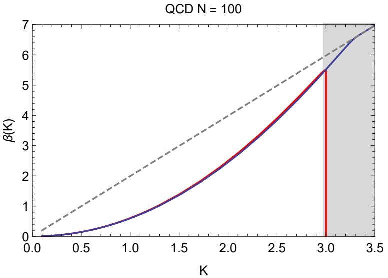
# A critical look

To summarize:

- summing along the columns removes all (negative) singularities; this statement does not require the exact knowledge of $\omega_n$'s
- looking at the rows, singularities will be encountered;

the comparison tells where one *should not* trust the calculation: higher orders are not subleading

| | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\dots$ |
|---|---|---|---|---|
| $F_1$ | $F_1^{(1)}$ | | | |
| $F_2$ | $F_2^{(1)}$ | $F_2^{(2)}$ | | |
| $F_3$ | $F_3^{(1)}$ | $F_3^{(2)}$ | $F_3^{(3)}$ | |
| $\vdots$ | | | | $\ddots$ |



# Conclusion

- we have provided a dictionary between bubble-resummation and critical point methods for RG-functions at large $N_f$
- for one-coupling theory, the critical exponent $\omega$ contains all the information about $\beta$, this is no longer true for two-coupling
- we have discussed the structure of higher orders in the $\beta$-function and quantitatively shown that they are not subleading
- these terms can be resummed and the original singularity is removed: no hint for a fixed point
- what is the connection with renormalons? $n!$ behaviour found in the finite parts