Towards the gauge beta function at $\mathcal{O}(1/N_f^2)$ and $\mathcal{O}(1/N_f^3)$

Manuel Reichert

Bridging perturbative and non-perturbative physics, Primosten, 07. October 2019

CP^3-Origins, SDU Odense, Denmark

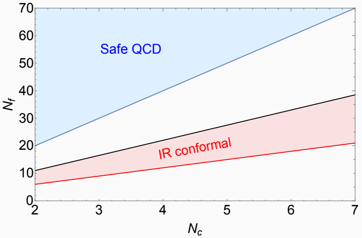
Nicola Dondi, Gerald Dunne, MR, Francesco Saninno: arXiv:1903.02568
Nicola Dondi, MR, Francesco Saninno: in preparation

Which matter systems are asymptotically safe in $d = 4$?

- Gauge-Yukawa theories at large N_f & N_c (perturbatively) [Litim, Sannino '14]
- How far does this extend to small N_c?
- Test gauge theories at large N_f non-perturbatively

Standard QCD picture:

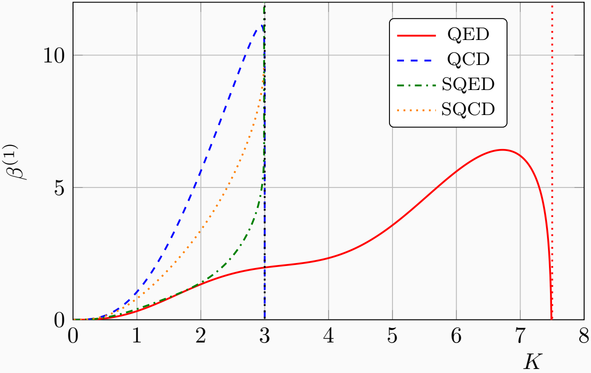
- Small N_f: asymptotic freedom & confinement in the IR
- Medium N_f: asymptotic freedom & IR Banks-Zaks fixed point
- Large N_f: asymptotic freedom lost $\rightarrow$ asymptotic safety?



[Antipin, Sannino '17]

Beta functions of (S)QED and (S)QCD

$$\beta(K) = \beta^{(0)}(K) + \frac{\beta^{(1)}(K)}{N_f} + \dots$$



UV fixed point for QED & QCD

Landau pole for SQED & SQCD

How physical are these fixed points?

- The fermion mass anomalous dimension goes to zero in QCD and to infinity in QED
  [Antipin, Sannino '17]
- Hints for FP in QCD at medium N_f from resummations with Meijer G-functions
  [Antipin, Maiezza, Vasquez '18]
- Lattice studies inconclusive so far
  [Leino, Rindlisbacher, Rummukainen, Sannino, Tuominen '19]
- Poles might be resummable within the $1/N_f$ series
  [Alanne, Blasi, Dondi '19]

How to go beyond $1/N_f$

- The next orders in the $1/N_f$ expansion would test the physical nature of the FP
- No known resummation formula for two bubble-chains, needed for $1/N_f^2$ and higher orders
- Can we extract the location of the pole, the residuum, etc., with a finite amount of coefficients?

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Two methods:

- Large-order behaviour of expansion coefficients
- Padé approximants

Large-order behaviour: Darboux's Theorem

The nearby singularity determines the large-order growth of the expansion coefficients a_n. E.g. for expansion around $z = 0$

- pole of order p at z_0 ($f(z) \sim \phi(z)(1 - z/z_0)^p + \text{finite}$)

$$a_n \sim \frac{1}{z_0^n} \binom{n+p-1}{n} \phi(z_0) + \ldots$$

- logarithmic branch cut at z_0 ($f(z) \sim \phi(z) \ln(1 - z/z_0) + \text{finite}$)

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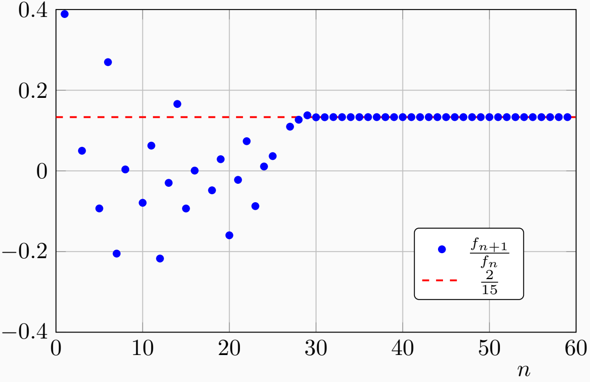
Expectation for QED $F_{\text{QED}} = \sum_n f_n x^n$

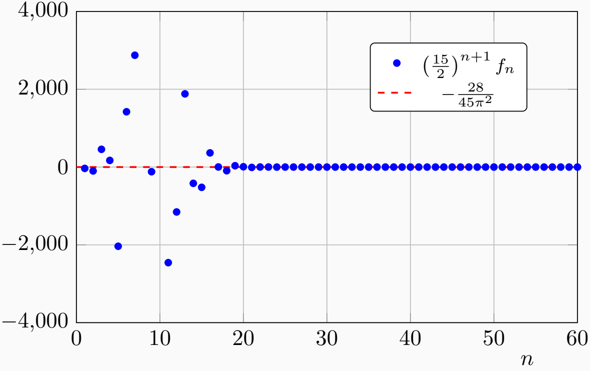
$$f_n \sim \left[R_0 \left(\frac{2}{15} \right)^n + R_1 \left(\frac{2}{21} \right)^n + R_2 \left(\frac{2}{27} \right)^n + \dots \right]$$

Large-order behaviour of F_{QED}



Ratio test $\frac{f_{n+1}}{f_n}$ reveals location of the first pole

Large-order behaviour of F_{QED}



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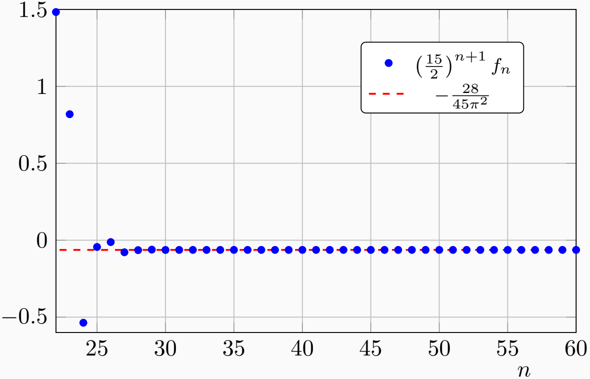
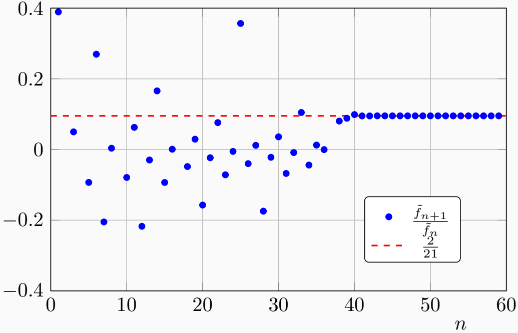


With the knowledge of the pole the residuum is computable

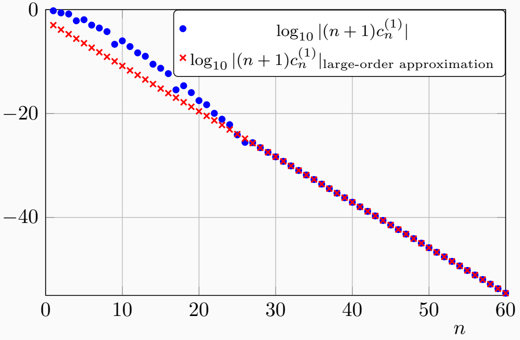
Large-order behaviour of F_{QED}



Subtracting the first pole reveals the second pole

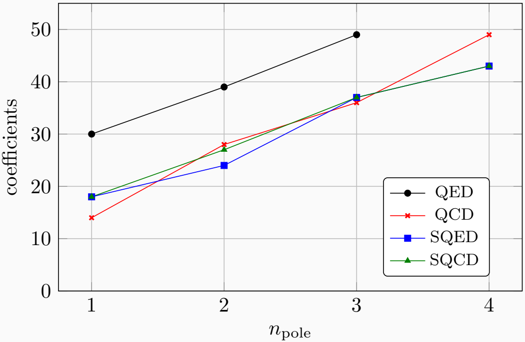
$$\tilde{f}_n = f_n + \frac{28}{45\pi^2}\left(\frac{15}{2}\right)^{-n-1}$$

Large-order behaviour of F_{QED}



After ~ 30 terms the large-order behaviour sets in
(for subleading behaviour later)

How many coefficients are needed?



"Closer" to the origin → less coefficients are needed

Padé methods

Analytic continuation of truncated Taylor series by ration of two polynomials

$$F_{\mathrm{QED}}(x) \approx \sum_{n=0}^{M} f_n \, x^n \quad \longrightarrow \quad \mathcal{P}^{[R,S]}(x) = \frac{P_R(x)}{Q_S(x)}$$

with $R + S = M$.

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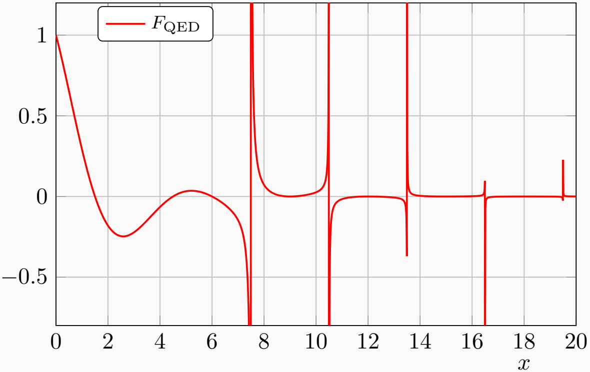
Rewriting of resummed $F_{\mathrm{QED}}(x)$

$$F_{\mathrm{QED}}(x) \sim \frac{\Gamma(1+\frac{x}{3})}{\Gamma(\frac{1}{2}+\frac{x}{3})} \frac{\sin^2\left(\frac{\pi x}{3}\right)}{\cos\left(\frac{\pi x}{3}\right)}$$

Padé approximant with $2R \approx S$ should lead to best results.

Padé approximants of F_{QED}



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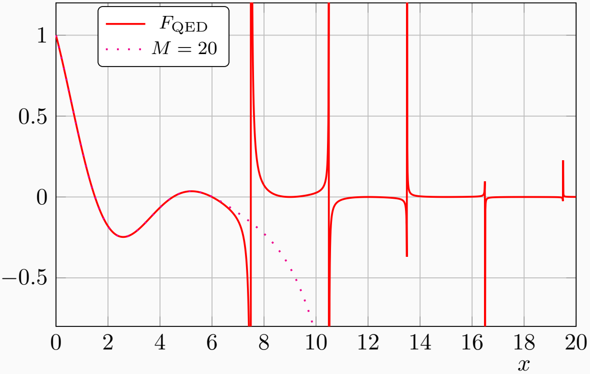

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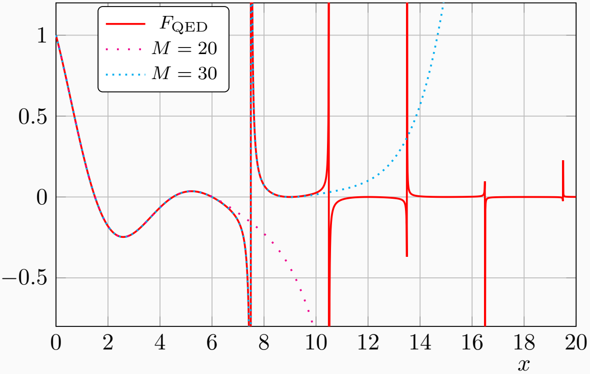

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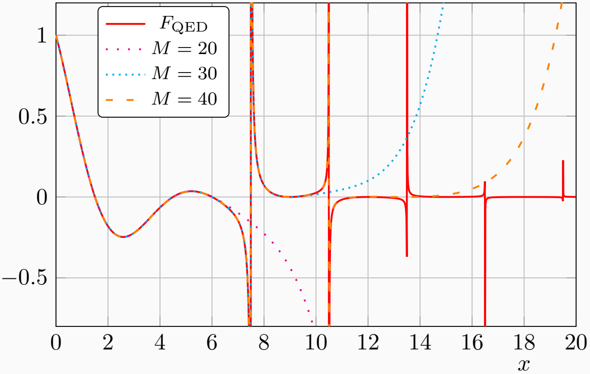

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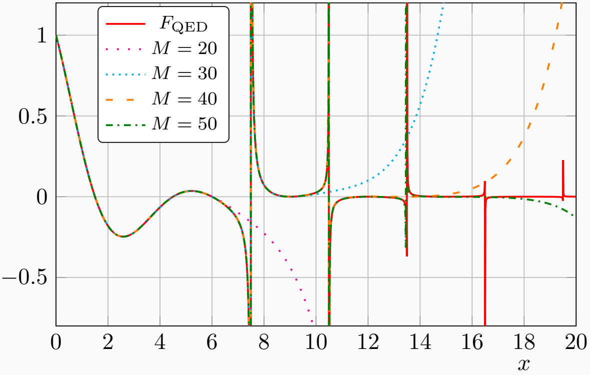
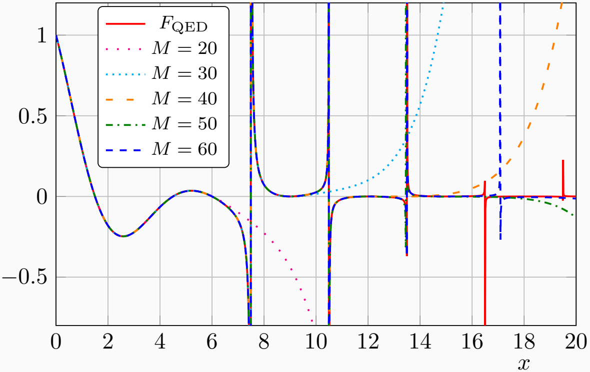
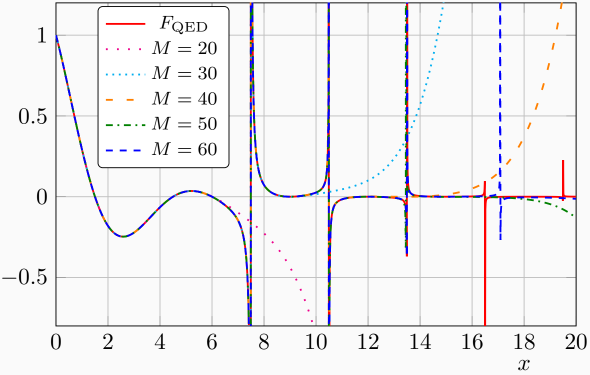




F_{QED}
$M = 20$
$M = 30$
$M = 40$
$M = 50$
$M = 60$
1
0.5
0
−0.5
0
2
4
6
8
10
12
14
16
18
20
x

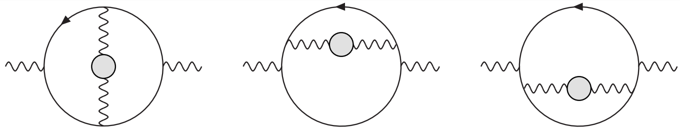
Padé approximants of F_{QED}



- Need ~ 30 coefficients to resolve first singularity (similar to large-order growth analysis)
- Can resolve function beyond the first singularity

QED beta function

$1/N_f$:



QED beta function

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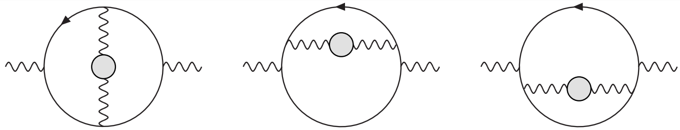
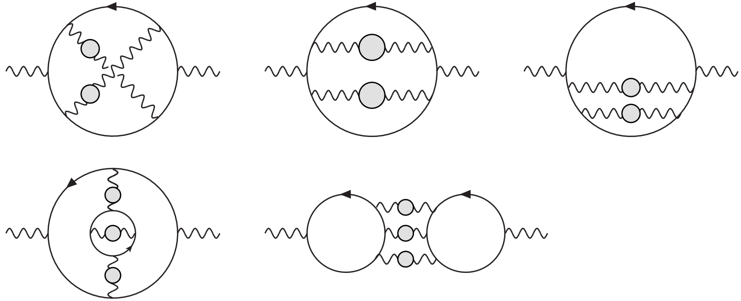


$1/N_f^2$ (subset):



QED beta function

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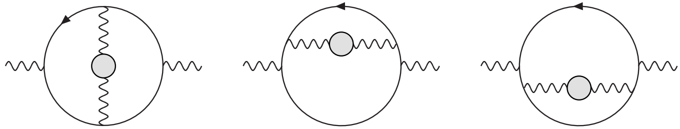

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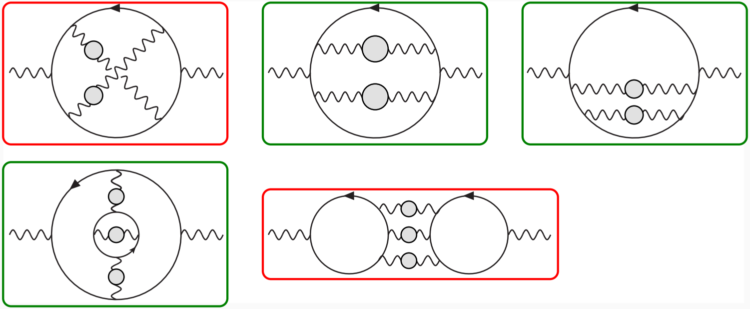
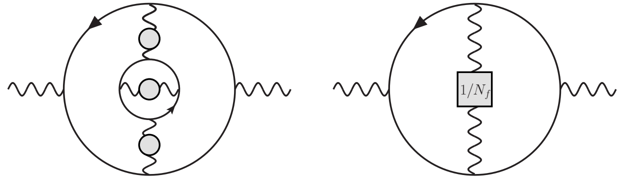


Master integral known / not know

Beyond $1/N_f$: nested diagrams

Nested sub-part of beta function: gauge & RG scale independent



Computation up to K^{44}

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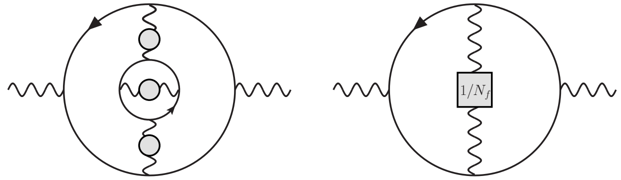
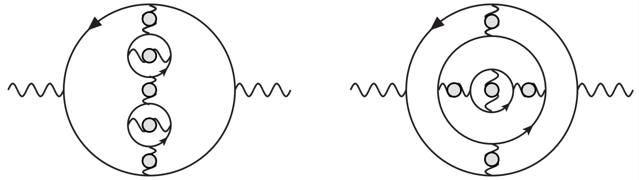


Computation up to K^{44}

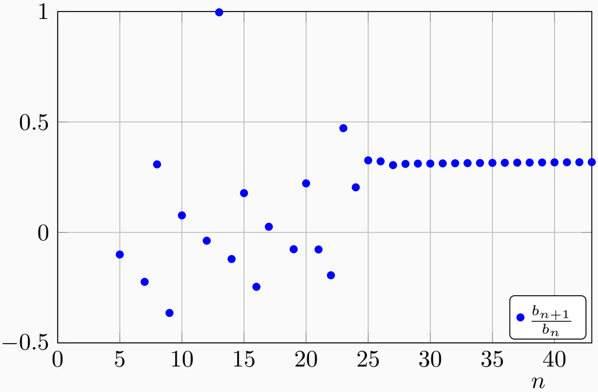
At $\mathcal{O}(1/N_f^3)$



Computation up to K^{32}

Ratio test at $\mathcal{O}(1/N_f^2)$

$$\beta^{(2)}_{\text{nested}} = \sum_n b_n K^n$$



New finite radius of convergence

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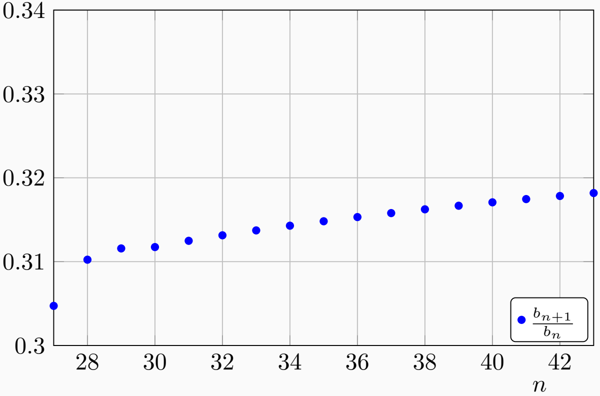

New finite radius of convergence
but extreme slow convergence

Richardson extrapolation

Enhance the convergence of the series

$$a_n = s + \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n^3} + \dots$$

Richardson extrapolation

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First Richardson ($B = C = \ldots = 0$)

$$\mathrm{R}^{(1)} a_n \equiv s = (n+1)a_{n+1} - n a_n$$

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Second Richardson ($C = \ldots = 0$)

$$\mathrm{R}^{(2)} a_n \equiv s = \frac{1}{2}\left((n+2)^2 a_{n+2} - 2(n+1)^2 a_{n+1} + n^2 a_n\right)$$

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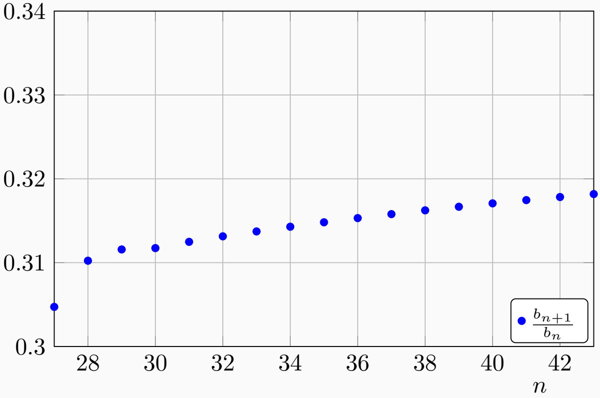
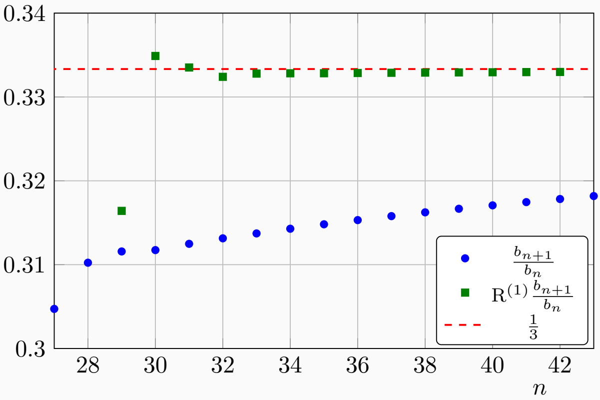
For oscillating series: Shanks transformation

Ratio test at $\mathcal{O}(1/N_f^2)$



Bare series: $K^* = 3.14$

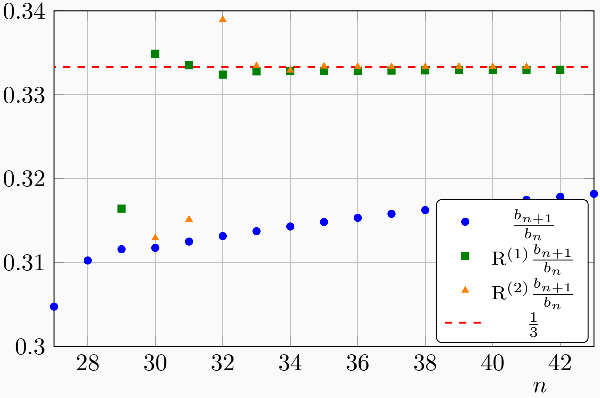
Ratio test at $\mathcal{O}(1/N_f^2)$



Bare series: $K^* = 3.14$

First Richardson: $K^* = 3.003$

Ratio test at $\mathcal{O}(1/N_f^2)$

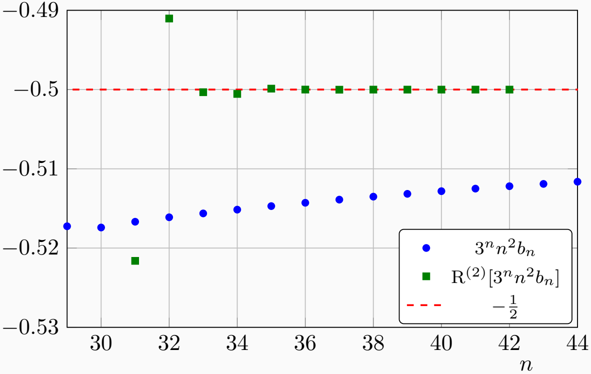


Bare series: $K^* = 3.14$

First Richardson: $K^* = 3.003$

Second Richardson: $K^* = 3.00008$

Residue at $\mathcal{O}(1/N_f^2)$



Bare series: $3^n n^2 b_n = -0.512$

Second Richardson: $3^n n^2 b_n = -0.500007$

Subleading behaviour

$$\tilde{b}_n = b_n + \frac{1}{2}\frac{1}{3^n}\frac{1}{n^2}$$

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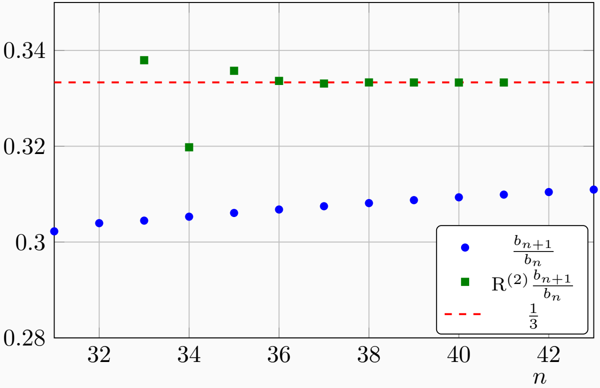
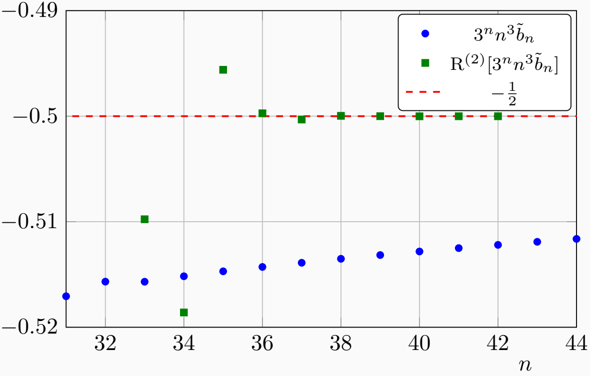


Bare series: $K^* = 3.215$

Second Richardson: $K^* = 3.0003$

Subleading behaviour

$$\tilde{b}_n = b_n + \frac{1}{2}\frac{1}{3^n}\frac{1}{n^2}$$



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Second Richardson: $3^n n^3 \tilde{b}_n = -0.500007$

Large-order behaviour

$$\begin{aligned} b_n &\sim -\frac{1}{2}\frac{1}{3^n}\left(\frac{1}{n^2}+\frac{1}{n^3}+\dots\right)+\mathcal{O}\left(\frac{1}{(x>3)^n}\right) \\ &= -\frac{1}{2}\frac{1}{3^n}\frac{1}{n(n-1)} \end{aligned}$$

Large-order behaviour

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$$= -\frac{1}{2}\frac{1}{3^n}\frac{1}{n(n-1)}$$

Resummation

$$\sum_{n=4}^{\infty} b_n K^n \sim \frac{1}{6}(K-3)\ln\left(1-\frac{K}{3}\right)+\text{finite}$$

Large-order behaviour

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Resummation

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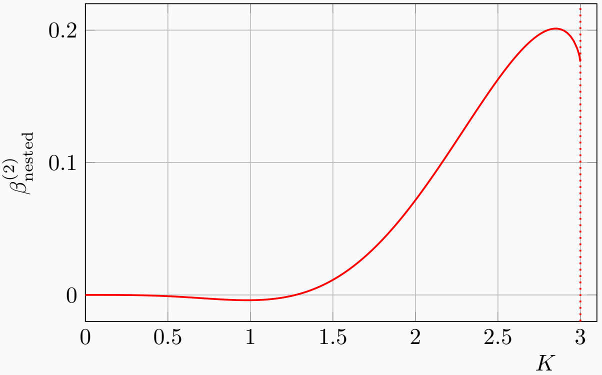
Logarithmic branch cut but no pole!

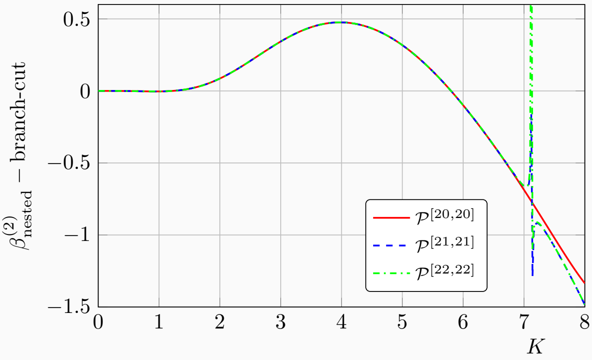
Nested beta function at $1/N_f^2$



"Exact" nested beta function up to $K = 3$

Beta function ambiguous beyond $K = 3$ or magic cancellation needed

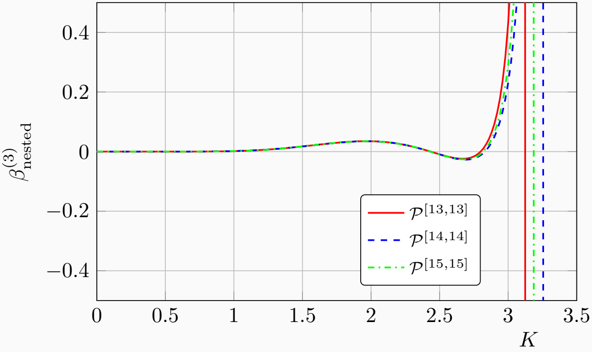
Nested beta function at $1/N_f^2$ beyond the first branch cut



No singularity before $K = 15/2$

Positive pole at $K = 15/2$?

Nested beta function at $1/N_f^3$



No singularity before $K = 3$

Branch cut at $K = 3$?

Outlook 1: Conformal Padé

[Costin, Dunne '19]

Can we do more with the coefficients that we have?

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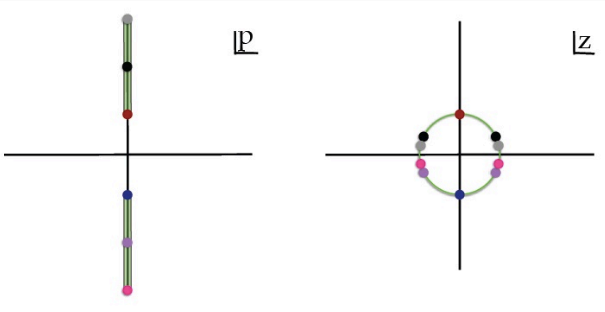
[Costin, Dunne '19]

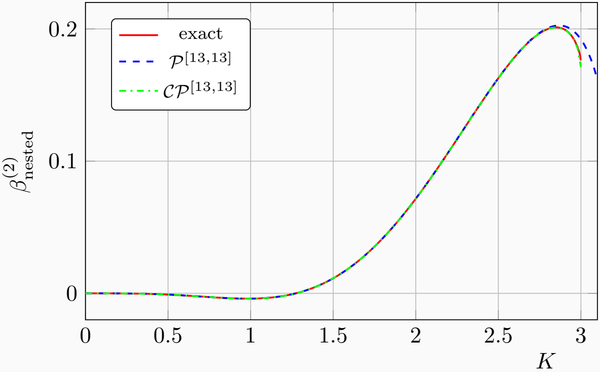
Can we do more with the coefficients that we have?

Conformal map:

$$K = \frac{6z}{1+z^2} \qquad \longleftrightarrow \qquad z = \frac{K/3}{1+\sqrt{1-K^2/9}}$$



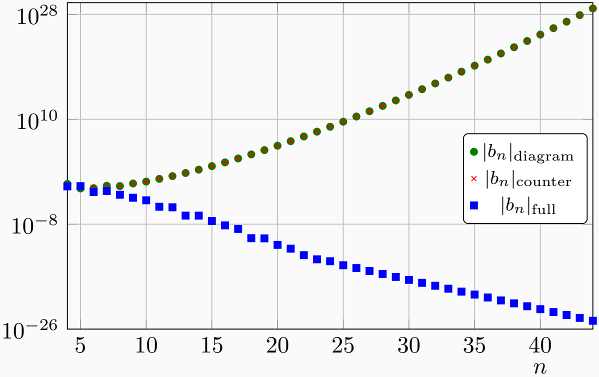
Outlook 1: Conformal Padé



Improvement over standard Padé

Requires knowledge on the location of the branch cut

Outlook 2: Renormalons

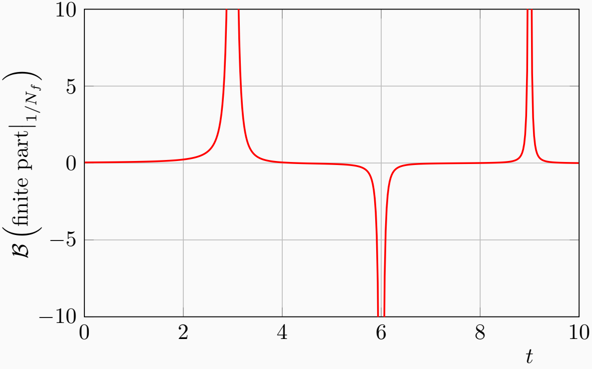


Two factorially divergent contributions but the sum goes to zero

Are we picking up renormalon contributions?

Outlook 2: Renormalons

Borel transform of the finite part of $1/N_f$



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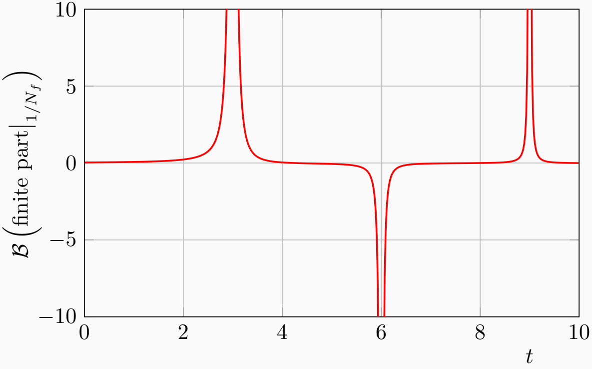
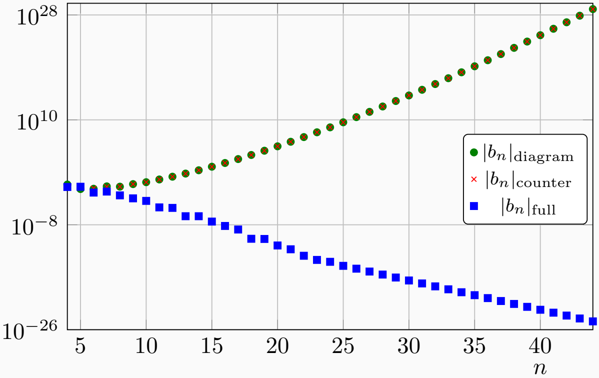


Do we pick up the renormalon at $t = 3$?

Why not the renormalon at $t = 6$?

Outlook 2: Renormalons



Large-order behaviour

$$a_n \sim \frac{n!}{n^3\,3^n}\left(-3 - 9\frac{1}{\ln(n)^3}\right) + \dots$$

Summary and outlook

- Large-order behaviour & Padé methods constitute powerful tools
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Thank you for your attention