Exploring the Large-N a-theorem through Dilaton Scattering

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- A-theorem is a constraining tool for RG flows, also beyond perturbation theory.
- Large-N methods can produce new results concerning conformal anomalies.
- Usually, the a-theorem is valid within perturbative RG flows. We aim at find non-trivial counter-examples in Large-N models.

Based on:

O. Antipin, NAD, F. Sannino, A. E. Thomsen [1808.00482]
NAD, F. Sannino, A. E. Thomsen... [Ongoing]

- *weak version*: It exists a quantity "a" defined at a CFT such that every RG flow between two CFTs (IR/UV) satisfies $a_{UV} > a_{IR}$.

[Cardy, '88] [Komargodski, Schwimmer '11]

- *strong version*: For every QFT it exist a function of the couplings $\tilde{a}(g)$ such that
  - $\tilde{a}(g)$ monotonically decreases along RG flows
  - at a fixed point it satisfies $\tilde{a}(g^*) = a$

[Komargodski, Schwimmer '11] [Jack, Osborn '90]

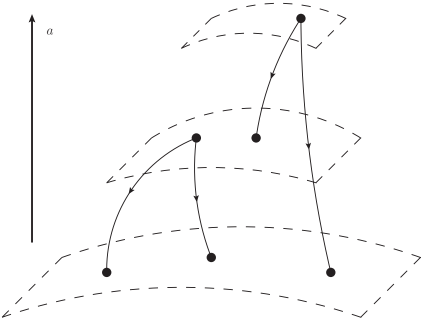
- *gradient flow*: the quantity $\tilde{a}$ satisfies an equation of the form:

$$\partial_i \tilde{a} = \chi_{ij} \beta^j \implies \mu \frac{d\tilde{a}}{d\mu} = \beta_i \partial_i \tilde{a} = \chi^g_{ij} \beta_i \beta_j$$

the strong version: χ_{ij} to be symmetric and positive definite.

[Jack, Osborn '90]

"a" is a measure of effective degrees of freedom in a CFT: RG flows are irreversible

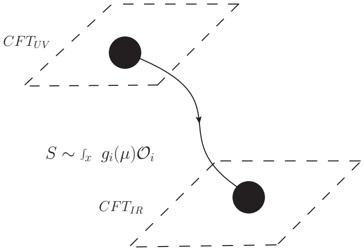


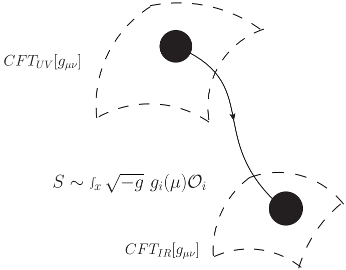
- $D.o.F(UV) > D.o.F(IR)$
- Theory space has a foliated structure
- Weyl consistency conditions

[Jack, Poole '14]

[Antipin *et al.* '13]

[Poole, Thomsen '19]





$T^\mu_\mu \supset \beta_i \mathcal{O}^i$ $\qquad$ $T^\mu_\mu \supset \mathcal{A}\,\mathbb{I} + \beta_i \mathcal{O}^i$

Weyl symmetry is broken by an additional c-number anomaly:

$$\mathcal{A} = c\,W^2 - a\,E_4; \quad a \sim \int_{S^d} d^d x \langle T^\mu_\mu \rangle$$

Proven to satisfy $a_{UV} > a_{IR}$.

Motivation for local renormalization group (LRG):

- It gives $\tilde{a}$ as the loop-corrected E_4 coefficient.
- $\tilde{a}$ automatically satisfies a gradient flow equation.
- It relates curved space anomalies to flat space divergences of correlators.

[Jack, Osborn ’90 ’13]

[Baume, Keren-Zur, Rattazzi, Vitale ’14]

Consider the connected vacuum functional $\mathcal{W}$:

$$e^{i\mathcal{W}[\gamma,g]} = \mathcal{N}\int \mathcal{D}\phi\, e^{iS_{CFT}[\gamma_{\mu\nu},\phi]+i\int_x g_0(x)\cdot\mathcal{O}[\phi]}, \quad \int_x \equiv \int \mathrm{d}^4x\sqrt{-\gamma}.$$

The theory is regularized in $d = 4-\epsilon$, and all CTs are in MS scheme

$$g_0(x) \to g(x,\mu) \qquad S \to S + S_{c.t.}[\gamma, g].$$

If correclty renormalised, this generates connected green's functions:

$$\mathcal{W}[\gamma, J, \mu] = \sum_n \frac{1}{n!}\int_{\{x_i\}} g(x_1,\mu)...g(x_n,\mu)\langle \mathbf{T}\{\mathcal{O}(x_1)...\mathcal{O}(x_n)\}\rangle_{R,\gamma}.$$

$$\text{Exact} \quad SO(d,2) \xrightarrow{\eta_{\mu\nu}\to\gamma_{\mu\nu}} \text{Diff}(d)\times\text{Weyl} \quad \sim \text{Anomalous}$$

Each symmetry acts on metric and CFT operators as:

$$\text{Weyl:} \quad \gamma_{\mu\nu} \to e^{-2\sigma(x)}\gamma_{\mu\nu}, \quad \mathcal{O} \to e^{\sigma(x)\Delta}\mathcal{O},$$
$$\text{Diff:} \quad \gamma_{\mu\nu}(x) \to \partial_\mu\xi^\sigma\partial_\nu\xi^\rho\gamma_{\sigma\rho}(\xi^{-1}(x)), \quad \mathcal{O}(x) \to \mathcal{O}(\xi^{-1}(x)).$$

Of course, we avoid Diff(d) anomalies,

$$\Delta_\sigma W = \int_x \mathcal{A}_\sigma[\gamma, g]$$
$$\Delta_\xi W = 0.$$

Notice: Weyl anomalies are related to scale anomalies since

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}W = \int_x \mathcal{A}_{\sigma=-1}[\gamma,g] \begin{cases} = 0 \quad \text{and} \quad \mathcal{A}_{\sigma=-1} = 0 \implies \text{no anomaly} \\ = 0 \quad \text{and} \quad \mathcal{A}_{\sigma=-1} \neq 0 \implies \text{Type A anomaly} \\ \neq 0 \quad \implies \text{Type B anomaly} \end{cases}$$

[S. Deser, A. Schwimmer '93]

The source transformation implementing Weyl is fixed:

$$\Delta_\sigma \gamma_{\mu\nu} = -2\sigma\gamma_{\mu\nu}, \quad \Delta_\sigma g_i = -\sigma\hat{\beta}^i \quad \text{where} \quad \hat{\beta}^i = -\rho^i g^i \epsilon + \beta^i(g)$$

This leaves the action invariant, apart from $S_{c.t}$:

$$\Delta_\sigma W[g_i, \gamma_{\mu\nu}] = \int_x \sigma \left(2\gamma^{\mu\nu} \frac{\delta}{\delta\gamma^{\mu\nu}} - \hat{\beta}^i \frac{\delta}{\delta g_i} \right) S_{c.t.} \equiv \int_x \mathcal{A}_\sigma[\gamma_{\mu\nu}, g_i]$$

At the starting CFT the anomaly reads:

$$\mathcal{A}_\sigma[\gamma_{\mu\nu}, 0] = \sigma\{c\, W^2 - a\, E_4\} + O(\partial\sigma).$$

We can consider an ansatz for $S_{c.t}$ containing all possible tensor structures:

$$S_{c.t} \supset \int_x \sqrt{\gamma}\, \mu^{-\epsilon} \Big\{ \lambda_a E_4 + \tfrac{1}{2} \mathcal{G}_{ij} \partial_\mu g^i \partial_\nu g^j G^{\mu\nu} + \tfrac{1}{2} \mathcal{A}_{ij}\, \nabla^2 g^i \nabla^2 g^j \\ + \tfrac{1}{2} \mathcal{B}_{ijk}\, \partial_\mu g^i \partial^\mu g^j \nabla^2 g^k \Big\} .$$

This can be used to obtain a formal expression for $\Delta_\sigma W$. A similar expansion has to be present for the RHS:

$$\mathcal{A}_\sigma[\gamma_{\mu\nu}, g_i] \supset \tilde{a} E_4 + \tfrac{1}{2} \chi^g_{ij} \partial_\mu g^i \partial_\nu g^j G^{\mu\nu} + \tfrac{1}{2} \chi^a_{ij}\, \nabla^2 g^i \nabla^2 g^j \\ + \tfrac{1}{2} \chi^b_{ijk}\, \partial_\mu g^i \partial^\mu g^j \nabla^2 g^k + O(\partial\sigma) .$$

anomaly coefficient need to match some c.t. combinations. On top of that, the anomaly is by definition a *finite* functional!

- Matching $O(\sigma)$:

$$\chi^g_{ij} = \left(\epsilon - \hat{\beta}^\ell \partial_\ell\right) \mathcal{G}_{ij} - \mathcal{G}_{\ell j}\partial_i \hat{\beta}^\ell - \mathcal{G}_{i\ell}\partial_j \hat{\beta}^\ell$$
$$\chi^a_{ij} = \left(\epsilon - \hat{\beta}^\ell \partial_\ell\right) \mathcal{A}_{ij} - \mathcal{A}_{\ell j}\partial_i \hat{\beta}^\ell - \mathcal{A}_{i\ell}\partial_j \hat{\beta}^\ell$$
$$\chi^b_{ijk} = \left(\epsilon - \hat{\beta}^\ell \partial_\ell\right) \mathcal{B}_{ijk} - \mathcal{B}_{\ell jk}\partial_i \hat{\beta}^\ell - \mathcal{B}_{i\ell k}\partial_j \hat{\beta}^\ell - \mathcal{B}_{ij\ell}\partial_k \hat{\beta}^\ell - 2\partial_i\partial_j \hat{\beta}^\ell \mathcal{A}_{\ell k}, \quad [...]$$

- Matching $O(\partial\sigma)$:

$$8\mu \tfrac{d\tilde{a}}{d\mu} = 8\beta_i \partial_i \tilde{a} = \chi^g_{ij}\beta_i\beta_j \ ,$$

$$\chi^g_{ij} = -2\chi^a_{ij} + \bar{\chi}^a_{ijk}\beta^k - \beta^\ell \partial_\ell V_{ij} - \partial_i \beta^\ell V_{\ell j} - \partial_j \beta^\ell V_{i\ell} \quad [...]$$

And every quantity on the RHS is written in term of $\mathcal{A}_{ij}, \mathcal{B}_{ijk}$.

Take-home message:

We have a gradient flow equation and a calculation prescription for $\tilde{a}$ in terms of flat spacetime CTs $\mathcal{A}_{ij}$, $\mathcal{B}_{ijk}$ of marginal operators.

Find which Green's function $\mathcal{A}_{ij}$, $\mathcal{B}_{ijk}$ renormalises. Using

$$\frac{\delta}{\delta g_i(x)} W[g_i] = \langle [\mathcal{O}_i(x)] \rangle,$$

applying an appropriate number of derivatives in the limit of flat space/sources one gets:

$$\langle [\mathcal{O}_i(p)][\mathcal{O}_j(q)] \rangle_R = \langle [\mathcal{O}_i(p)][\mathcal{O}_j(q)] \rangle + \mu^{-\epsilon} \mathcal{A}_{ij} p^2 q^2 \delta(p+q),$$

$$\begin{aligned} \langle [\mathcal{O}_i(p)][\mathcal{O}_j(q)][\mathcal{O}_j(r)] \rangle_R = & \langle [\mathcal{O}_i(p)][\mathcal{O}_j(q)][\mathcal{O}_j(r)] \rangle + ... \\ & ... + \mu^{-\epsilon} (\mathcal{B}_{ijk} p \cdot q r^2 + \mathcal{B}_{ikj} p \cdot r q^2 + \mathcal{B}_{jki} q \cdot r p^2). \end{aligned}$$

$\Longrightarrow$ build some perturbative expansion for the 2,3-pt functions as well as β-functions.

Application: Large N_f gauge theories (see Simone's and Manuel's talk).

$$\mathcal{L} = i \sum_{i=1}^{N} \bar{\psi}_i \not{D} \psi_i - \frac{1}{4g^2} F^2 + \mathcal{L}_{ghost} + \mathcal{L}_{g.f}$$

Of course, we will use a normalisation different from the literature:

$$\kappa = \frac{\beta_0 \alpha}{\pi} = \frac{S_2(R_\phi) N_f g^2}{6\pi^2} \implies \beta_\kappa = \kappa^2 + \mathcal{O}(1/N) .$$

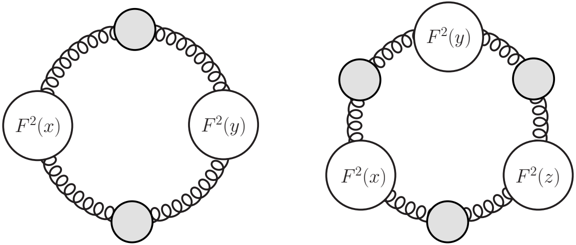
To have a feeling: now the (in)famous pole at $15/2$ is found at $\kappa = 5$. We restrict ourselves to LO, where the RG flow features a one-loop landau pole:

$$\kappa(\mu) = \log\left(\frac{\Lambda}{\mu}\right)^{-1}, \quad \Lambda = \mu_0 e^{1/\kappa(\mu_0)} .$$

The marginal operator driving the flow is

$$[\mathcal{O}_\kappa] = \frac{\delta S}{\delta \kappa(x)} = \frac{\beta_0}{16\pi^2 \kappa^2} F^2 + \text{g.f. terms} + \mathcal{O}(1/N)$$

- We study a one-coupling theory: all coupling indexes are suppressed.
- The CTs $\mathcal{A}, \bar{\mathcal{A}}$ renormalize divergences in 2 and 3-pt function of F^2: $\langle F^2 F^2 \rangle$, $\langle F^2 F^2 F^2 \rangle$ when insertion points merge.
- We calculate the LO resummation of the metric and $\tilde{a}$-function.



The final result reads:

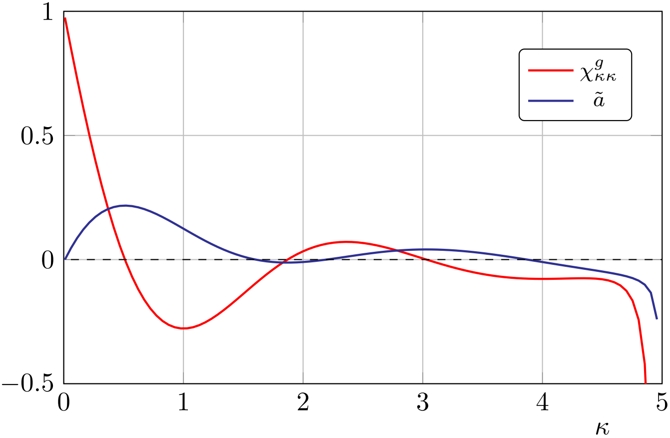
$$\chi^{g}_{\kappa\kappa} = \frac{d(G)}{16\pi^2\kappa^2}\partial_\kappa\left[\kappa\, H^a(\kappa) - \frac{1}{6}\kappa^2\, \bar{H}^a(\kappa)\right],$$

where two resummed functions appear:

$$H^a(x) = \frac{(1-\frac{x}{3})(240-240x+90x^2-15x^3+x^4)\Gamma(4-x)}{60(4-x)(6-x)\Gamma(1+\frac{x}{2})\Gamma^3(2-\frac{x}{2})},$$

$$\bar{H}^a(x) = \frac{(80-60x+13x^2-x^3)x\,\Gamma(4-x)}{120(4-x)\Gamma(1+\frac{x}{2})\Gamma^3(2-\frac{x}{2})}.$$

we have poles at $x = 5+n, n \in \mathbb{N}$, the $1/N_f$ expansion is broken there.



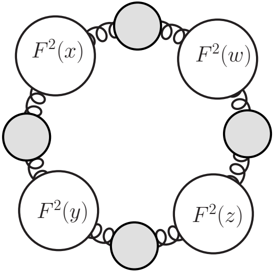
violation of metric positivity at $\kappa^* \sim 0.51, \implies \mu^* \sim 0.14\,\Lambda$

4-point function contribution

- We added to the action marginal scalar primaries only, but those are not the only ones appearing in the trace anomaly equation,

$$S_{CFT}[\gamma,\phi] + \int_x \left\{ g^i \mathcal{O}_i + A_\mu^A J_A^\mu + m^a \mathcal{O}_a \right\}.$$

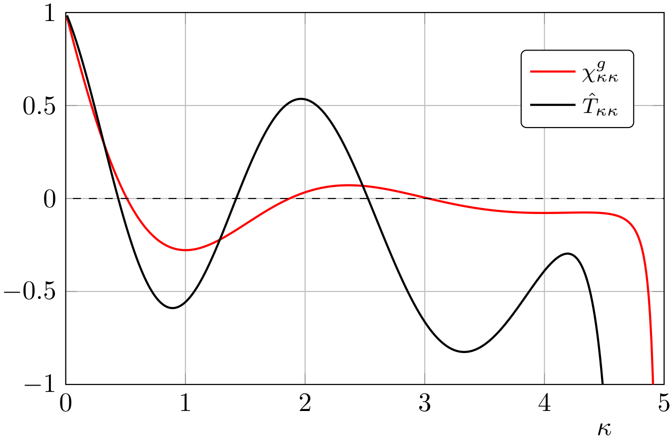
- Turns out that the J_A^μ contribution modifies the metric definition $\chi^g_{\kappa\kappa} \to \hat{T}_{\kappa\kappa}$. Gets contribution from 4-pt functions:



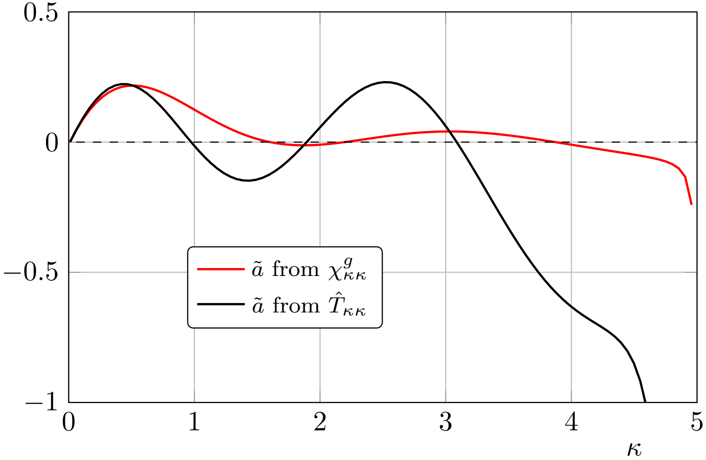
$+2\ perm.$

[Baume, Keren-Zur, Rattazzi, Vitale '14]

Metric comparisons



$\tilde{a}$ comparison



violation of metric positivity at $\kappa^* = 0.51\,(n.imp.) \;-\; 0.43\,(imp.)$.

Caveat(?): The gradient flow equation is invariant under:

$$\tilde{a} \to \tilde{a} + c_{ij}\beta^i\beta^j, \quad \chi_{ij} \to \chi_{ij} + \mathcal{L}_\beta c_{ij}$$

The strong theorem is in principle valid as long as it exist a choice for c_{ij} such that χ_{ij} is positive definite.

$\Longrightarrow$ Can we find a more "physical" scheme?

- The Dilaton effective action is defined by:

$$\Gamma[\bar{\gamma},\tau,\mu] = W[\gamma_{\mu\nu} = e^{2\tau}\bar{\gamma}_{\mu\nu}, g^i(\mu)].$$

where g^i are now spacetime independent!

- This effective action generates correlators of T^μ_μ:

$$\Gamma[\bar{\gamma},\tau,\mu] = \sum_n \frac{i^{n-1}}{n!} \int_{\{x_i\}} \tau(x_1)...\tau(x_n)\langle \mathbf{T}\{T(x_1)...T(x_n)\}\rangle_{\bar{\gamma},\mu}$$

- Work with on-shell condition:

$$R(e^{2\tau}\bar{\gamma}_{\mu\nu}) = 0$$

- The action can be split in two distinct contributions:

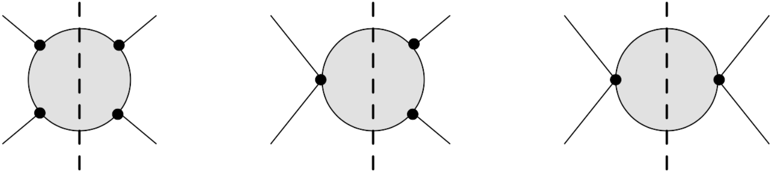
$$\Gamma = \Gamma_{loc} + \Gamma_{n.loc}$$

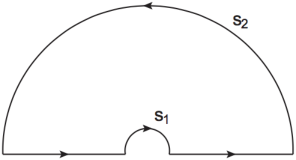
- The $2 \to 2$ dilaton scattering amplitude has an analogous splitting $A(s,t) = A_{loc}(s,t) + A_{n.loc}(s,t)$.
- The non-local contribution can be obtained from the effective coupling in the action for canonical dilatons $e^{-\tau} = 1 + \phi$,

$$\mathcal{L}_{\text{eff}} = g^i(\mu e^{\tau})\mathcal{O}_i = \left(g^i - \phi\beta^i + \frac{\phi^2}{2}\beta^j[\delta^i_j + \partial_j\beta^i] + ... \right)\mathcal{O}_i.$$

- We look at the absorptive part in the specifical kinematic region $t = 0$ where:

$$A(s,0) = -8s^2\alpha(s) \quad \text{so that} \quad \text{Im}A(s,0) > 0 \iff \text{Im}\alpha(s) < 0$$





Why do we use this definition? Because this amplitude satisfies:

$$A(s,0) = A(-s,0) \quad , A(s,0)^* = A(s^*,0), \quad \int_C \mathrm{d}s \frac{A(s,0)}{s^3} = 0,$$

this conditions lead to the definition of a monotonically decreasing function:

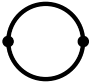
$$\bar{\alpha}(s) = \int_0^\pi \mathrm{d}\theta \alpha(se^{i\theta}), \quad \bar{\alpha}(s_2) - \bar{\alpha}(s_1) = -\frac{2}{\pi}\int_{s_1}^{s_2} \frac{\mathrm{d}s}{s} \mathrm{Im}\alpha(s) > 0\,(?)$$

- To compute $\alpha(s)$ we need finite parts of F^2 correlators.
- It possible to show that these multiple-chain diagrams are factorially divergent: as a computational trick we consider renormalised chains for which the Borel transform reads:

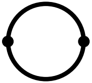
$$\mathcal{B}\left[\frac{1}{1-\Pi_R(k)}\right] = e^{-Ct/2}\left(\frac{k^2}{\mu^2}\right)^{t/2}$$

- For a double chain integral we use the convolution property:

$$\mathcal{B}\Big[\sum_{n,m}$$



$$\Big] \sim \int_{[u_i]}$$



$$, \quad \int_{[u_i]} = \int \mathrm{d}u_1\,\mathrm{d}u_2\,\delta\big(\textstyle\sum u_i - t/2\big)$$

Applying this procedure to the $\langle F^2 F^2 \rangle$ correlator one gets

$$\mathcal{B}[\langle F^2(p) F^2(-p) \rangle] = -\frac{2ie^{-Ct/2}}{32\pi^2} \left(\frac{4\pi}{\beta_0}\right)^2 (-s)^{2-t/2} \times \int_{\{u_i\}} G(1-u_1, 1-u_2)\,[1+\mathcal{P}(u_1,u_2)]$$

with a polynomial coming from the numerator structure,

$$\mathcal{P}(u_1,u_2) = -\frac{2u_2(1+u_2)(10+7u_2+8u_1+4u_1u_2+u_2^2)}{(1+u_1+u_2)(2+u_1+u_2)^2(3+u_1+u_2)}$$

and a loop integral contribution:

$$G(1-u_1,1-u_2) = \frac{\Gamma(-t/2)}{\Gamma(2+t/2)} \frac{\Gamma(1+u_1)}{\Gamma(1-u_1)} \frac{\Gamma(1+u_2)}{\Gamma(1-u_2)}.$$

$\Longrightarrow$ This correlator is not Borel summable! Renormalon poles at $t = 2n, n \geq 1$.

To conclude:

- We showed that "counterterm-derived" definitions of $\tilde{a}$ are not monotonic at LO in large N_f.
- We are extending the Large N_f methods to dilaton cross-sections.

Coming next:

- Verify wether positivity is lost in the dilaton cross section.
- Adress renormalon issues in the dilaton cross-sections and their relation to non-perturbative contributions.

Thank you!