Remarks on the RG and scale anomalies

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Outline

Scale transformations

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$$

locally the same as

$$x^\mu \to \Omega x^\mu$$

Choose *dimensionless coordinates*, and $\hbar = c = 1$.

Dimensional analysis vs. scale transformations

Invariance under

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \ , \quad \psi_a \to \Omega^{w_a} \psi_a \ , \quad g_i \to \Omega^{w_i} g_i$$

$$S(g_{\mu\nu}, \psi_a | g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{w_a} \psi_a | \Omega^{w_i} g_i)$$

fixes the weights $w_a = -d_a$, $w_i = -d_i$. Always true.

Invariance under *global scale transformations*

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \ , \quad \psi_a \to \Omega^{w_a} \psi_a \ , \quad g_i \to g_i$$

$$S(g_{\mu\nu}, \psi_a | g_i) = S(\Omega^2 g_{\mu\nu}, \Omega^{w_a} \psi_a | g_i)$$

not always true: requires $w_i = 0$.

Example: scalar field

e.g. for

$$S[\phi] = \int d^4x \left[-\frac{1}{2}(\partial\phi)^2 - V(\phi) \right]$$

we must have

$$V(\phi) = \frac{\lambda}{4!}\,\phi^4\,,$$

Invariance under infinitesimal scale transformation $\Omega = 1 + \epsilon$

$$\delta_\epsilon g_{\mu\nu} = 2\epsilon g_{\mu\nu} \; ; \qquad \delta_\epsilon \phi = -\frac{d-2}{2}\epsilon\phi$$

implies

$$\delta_\epsilon S = \epsilon \int_x T^\mu{}_\mu = 0$$

The anomaly in perturbatively renormalizable QFT

However

$$\delta_\epsilon \Gamma = \epsilon \int_x \langle T^\mu{}_\mu \rangle \equiv -\mathcal{A}(\epsilon)$$

$$\mathcal{A}(\epsilon) = \epsilon\, \beta \int_x \frac{1}{4!}\phi^4 \ ,$$

$$\beta = \frac{3\lambda^2}{16\pi^2} \, .$$

Effective Average Action in flat spacetime

$$e^{W_k[J]} = \int (D\phi) e^{-S-\Delta S_k + \int J\phi)}$$

$$\Delta S_k(\phi) = \frac{1}{2} \int d^4 q \, \phi(-q) R_k(q^2) \phi(q)$$

$$\Gamma_k[\phi] - = W_k[J] + \int J\phi - \Delta S_k$$

Effective Average Action in flat spacetime

Satisfies the Wetterich equation

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\varphi\delta\varphi} + R_k\right)^{-1} k\frac{dR_k}{dk}$$

One loop EA

$$\Gamma = S + \frac{1}{2}\mathrm{Tr}\log\left(\frac{\delta^2 S}{\delta\varphi\delta\varphi}\right)$$

One loop EAA

$$\Gamma_k = S + \Delta S_K + \frac{1}{2}\mathrm{Tr}\log\left(\frac{\delta^2(S+\Delta S_k)}{\delta\varphi\delta\varphi}\right) - \Delta S_k$$

One loop RG

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2 S}{\delta\varphi\delta\varphi} + R_k\right)^{-1} k\frac{dR_k}{dk}$$

Anomalous WI of scale transformations

Assume that S is scale invariant. What is the anomaly for Γ_k?
For the EAA

$$\begin{aligned}\delta_\epsilon \Gamma_k &= -\mathcal{A}(\epsilon) + \frac{1}{2}\epsilon \,\mathrm{Tr}\left(\frac{\delta^2\Gamma_k}{\delta\varphi\delta\varphi} + R_k\right)^{-1} k\frac{\delta R_k}{dk} \\ &= -\mathcal{A}(\epsilon) + \epsilon\,\partial_t \Gamma_k \,,\end{aligned}$$

[B. Delamotte, M. Tissier and N. Wschebor, Phys. Rev. E **93** (2016) 012144, arXiv:1501.01776 [cond-mat.stat-mech]]
[T.Morris, R.P. Phys.Rev. D99 (2019) 105007 arXiv:1810.09824 [hep-th]]

For a monomial

$$\Gamma_k = \lambda(k)\,\mathcal{O}(\phi)$$

If $\mathcal{O}$ has mass dimension Δ. we can also write $\lambda\,\mathcal{O} = \tilde{\lambda}\,\tilde{\mathcal{O}}$ where

$$\tilde{\lambda} = k^{\Delta}\lambda \; ; \qquad \tilde{\mathcal{O}} = k^{-\Delta}\mathcal{O}$$

The l.h.s. of the WI is

$$\delta_\epsilon \left(\lambda \, \mathcal{O}\right) = \lambda \delta_\epsilon \mathcal{O} = -\epsilon \, \Delta \, \lambda \, \mathcal{O}$$

On the other hand

$$\partial_t \left(\lambda \, \mathcal{O}\right) = \partial_t \lambda \, \mathcal{O} \ ,$$

Thus the WI gives

$$-\epsilon \Delta \, \lambda \mathcal{O} = -\mathcal{A}(\epsilon) + \epsilon \partial_t \lambda \, \mathcal{O}$$

Bringing the l.h.s. to the r.h.s. it reconstructs the derivative of $\tilde{\lambda}_i$, times $k^{-\Delta}$, which can be rewritten as

$$\mathcal{A}(\epsilon) = \epsilon \, \partial_t \tilde{\lambda} \, \tilde{\mathcal{O}} = \epsilon \, \tilde{\beta} \, \tilde{\mathcal{O}} \ .$$

For local expansion of EAA

$$\Gamma_k = \sum_i \lambda_i(k)\, \mathcal{O}_i(\phi)$$

the WI implies

$$\mathcal{A}(\epsilon) = \epsilon \sum_i \tilde{\beta}_i\, \tilde{\mathcal{O}}_i$$

The anomaly vanishes at a FP

This does not imply that Γ_k is scale invariant according to the definition given above. Instead

$$\delta_\epsilon \Gamma_k = -\epsilon \sum_i \Delta_i\, \lambda_i\, \mathcal{O}_i$$

Quantum scale invariance

However, define $\hat{\delta}_\epsilon$

$$\hat{\delta}_\epsilon k = -\epsilon\, k \ ,$$

and the same as the action of δ_ϵ on all other quantities. Then

$$\begin{aligned}\hat{\delta}_\epsilon \Gamma_k &= \delta_\epsilon \Gamma_k + \hat{\delta}_\epsilon k \partial_k \Gamma_k \\ &= \delta_\epsilon \Gamma_k - \epsilon \partial_t \Gamma_k \ = \ \mathcal{A}(\epsilon)\end{aligned}$$

At a fixed point one has scale invariance *in the sense of* $\hat{\delta}_\epsilon$

Another point of view

We used $\partial_t \phi = 0$, so $\partial_t \mathcal{O} = 0$ and we obtained

$$\partial_t(\lambda \mathcal{O})|_{\mathcal{O}} = (\tilde{\beta} - \Delta\tilde{\lambda})\tilde{\mathcal{O}}$$

Consider instead

$$\partial_t(\lambda \mathcal{O})\big|_{\tilde{\mathcal{O}}} = \tilde{\beta}\tilde{\mathcal{O}}$$

This would give

$$\epsilon \partial_t \Gamma_k\big|_{\tilde{\mathcal{O}}} = \epsilon \partial_t \Gamma_k\big|_{\mathcal{O}} - \delta_\epsilon \Gamma_k$$

This is an infinitesimal Wilsonian RG transformation
Comparing with the WI

$$\mathcal{A} = \epsilon \partial_t \Gamma_k\big|_{\mathcal{O}} - \delta_\epsilon \Gamma_k$$

we see that the anomaly is the Wilsonian RG

Local scale/Weyl transformations

Interpret global rescalings of the metric as changes of the unit of length. Allow choice of unit to depend on position.

"It is evident that two rods side by side, stationary with respect to each other, can be intercompared....this cannot be done for....rods with either a space- or time-like separation".

"A statement such as "a hydrogen atom on Sirius has the same diameter as one on the Earth" is either a definition or else meaningless" (Dicke 1962)

Physics must be formulated in a way that is invariant under local changes of units, i.e. under *local* rescalings of the metric.
Allow parallel transport to affect norm of vectors.

Various routes to Weyl invariance

Global scale invariance is quite restrictive, local scale invariance even more so.

Construct Weyl invariant theories from non-invariant ones.

- Stückelberg gauging - can be applied to any theory
- Weyl gauging - can be applied to scale-invariant theories
- Ricci gauging - can be applied in special cases

Stückelberg gauging ("Fake" Weyl invariance)

Turn dimensionful couplings into fields.

Minimal version: one "dilaton" χ of weight -1: $\chi \to \Omega^{-1}\chi$
Write $\hat{g}_{\mu\nu} = \chi^2 g_{\mu\nu}$, $\hat{\psi}_a = \chi^{w_a}\psi_a$ $\hat{g}_i = \chi^{w_i} g_i$

$$S(g_{\mu\nu}, \psi_a | g_i) \to S(\hat{g}_{\mu\nu}, \hat{\psi}_a | \hat{g}_i) := \hat{S}(g_{\mu\nu}, \psi_a, \chi | \hat{g}_i)$$

$\hat{S}$ is Weyl invariant by construction.

E.g.: $S = \int d^4x \sqrt{g} m^2 \phi^2 \to \hat{S} = \int d^4x \sqrt{g} \hat{m}^2 \chi^2 \phi^2$

(Note: there is a Weyl gauge where $\chi = \mu$ (constant).)

This is a local version of the transformations of dimensional analysis.

Weyl calculus

Abelian gauge field transforming as $b_\mu \mapsto b_\mu + \Omega^{-1}\partial_\mu \Omega$.
For scalar field ϕ of weight w

$$D_\mu \phi = \partial_\mu \phi - w b_\mu \phi$$

More generally

$$\hat{\Gamma}_\mu{}^\lambda{}_\nu = \Gamma_\mu{}^\lambda{}_\nu - \delta^\lambda_\mu b_\nu - \delta^\lambda_\nu b_\mu + g_{\mu\nu} b^\lambda$$

is invariant under local Weyl transformations, hence for a tensor of weight w

$$D_\mu t = \hat{\nabla}_\mu t - w b_\mu t$$

is diffeomorphism and Weyl covariant.
Note $\hat{\nabla}_\lambda g_{\mu\nu} = 2 b_\lambda g_{\mu\nu}$ but $D_\lambda g_{\mu\nu} = 0$

Weyl curvature

$$[D_\mu, D_\nu]v^\rho = \mathcal{R}_{\mu\nu}{}^\rho{}_\sigma v^\sigma$$

$$\begin{aligned}\mathcal{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} + (w-1)F_{\mu\nu}g_{\rho\sigma} \\ &+ g_{\mu\rho}\left(\nabla_\nu b_\sigma + b_\nu b_\sigma\right) - g_{\mu\sigma}\left(\nabla_\nu b_\rho + b_\nu b_\rho\right) \\ &- g_{\nu\rho}\left(\nabla_\mu b_\sigma + b_\mu b_\sigma\right) + g_{\nu\sigma}\left(\nabla_\mu b_\rho + b_\mu b_\rho\right) \\ &- \left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) b^2\end{aligned}$$

$$\mathcal{R} = R + 2(d-1)\nabla^\mu b_\mu - (d-1)(d-2)b^2$$

$$F_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$$

Weyl gauge field is generally non-integrable.

Weyl gauging

Starting from any globally scale invariant action $S(g_{\mu\nu}, \psi_a | g_i)$ (all g_i dimensionless) replace $\nabla \to D$ and $R \to \mathcal{R}$.

Resulting action $\hat{S}(g_{\mu\nu}, \psi_a, b_\mu | g_i)$ is Weyl invariant.

Integrable Weyl gauging=Stückelberg gauging

If $F_{\mu\nu} = 0$ we can write

$$b_\mu = -\chi^{-1}\partial_\mu \chi$$

where the field χ transforming as $\chi \to \Omega^{-1}\chi$ can be identified with the dilaton.

Starting from any action $S(g_{\mu\nu}, \psi_a | g_i)$,
replace $g_i \to \chi^{d_i}\hat{g}_i$, $\nabla \to D$ and $R \to \mathcal{R}$.

Resulting action $\hat{S}(g_{\mu\nu}, \psi_a, \chi | \hat{g}_i)$ agrees with Stückelberg gauging.

Ricci gauging

The Weyl variation of

$$R_{\mu\nu} - \frac{1}{2(n-1)} g_{\mu\nu} R$$

is the same as the Weyl variation of

$$-\nabla_\mu b_\nu - b_\mu b_\nu + \frac{1}{2} g_{\mu\nu} b^2$$

so if b_μ occurs in a Weyl-gauged action only in this combination, it can be replaced by a combination of Ricci tensors.

Ricci gauging

Works when the starting theory in flat space is conformal

Example

$$S[\phi] = \int d^d x \left[-\frac{1}{2}(\partial\phi)^2 - \frac{d-2}{4(d-1)} R \right]$$

[A. Iorio, L. O'Raifeartaigh, I. Sachs, C. Wiesendanger, Nucl. Phys. B495 (1997) 433-450, arxiv:hep-th/9607110]
[K. Farnsworth, M.A. Luty, V. Prilepina, JHEP 1710 (2017) 170, arXiv:1702.07079 [hep-th]]

Historical remark

If v^μ has $w = 0$, $|v|$ has weight 1, If v^μ parallel-trasported around loop ℓ,

$$0 = D_\mu |v| = \partial_\mu |v| - b_\mu |v|$$

and $\Delta|v| = e^{\int b}|v|$.

Weyl regarded b_μ as e.m. field.

Einstein's critique: spectral lines would be blurred.

Weyl's theory still viable if b_μ interpreted as a piece of the spacetime connection. The spacetime connection is massive and its curvature vanishes by a gravitational analog of the Meissner effect.

[D.M. Ghilencea, JHEP 1903 (2019) 049, arXiv:1812.08613 [hep-th]]

Remark

Dilaton belongs to an orbit of $\mathbb{R}^+$ on $\mathbb{R}^+$:

$$\chi > 0 \,, \qquad \chi = e^{\sigma}$$

Extending configuration space to $\mathbb{R}$ one finds new cosmological solutions.

e.g. [I.bars, P.Steinhardt, N.Turok, Phys.Rev. D89 (2014) 061302, arXiv:1312.0739 [hep-th]]

Main question

Can Weyl invariance be preserved in quantum theory?

Can Weyl invariance be preserved by renormalization group flow?

Conformal anomaly

Regularization breaks Weyl invariance, but its effect can be offset if dilaton is present

[Englert, C. Truffin and R. Gastmans, Nucl. Phys. B117, 407 (1976)]
[R.Floreanini and R. P., Nucl. Phys. B436, 141 (1995)]
[M. Shaposhnikov and I. Tkachev, Phys. Lett. B675, 403 (2009)]
[D.M. Ghilencea, Z. Lalak, P. Olszewski, Eur.Phys.J. C76 (2016) no.12, 656]
[D.M. Ghilencea, Z. Lalak, P. Olszewski, Phys.Rev. D96 (2017) no.5, 055034]
[Z. Lalak, P. Olszewski, Phys.Rev. D98 (2018) no.8, 085001,]

Example: matter in background metric

$$S(\phi; g_{\mu\nu}) = \frac{1}{2}\int d^4x\sqrt{g}\,\phi\Delta^{(1/6)}\phi\ , \qquad \Delta^{(1/6)} = -\Box + \frac{R}{6}$$

Standard scalar measure and path integral

$$(d\phi)^{\mathrm{I}} = \prod_x \frac{d\phi(x)}{\mu}$$

$$\Gamma^{\mathrm{I}}[g_{\mu\nu}] = -\log\int(d\phi)e^{-\int d^4x\sqrt{g}\phi\Delta^{(1/6)}\phi} = \frac{1}{2}\log\det\left(\frac{\Delta^{(1/6)}}{\mu^2}\right)$$

Trace anomaly

$$\delta_\epsilon \Gamma^{\mathrm{I}}[g_{\mu\nu}] = \int dx \sqrt{g}\, \epsilon(x) \langle T^\mu{}_\mu \rangle := -\mathcal{A}(\epsilon)$$

$$\langle T^\mu{}_\mu \rangle = \frac{2}{\sqrt{g}} g^{\mu\nu} \frac{\delta \Gamma^{\mathrm{I}}}{\delta g^{\mu\nu}} = c\, C^2 + aE$$

$$E = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$$
$$C^2 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$$

$$c = \frac{1}{120(4\pi)^2} \left(n_S + 6 n_D + 12 n_M \right)$$

$$a = -\frac{1}{360(4\pi)^2} \left(n_S + 11 n_D + 62 n_M \right)$$

The Weyl-invariant scalar measure

In presence of dilaton can define

$$(d\phi)^{\mathrm{II}} = \prod_x \frac{d\phi(x)}{\chi(x)}$$

$$\Gamma^{\mathrm{II}}[g_{\mu\nu},\chi] = -\log \int (d\phi) e^{-\int d^4x \sqrt{g}\phi \Delta^{(1/6)}\phi} = \frac{1}{2}\log\det\left(\frac{1}{\chi^2}\Delta^{(1/6)}\right)$$

$$\frac{1}{\Omega^{-2}\chi^2}\,\Delta^{(1/6)}_{\Omega^2 g}(\Omega^{-1}\phi) = \Omega^{-1}\left(\frac{1}{\chi^2}\Delta^{(1/6)}\phi\right)$$

eigenvalues are Weyl invariant $\Rightarrow$ det $\left(\frac{1}{\chi^2}\Delta^{(1/6)}\right)$ is Weyl invariant

Weyl invariant quantization

μ has been promoted to a field
dilaton acts as Stückelberg field in the *quantum* effective action

Note: trace of energy-momentum tensor still nonzero:

$$
\begin{aligned}
0 = \delta_\epsilon \Gamma^{\mathrm{II}}[g_{\mu\nu}, \chi] &= \int dx \sqrt{g} \left(\frac{\delta \Gamma^{\mathrm{II}}}{\delta g_{\mu\nu}} \delta_\epsilon g_{\mu\nu} + \frac{\delta \Gamma^{\mathrm{II}}}{\delta \chi} \delta_\epsilon \chi \right) \\
&= \int dx \sqrt{g}\, \epsilon(x) \left(\langle T^\mu{}_\mu \rangle - \frac{\delta \Gamma^{\mathrm{II}}}{\delta \chi} \chi \right) \\
&= -\mathcal{A}(\epsilon) - \int dx \sqrt{g}\, \epsilon(x) \frac{\delta \Gamma^{\mathrm{II}}}{\delta \chi} \chi
\end{aligned}
$$

Another point of view

$$\Gamma^{\mathrm{I}}(g^{\chi}_{\mu\nu}) - \Gamma^{\mathrm{I}}(g_{\mu\nu}) = \Gamma_{WZ}(g_{\mu\nu}, \chi)$$

Wess-Zumino consistency condition:

$$\Gamma_{WZ}(g^{\Omega}_{\mu\nu}, \chi^{\Omega}) - \Gamma_{WZ}(g_{\mu\nu}, \chi) = -\Gamma_{WZ}(g_{\mu\nu}, \Omega)$$

where $g^{\Omega} = \Omega^2 g$, $\chi^{\Omega} = \Omega^{-1}\chi$
If we identify $\Gamma^{\mathrm{I}}(g_{\mu\nu})$ with $\Gamma^{\mathrm{II}}(g_{\mu\nu}, \chi = \mu)$,

$$\Gamma^{\mathrm{II}}(g_{\mu\nu}, \chi) = \Gamma^{\mathrm{I}}(g_{\mu\nu}) + \Gamma_{WZ}(g_{\mu\nu}, \chi)$$

The WZ action contains the effect of the anomaly
Solving the EOM of Γ gives the gravitational field including backreaction of quantum matter.

Summary so far

"Stückelberg gauging commutes with quantization"

$$\begin{array}{ccc} S(g_{\mu\nu}, \psi_a | g_i) & \longrightarrow & \Gamma(g_{\mu\nu}, \psi_a | g_i) \\ \downarrow & & \downarrow \\ \hat{S}(g_{\mu\nu}, \chi, \psi_a | \hat{g}_i) & \longrightarrow & \hat{\Gamma}(g_{\mu\nu}, \chi, \psi_a | \hat{g}_i) \end{array}$$

Generalization to interacting theories

[R. P., New J. Phys. **13** 125013 (2011) arXiv:1110.6758 [hep-th]]

[A. Codello, G. D'Odorico, C. Pagani, R. P., Class. Quant. Grav. 30 (2013), arXiv:1210.3284 [hep-th]]

Use Functional RG

Assume there exists a dilaton field χ.
$S(\psi, \chi, g_{\mu\nu}|g_i)$ invariant under Weyl transformations.

Measure can be made invariant.

Construction of Weyl invariant flow

Assume that in general $k = k(x)$.
Make the cutoff term invariant by Stückelberg procedure:

$$\Delta S_k = \int dx \sqrt{g}\, \phi\, k^2 r\left(\frac{1}{k^2}\Delta\right)\phi = \int dx \sqrt{g}\, \chi^2\, \phi\, u^2 r\left(\frac{1}{u^2}\mathcal{O}\right)\phi$$

with

$$\mathcal{O} = \frac{1}{\chi^2}\Delta$$

RG scale measured in units of dilaton:

$$u = k/\chi$$

u dimensionless, Weyl-invariant.

Assume $u = k/\chi$ is constant. Think of the EAA as

$$\Gamma_k(\phi, g_{\mu\nu}, \chi) = \Gamma_u(\phi, g_{\mu\nu}, \chi)$$

It satisfies

$$u\frac{d\Gamma_u}{du} = \frac{1}{2}\mathrm{Tr}\left(\frac{\delta^2\Gamma_u}{\delta\varphi\delta\varphi} + R_u\right)^{-1} u\frac{\delta R_u}{du}$$

Beta functional is Weyl invariant

Structure of ERGE:

$$u\frac{d\Gamma_u}{du} = \frac{1}{2}\mathrm{Tr}\left(A^{-1}u\frac{dB}{du}\right)$$

with

$$A = \frac{\delta^2(\Gamma_u + \Delta S_u)}{\delta\phi\delta\phi}, \qquad B = \frac{\delta^2\Delta S_u}{\delta\phi\delta\phi}$$

If ϕ has weight w and Γ_u and ΔS_u are Weyl invariant, A and B are Weyl covariant:

$$A \mapsto \Omega^{-w}A\Omega^{-w}\ , \quad B \mapsto \Omega^{-w}B\Omega^{-w}\ .$$

$\Rightarrow$ r.h.s. of ERGE is Weyl invariant.
RG flow preserves Weyl invariance.
If UV action is Weyl invariant, Γ is Weyl invariant.

Interpretation

In Weyl theory χ^{-1} interpreted as unit of length

In RG k^{-1} used as unit of length

"Relational" form of RG depends only on invariant ratio $u = k/\chi$

There are no anomalies for the scale transformations of dimensional analysis

Nonintegrable case

[C. Pagani, R. P., Class.Quant.Grav. 31 (2014) 115005 arXiv:1312.7767 [hep-th]]

An action for non-integrable Weyl gravity.

$$S = \int d^4x\sqrt{g}\left[\frac{g_1}{2}D_\mu\chi D^\mu\chi + g_2\chi^4 + \frac{g_3}{4}F_{\mu\nu}F^{\mu\nu} - g_4\chi^2\mathcal{R}^{(b)}\right] .$$

$$D_\mu\chi = (\partial_\mu + b_\mu)\chi.$$

Can define $s_\mu = -\chi^{-1}\partial_\mu\chi$, $D^{(s)}_\mu\chi = 0$,

$$\mathcal{R}^{(b)} = \mathcal{R}^{(s)} + 6\chi^{-1}D^2\chi$$

where $\mathcal{R}^{(s)} = R - 6\chi^{-1}\nabla^2\chi$

Rewriting

$$S = \int d^4x\sqrt{g}\left[\frac{g_1 + 12g_4}{2} D_\mu\chi D^\mu\chi + g_2\chi^4 + \frac{g_3}{4} F_{\mu\nu}F^{\mu\nu} - g_4\chi^2\mathcal{R}^{(s)}\right] .$$

Unitary gauge $\chi = const$

$$S(g) = \int d^4x\sqrt{g}\left[\frac{g_1 + 12g_4}{32\pi G g_4} b_\mu b^\mu + \frac{g_3}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{16\pi G}(2\Lambda - R)\right] ,$$

If $g_1 + 12g_4 \neq 0$, Weyl field is massive.

Additional invariances

Weyl invariance

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu} \ , \qquad b'_\mu = b_\mu + \Omega^{-1}\partial_\mu \Omega \ , \qquad \chi' = \Omega^{-1}\chi \ ,$$

is always present, but for $g_1 + 12 g_4 = 0$, additional invariance

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu} \ , \qquad b'_\mu = b_\mu \ , \qquad \chi' = \Omega^{-1}\chi \ ,$$

or equivalently

$$g'_{\mu\nu} = g_{\mu\nu} \ , \qquad b'_\mu = b_\mu + g^{-1}\partial_\mu g \ , \qquad \chi' = \chi \ ,$$

RG invariance of massless subspace

Massless subspace is RG-invariant provided cutoff is of the form

$$R_k(-D^{(s)2})$$

but not in the case

$$R_k(-D^{(b)2})$$

Summary and conclusions

1) Global scale anomaly identical to Wilsonian RG

2) Weyl invariance can be preserved in case of Stückelberg gauging

3) If classical theory is Weyl invariant without a dilaton (e.g. Weyl gauging, Ricci gauging) it is necessary to introduce a dilaton to have Weyl invariance in the RG flow/quantum theory.

4) cutoff/renormalization scale can depend on position.
A theory with nonconstant cutoff is equivalent to a theory with constant cutoff but conformally related metric.