FACTORISATION AND SUBTRACTION BEYOND NLO

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Precision QCD@LHC - Hyderabad - 28/01/20

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
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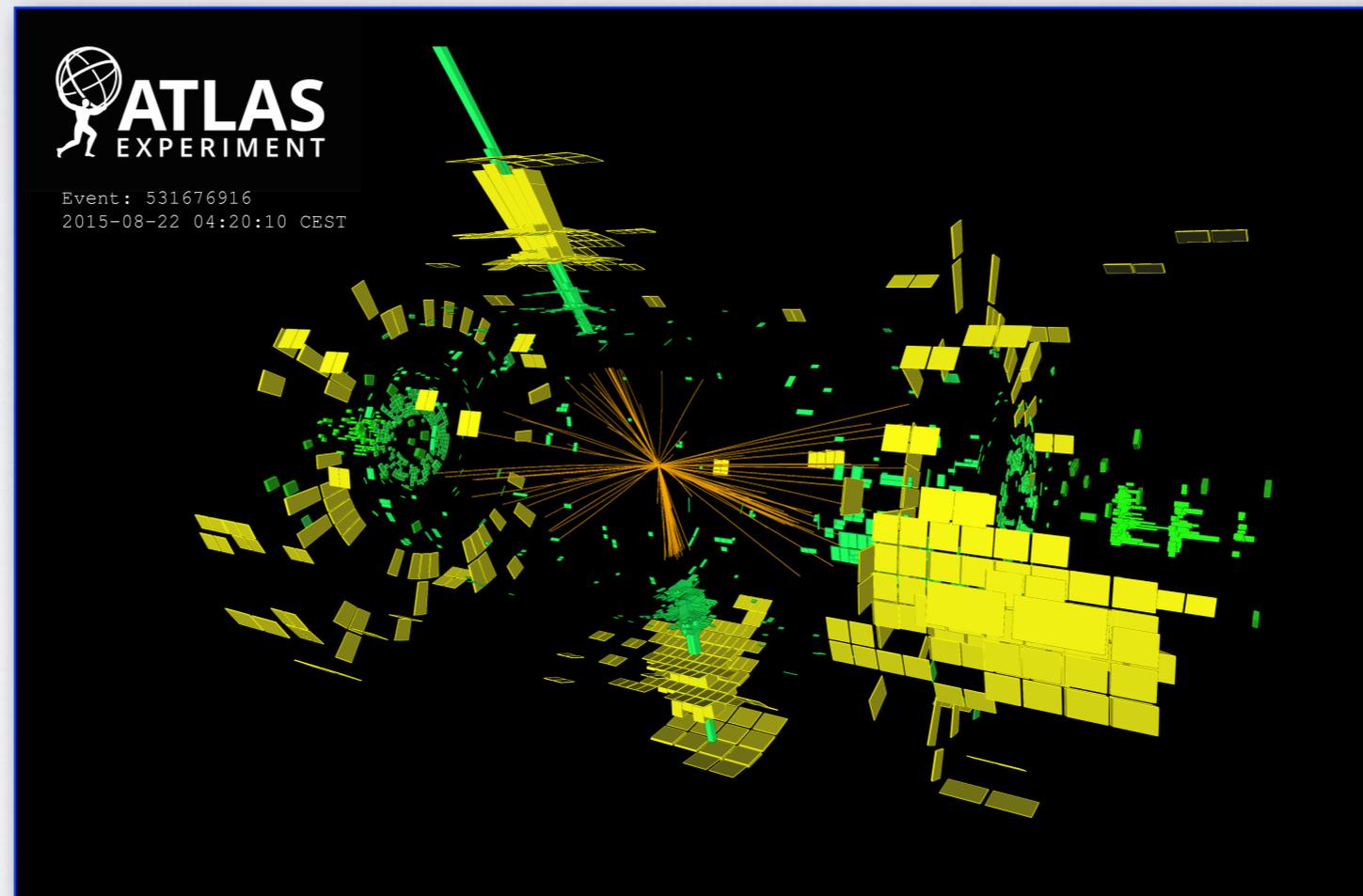
INFN

Outline

- Introduction

- Algorithms

- Factorisation

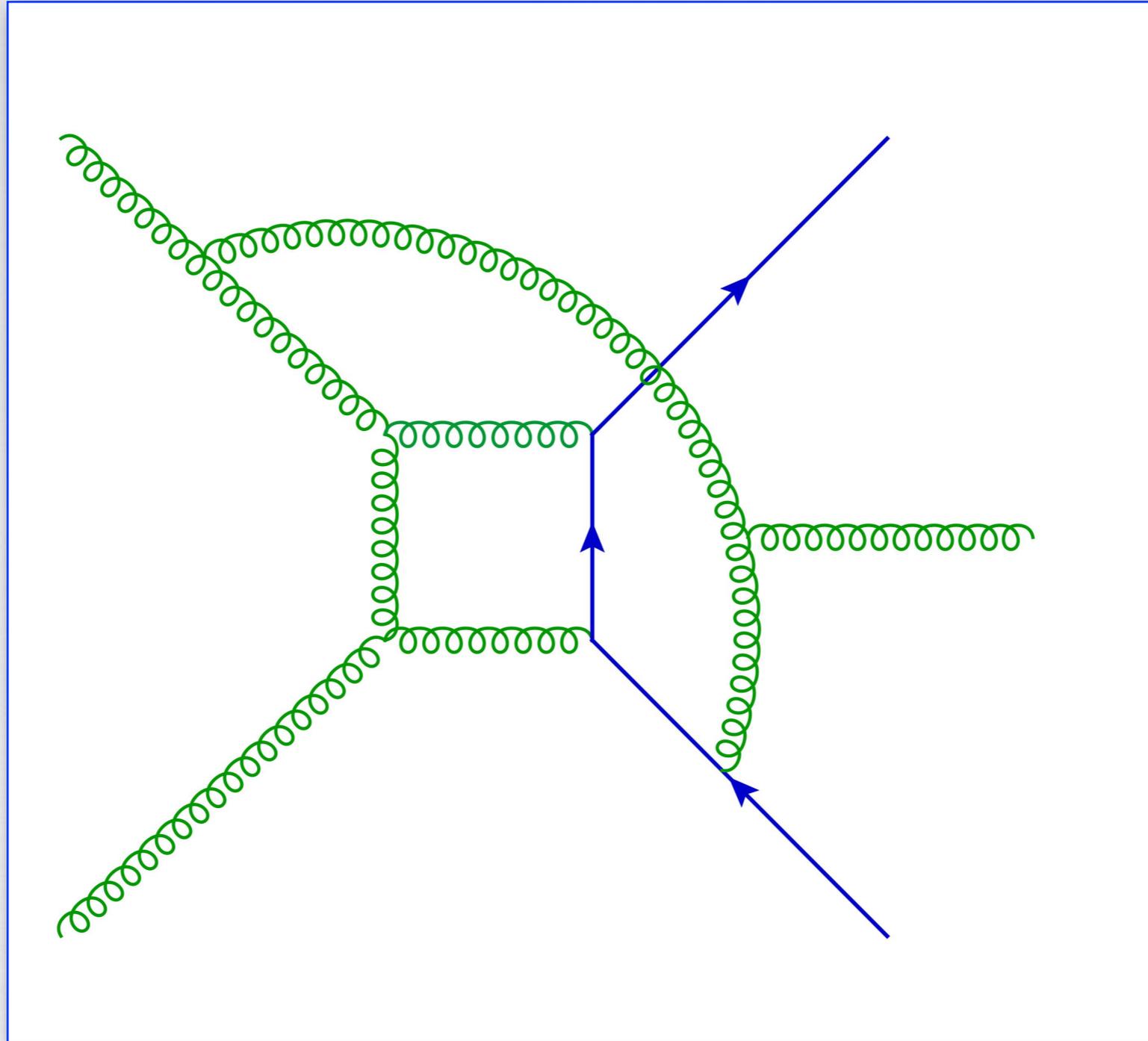- Counterterms

- Outlook

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In collaboration with
Ezio Maina,
Giovanni Pelliccioli
Chiara Signorile-Signorile
Paolo Torrielli
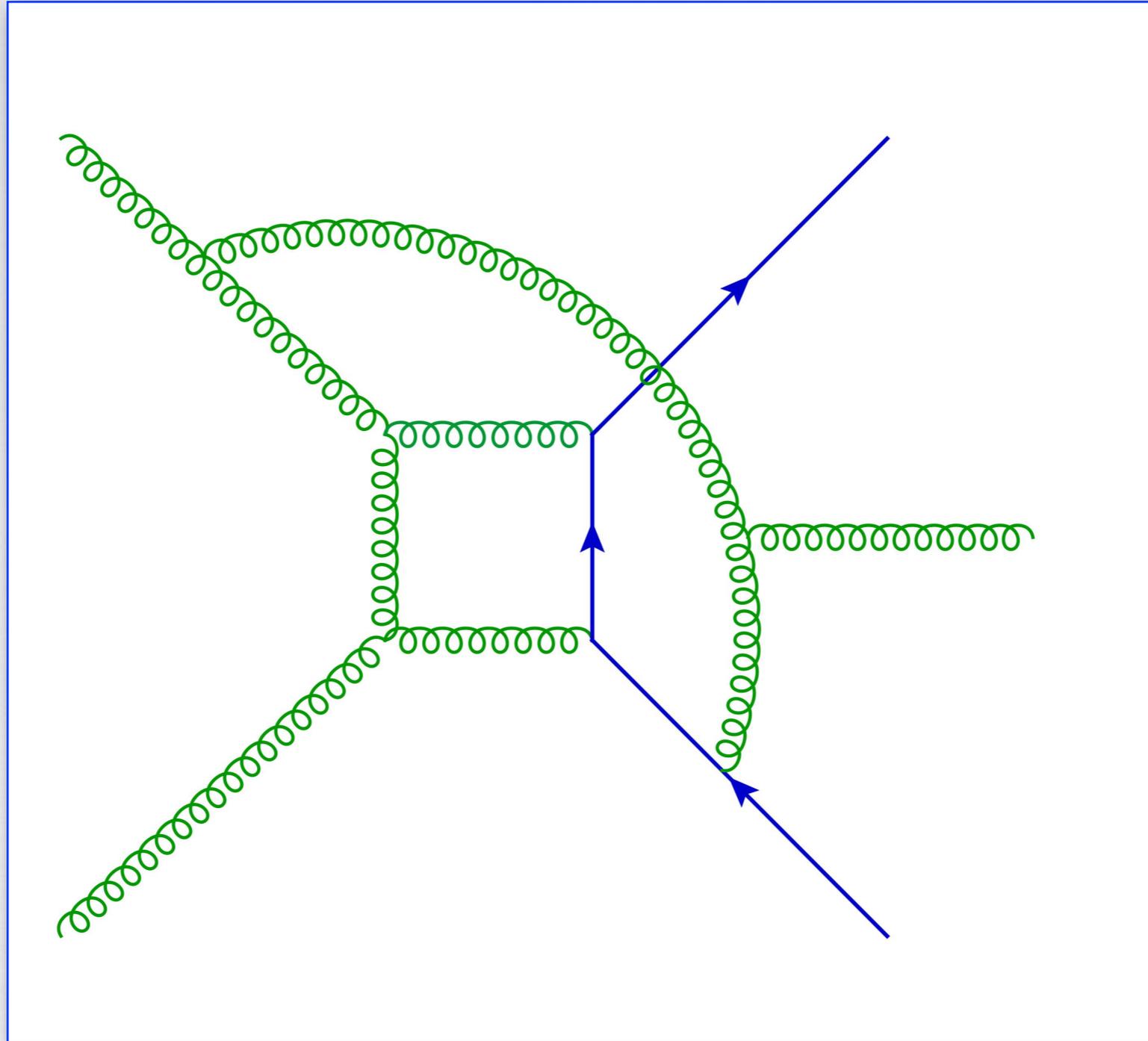Sandro Uccirati

INTRODUCTION

Pictorial infrared

A diagram contributing a double-virtual NNLO correction to t-tbar-jet production
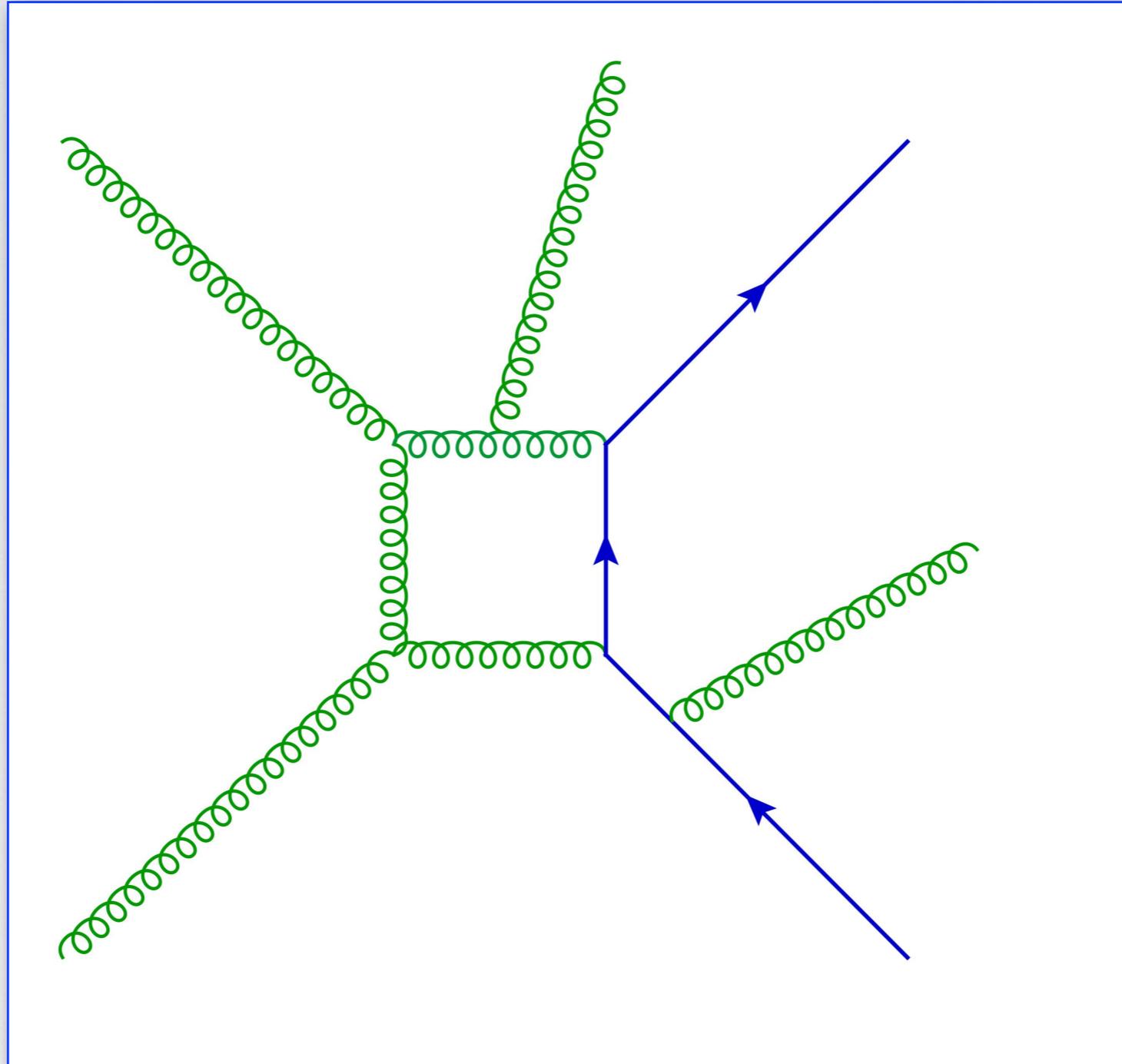
Pictorial infrared
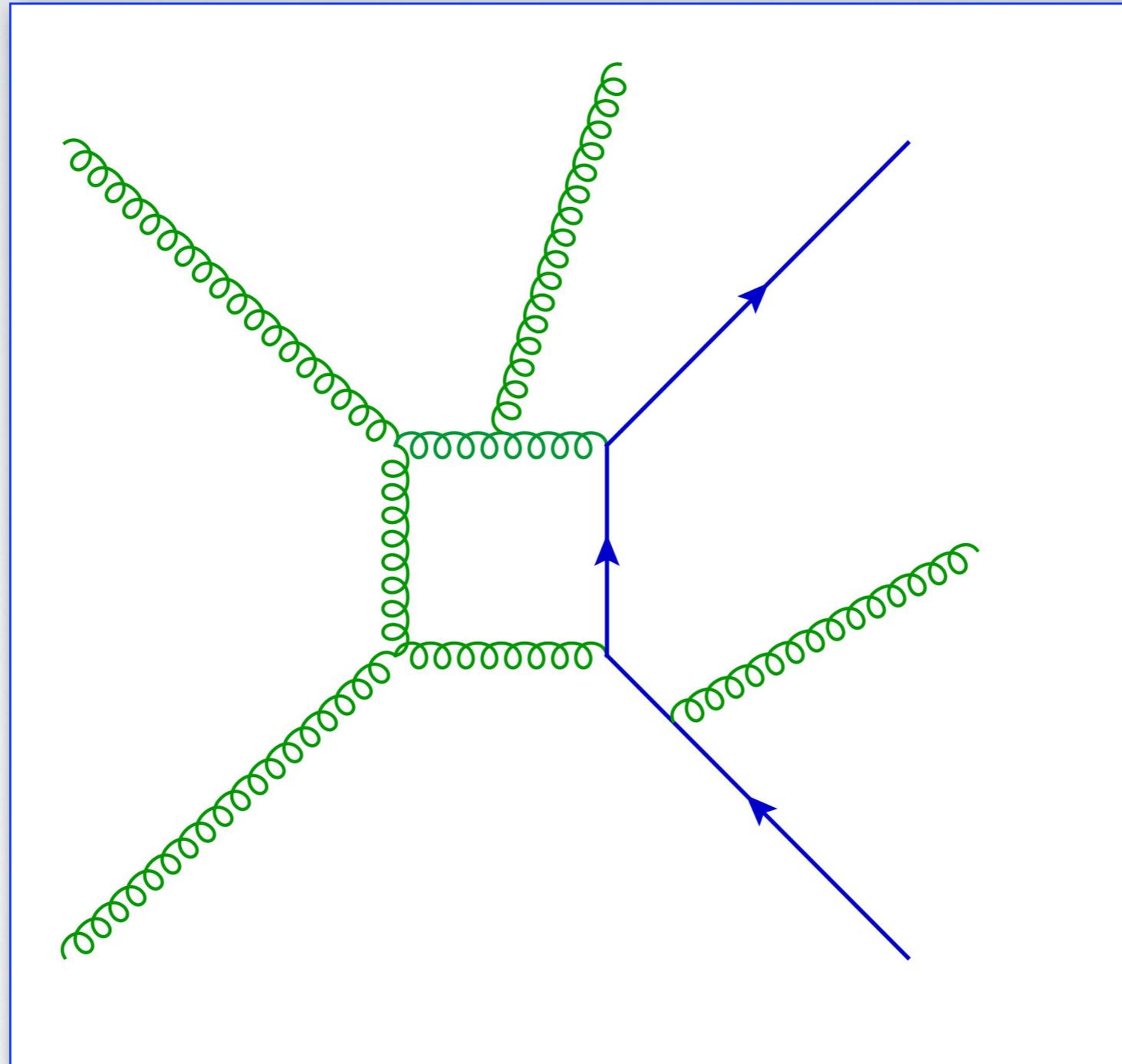

$$\frac{1}{\epsilon^4}$$

A diagram contributing a double-virtual NNLO correction to t-tbar-jet production

Pictorial infrared

A diagram contributing a real-virtual NNLO correction to t-tbar-jet production

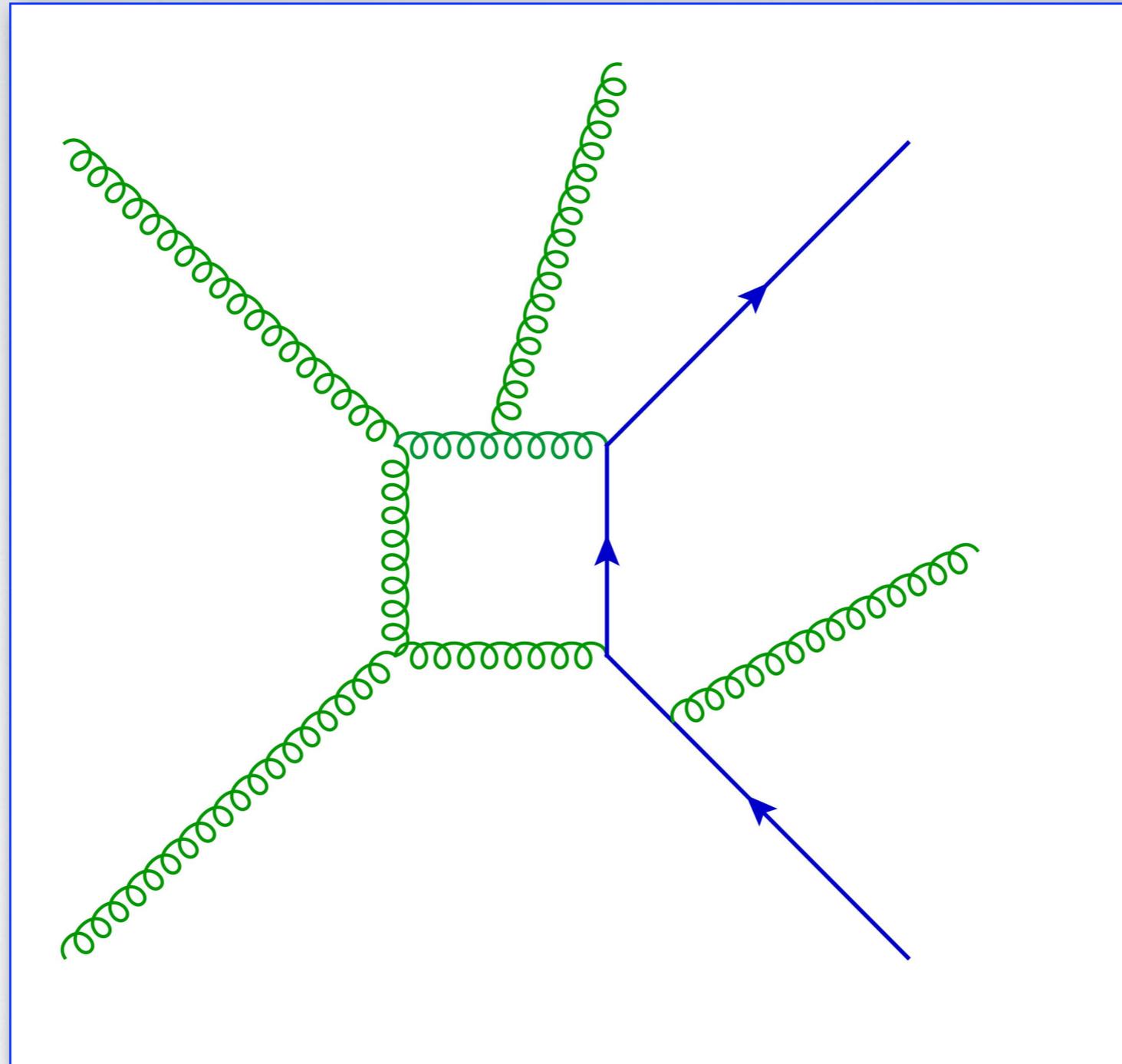# Pictorial infrared

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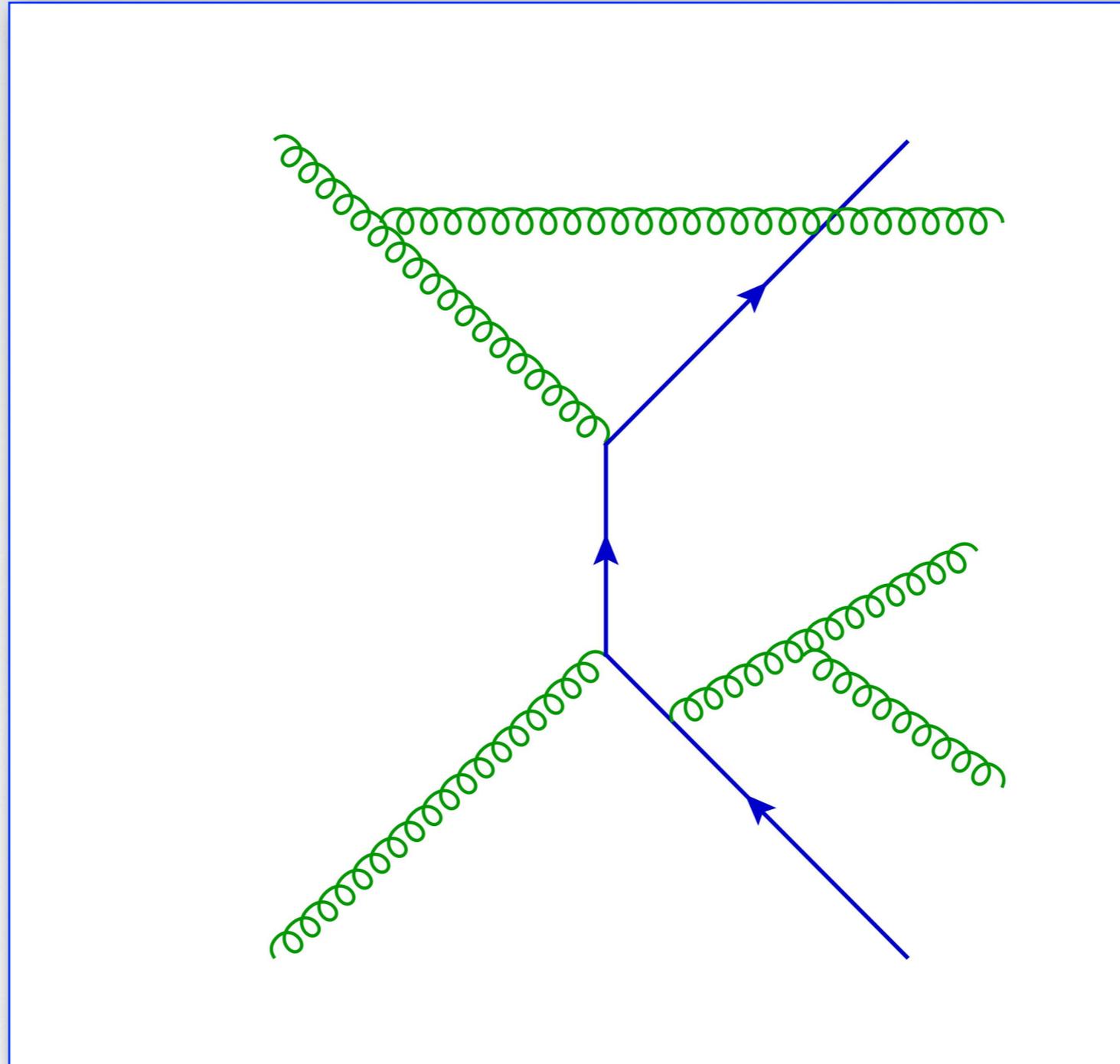
A diagram contributing a double-real NNLO correction to t-tbar-jet production

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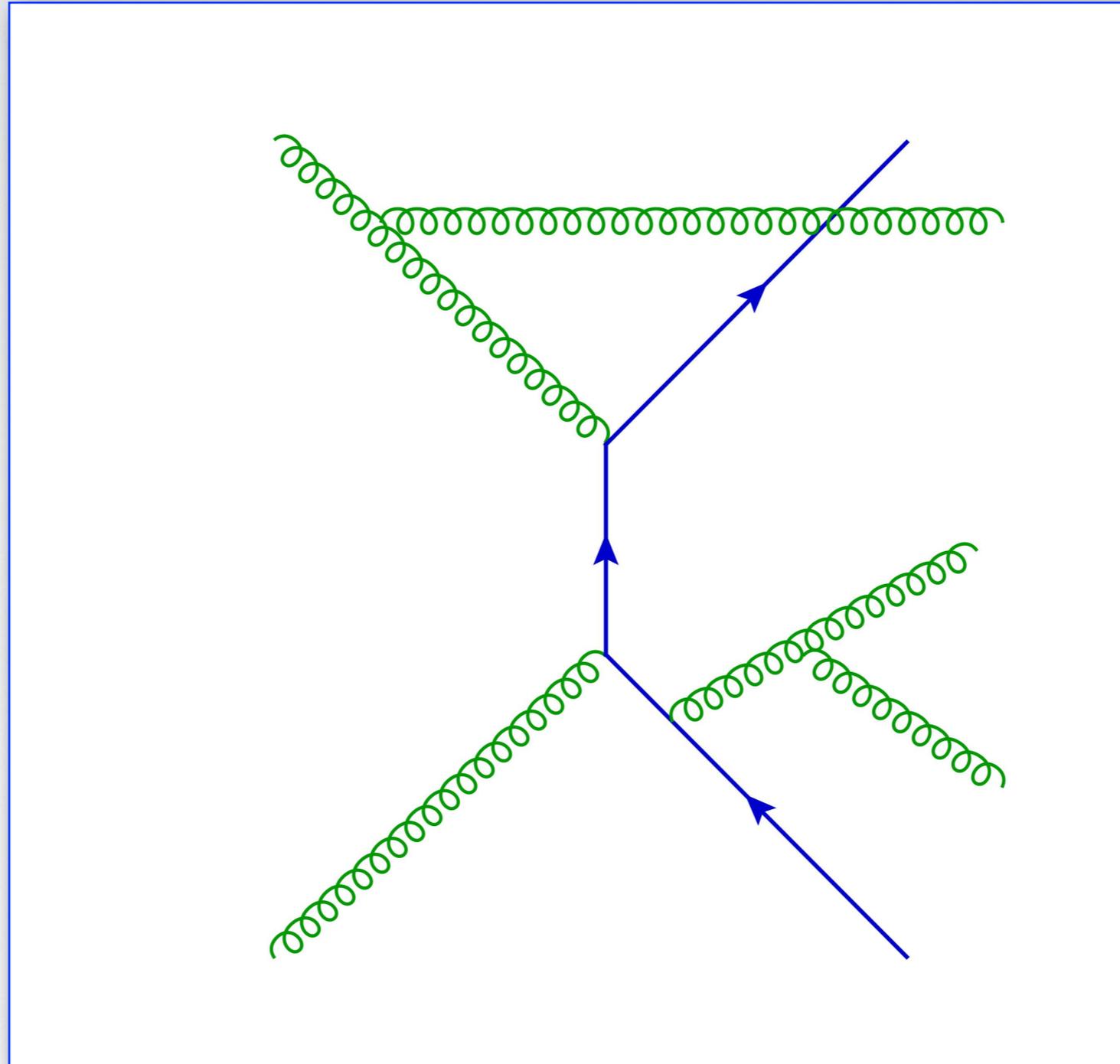
$$\left(\frac{dE}{E} \frac{dk_\perp}{k_\perp} \right)^2$$

A diagram contributing a double-real NNLO correction to t-tbar-jet production

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- Can one use the structure of virtual singularities as an organising principle for subtraction?

- Can the simplifying features of virtual corrections be exported to real radiation?

The subtraction problem at NLO is completely solved, with efficient algorithms applicable to any process for which matrix elements are known.

At NNLO after fifteen years of efforts several groups have working algorithms, successfully applied to `simple' process with up to four legs. Heavy computational costs.

- Antenna Subtraction.
- Stripper
- Nested Soft-Collinear Subtractions.
- ColourfulNNLO.
- N-Jettiness Slicing.
- Q_T Slicing.
- Projection to Born.
- Unsubtraction.
- Geometric Slicing …

ALGORITHMS
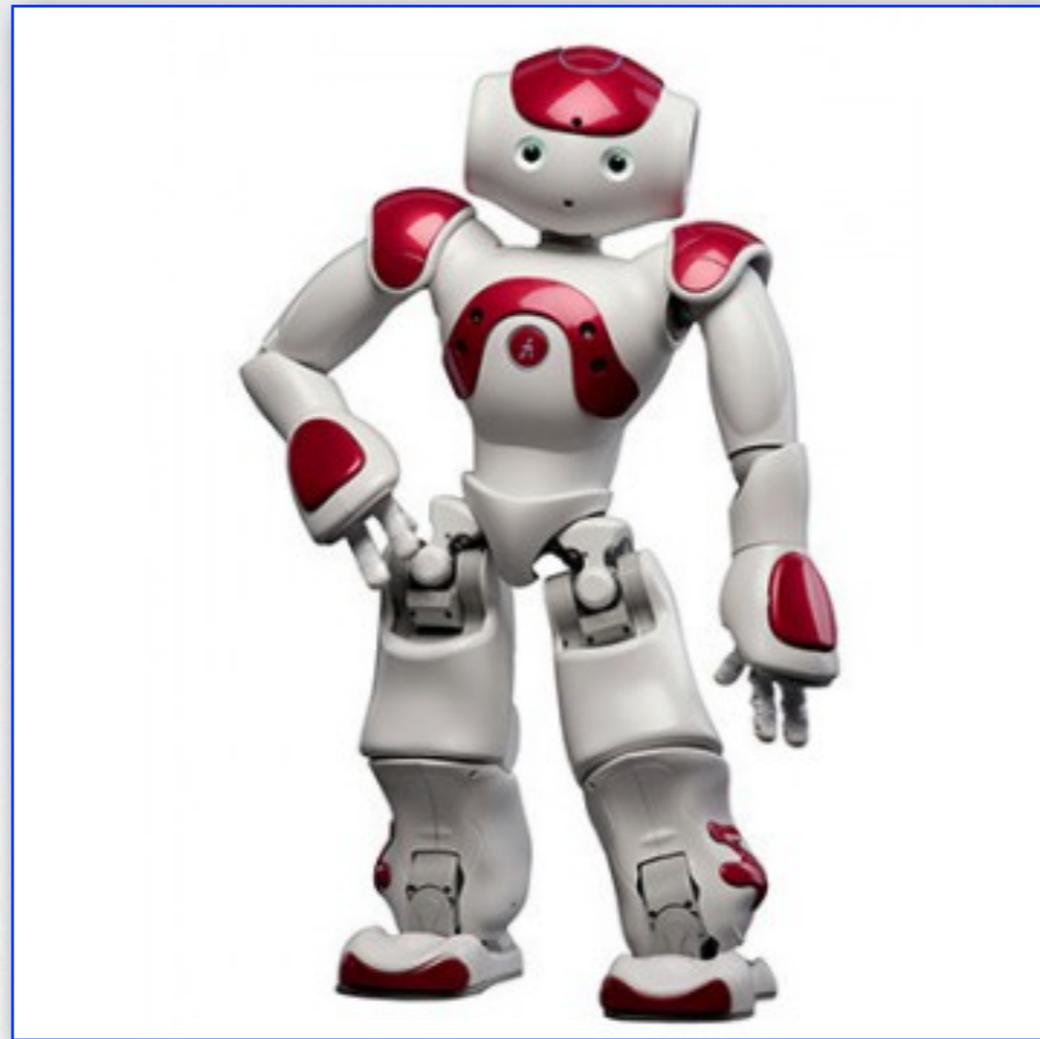
The computation of a generic IRC-safe observable at NLO requires the combination

$$\frac{d\sigma_{\mathrm{NLO}}}{dX} = \lim_{d \to 4} \left\{ \int d\Phi_n \, V_n \, \delta_n(X) + \int d\Phi_{n+1} \, R_{n+1} \, \delta_{n+1}(X) \right\},$$

The necessary numerical integrations require finite ingredients in d=4. Define counterterms

$$K_{n+1}^{(1)} = \mathbf{L}^{(1)} \, R_{n+1} \, .$$

$$I_n^{(1)} \equiv \int d\Phi_{\mathrm{r},1}^{n+1} \, K_{n+1}^{(1)} \, ,$$

Add and subtract the same quantity to the observable: each contribution is now finite.

$$\frac{d\sigma_{\mathrm{NLO}}}{dX} = \int d\Phi_n \left(V_n + I_n^{(1)} \right) \delta_n(X) + \int d\Phi_{n+1} \left(R_{n+1} \, \delta_{n+1}(X) - K_{n+1}^{(1)} \, \delta_n(X) \right),$$

Search for the simplest fully local integrand K_{n+1} with the correct singular limits.

The pattern of cancellations is more intricate at higher orders

$$\frac{d\sigma_{\rm NNLO}}{dX} = \lim_{d\to 4}\left\{\int d\Phi_n\, VV_n\, \delta_n(X) + \int d\Phi_{n+1}\, RV_{n+1}\, \delta_{n+1}(X) \right.$$

$$\left. + \int d\Phi_{n+2}\, RR_{n+2}\, \delta_{n+2}(X)\right\},$$

More counterterm functions need to be defined

$$K^{(1)}_{n+2} = \mathbf{L}^{(1)}\, RR_{n+2}\,, \qquad K^{(2)}_{n+2} = \mathbf{L}^{(2)}\, RR_{n+2}\,, \qquad K^{(12)}_{n+2} = \mathbf{L}^{(1)}\,\mathbf{L}^{(2)}\, RR_{n+2}\,, \qquad K^{(\mathbf{RV})}_{n+1} = \mathbf{L}^{(1)}\, RV_{n+1}\,.$$

$$I^{(1)}_{n+1} = \int d\Phi^{n+2}_{\mathrm{r},1}\, K^{(1)}_{n+2}\,, \quad I^{(12)}_{n+1} = \int d\Phi^{n+2}_{\mathrm{r},1}\, K^{(12)}_{n+2}\,, \quad I^{(2)}_{n} = \int d\Phi^{n+2}_{\mathrm{r},2}\, K^{(2)}_{n+2}\,, \quad I^{(\mathbf{RV})}_{n} = \int d\Phi^{n+1}_{\mathrm{r},1}\, K^{(\mathbf{RV})}_{n+1}\,.$$

A finite expression for the observable in d=4 must combine several ingredients

$$\frac{d\sigma_{\rm NNLO}}{dX} = \int d\Phi_n\left[VV_n + I^{(2)}_n + I^{(\mathbf{RV})}_n\right]\delta_n(X)$$

$$+ \int d\Phi_{n+1}\left[\left(RV_{n+1} + I^{(1)}_{n+1}\right)\delta_{n+1}(X) - \left(K^{(\mathbf{RV})}_{n+1} + I^{(12)}_{n+1}\right)\delta_n(X)\right]$$

$$+ \int d\Phi_{n+2}\left[RR_{n+2}\,\delta_{n+2}(X) - K^{(1)}_{n+2}\,\delta_{n+1}(X) - \left(K^{(2)}_{n+2} - K^{(12)}_{n+2}\right)\delta_n(X)\right]$$

A systematic generalisation to higher orders is possible. At three loops one finds

$$
\begin{aligned}
\frac{d\sigma_{\mathrm{N3LO}}}{dX} \;=\; & \int d\Phi_n \left[VVV_n + I_n^{(3)} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\,\mathbf{2})} \right] \delta_n(X) \\[2mm]
& + \int d\Phi_{n+1} \left[\left(RVV_{n+1} + I_{n+1}^{(2)} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{1})} \right) \delta_{n+1}(X) \right. \\[2mm]
& \qquad\qquad \left. - \left(K_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(23)} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{12})} \right) \delta_n(X) \right] \\[2mm]
& + \int d\Phi_{n+2} \left\{ \left(RRV_{n+2} + I_{n+2}^{(1)} \right) \delta_{n+2}(X) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(12)} \right) \delta_{n+1}(X) \right. \\[2mm]
& \qquad\qquad \left. - \left[\left(K_{n+2}^{(\mathbf{RRV},\,\mathbf{2})} + I_{n+2}^{(13)} \right) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{12})} + I_{n+2}^{(123)} \right) \right] \delta_n(X) \right\} \\[2mm]
& + \int d\Phi_{n+3} \left[RRR_{n+3}\, \delta_{n+3}(X) - K_{n+3}^{(1)}\, \delta_{n+2}(X) - \left(K_{n+3}^{(2)} - K_{n+3}^{(12)} \right) \delta_{n+1}(X) \right. \\[2mm]
& \qquad\qquad \left. - \left(K_{n+3}^{(3)} - K_{n+3}^{(13)} - K_{n+3}^{(23)} + K_{n+3}^{(123)} \right) \delta_n(X) \right],
\end{aligned}
$$

A general formula for $N^k LO$ subtraction is available, involving $p = 2^{(k+1)} - 2 - k$ counterterms.

NNLO status

- So far we have applied the formalism to massless final state radiation.

- For this case, at NLO we have a full-fledged subtraction formalism, and simple integrals.

- A simple proof-of-concept case (double-quark-pair production) has been completed.

- A complete set of NNLO sector functions with the desired sum rules is available.

- Flexible phase space mappings for single and double unresolved limits exist.

- Phase space mappings have been checked not to misalign nested limits.

- All integrals for final state radiation are done analytically, with no need for IBP techniques.

- The development of a differential code for NNLO subtraction is under way.

- Generalisation to initial state radiation requires work but no new concepts.

- More `interesting' integrals may arise with massive partons.

FACTORISATION
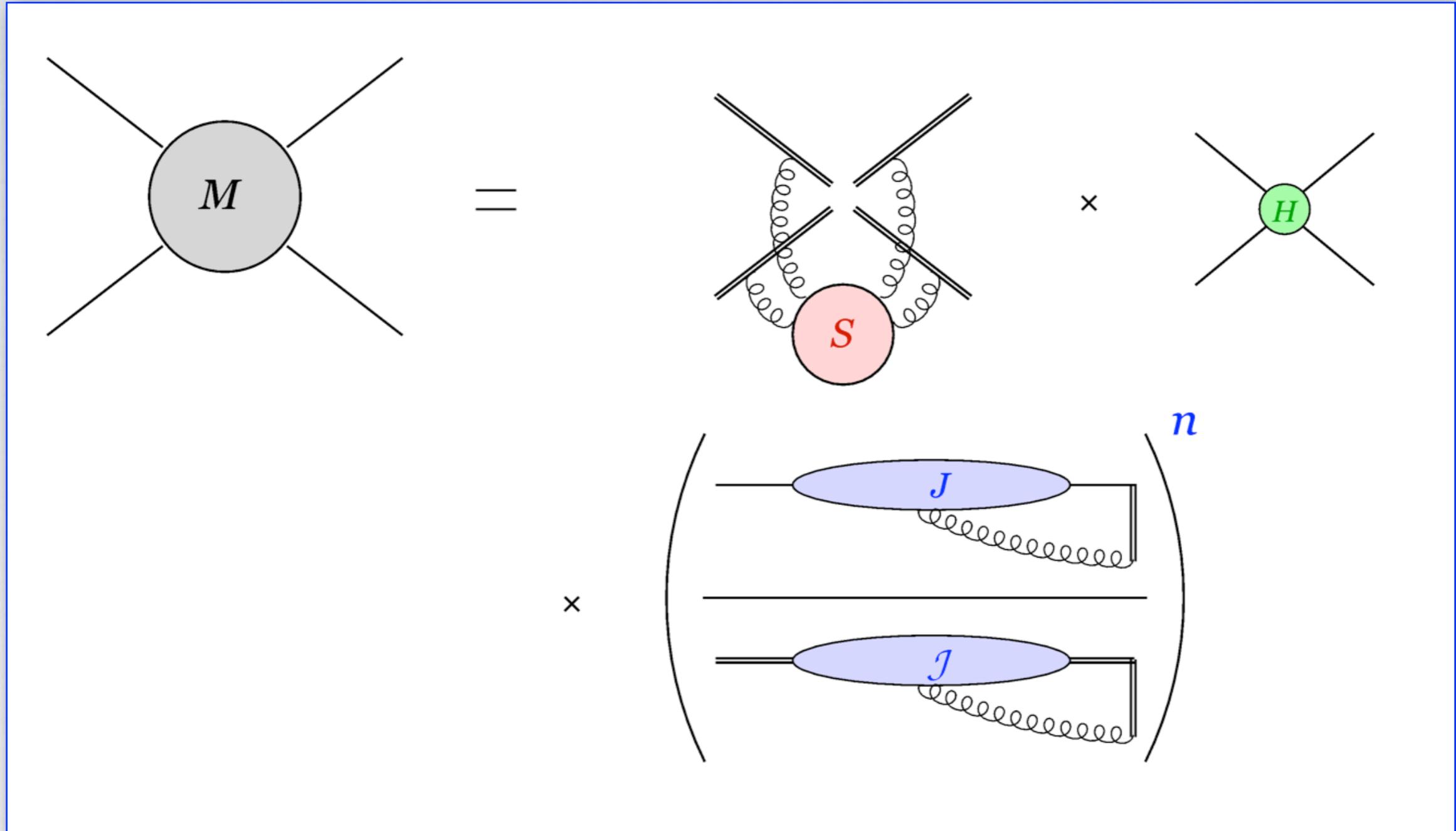
A pictorial representation of soft-collinear factorisation for fixed-angle scattering amplitudes
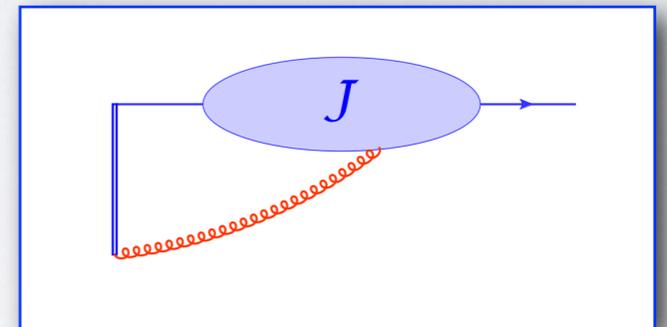
Operator Definitions

The precise **functional form** of this graphical factorisation is

$$
\mathcal{A}_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^{n}\left[\frac{\mathcal{J}_i\left((p_i\cdot n_i)^2/(n_i^2\mu^2)\right)}{\mathcal{J}_{E,i}\left((\beta_i\cdot n_i)^2/n_i^2\right)}\right]\,\mathcal{S}_n\left(\beta_i\cdot\beta_j\right)\,\mathcal{H}_n\left(\frac{p_i\cdot p_j}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}\right)
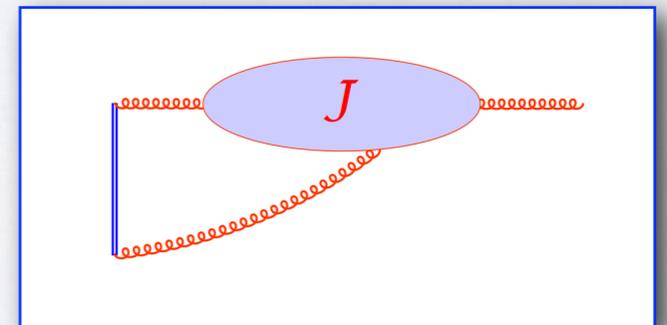$$

Here we introduced dimensionless **four-velocities** $\beta_i = p_i/Q$, and **factorisation vectors** n_i^μ , $n_i^2 \neq 0$ to define the jets in a **gauge-invariant** way. For **outgoing quarks**

$$
\bar{u}_s(p)\,\mathcal{J}_q\left(\frac{(p\cdot n)^2}{n^2\mu^2}\right) = \langle p,s\,|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle
$$

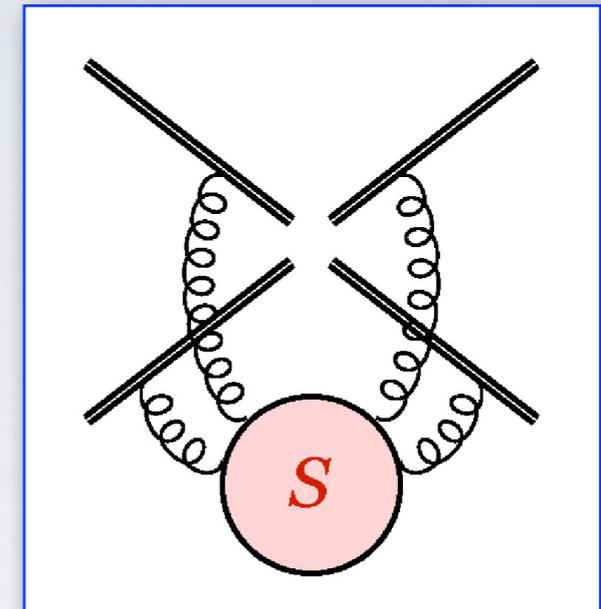where Φ_n is the **Wilson line** operator along the direction n. For **outgoing gluons**

$$
g_s\,\varepsilon_\mu^{*\,(\lambda)}(k)\,\mathcal{J}_g^{\mu\nu}\left(\frac{(k\cdot n)^2}{n^2\mu^2}\right) \equiv \langle k,\lambda|\left[\Phi_n(\infty,0)\,iD^\nu\,\Phi_n(0,\infty)\right]|0\rangle\ ,
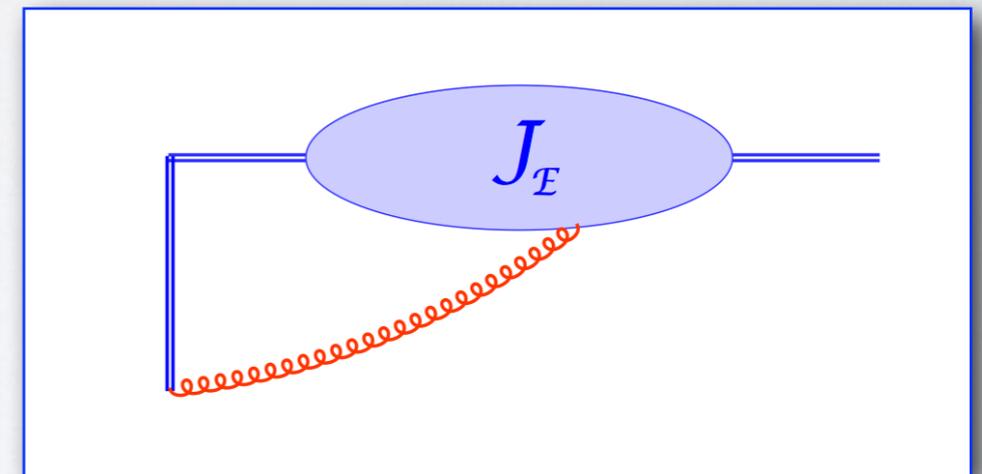$$

Becher, Bell 2010

Wilson line correlators

The soft function S is a color operator, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$\mathcal{S}_n \left(\beta_i \cdot \beta_j \right) = \langle 0| \prod_{k=1}^{n} \Phi_{\beta_k}(\infty, 0) |0\rangle$$

The soft jet function $J_{\mathcal{E}}$ contains soft-collinear poles: it is defined by replacing the field in the ordinary jet J with a Wilson line in the appropriate color representation.

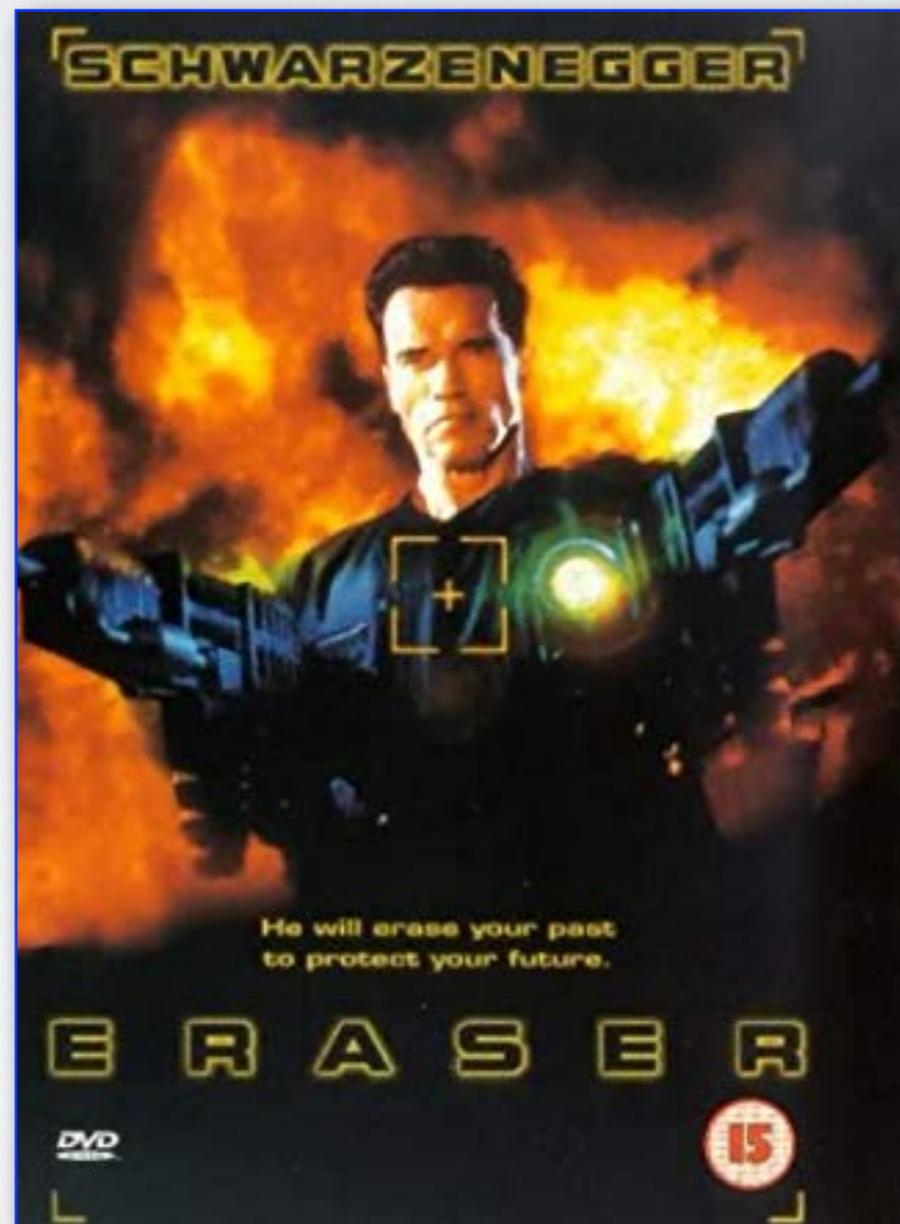$$\mathcal{J}_{\mathrm{E}} \left(\frac{(\beta \cdot n)^2}{n^2} \right) = \langle 0| \Phi_\beta(\infty, 0) \Phi_n(0, \infty) |0\rangle$$

Wilson-line matrix elements exponentiate non-trivially and have tightly constrained functional dependence on their arguments. They are known to three loops.

COUNTERTERMS

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At amplitude level poles factorise and exponentiate.
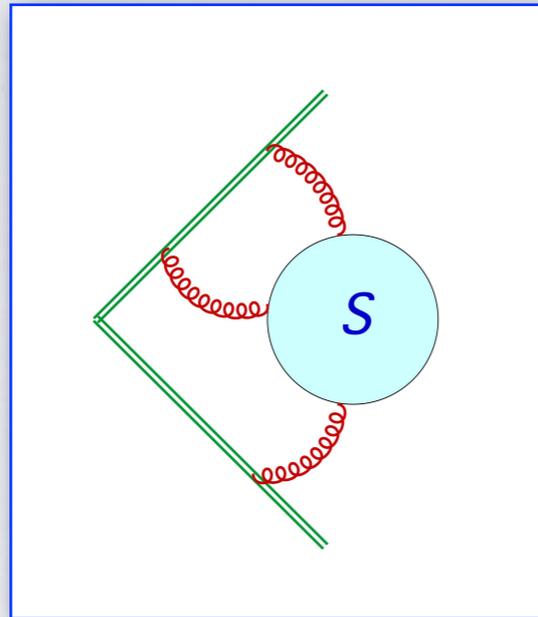
Soft cross sections: pictorial

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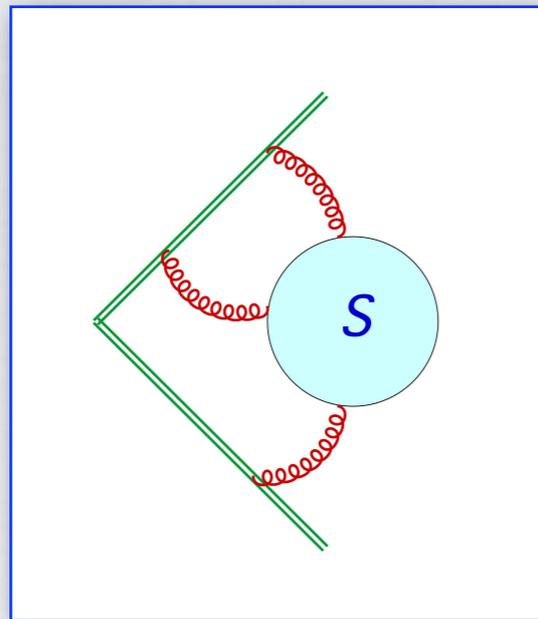

We need to build **cross-section level** quantities.
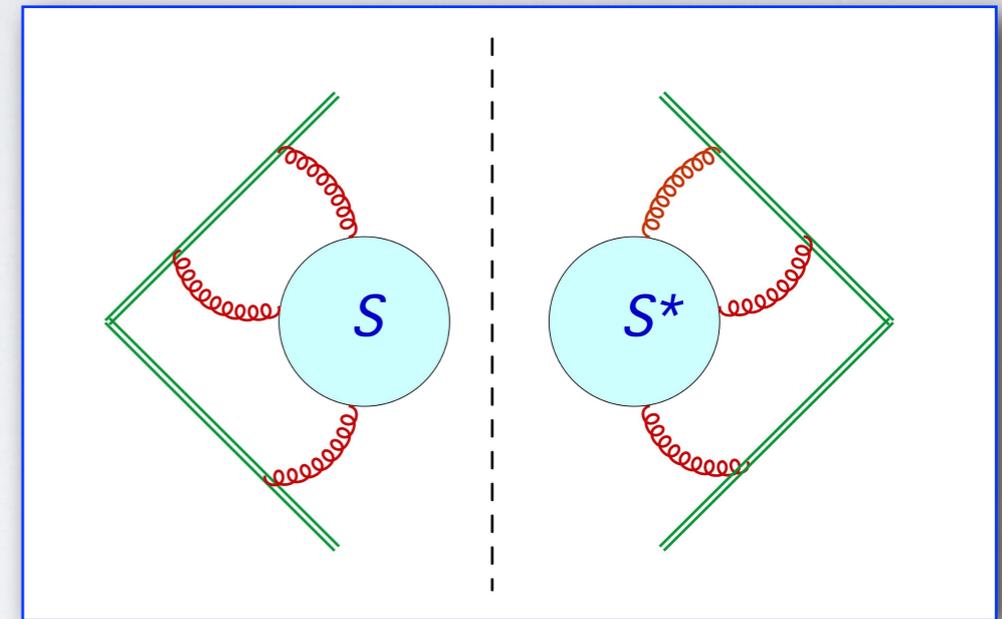
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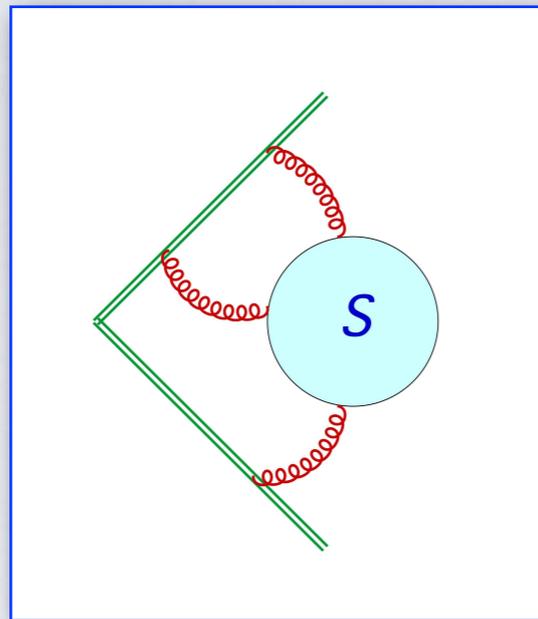
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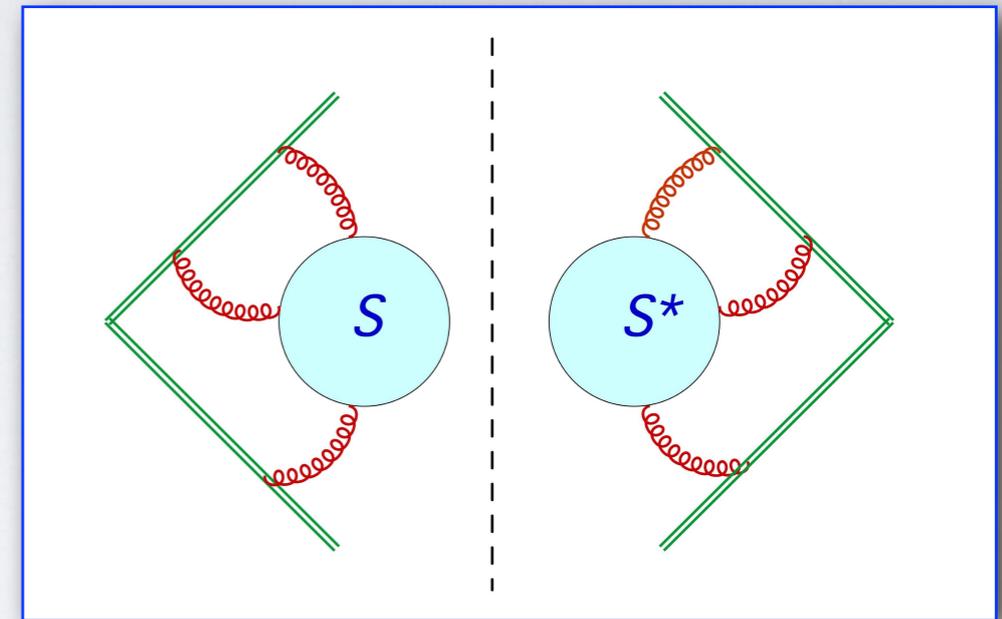
- Inclusive eikonal cross sections are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local IR counterterms.
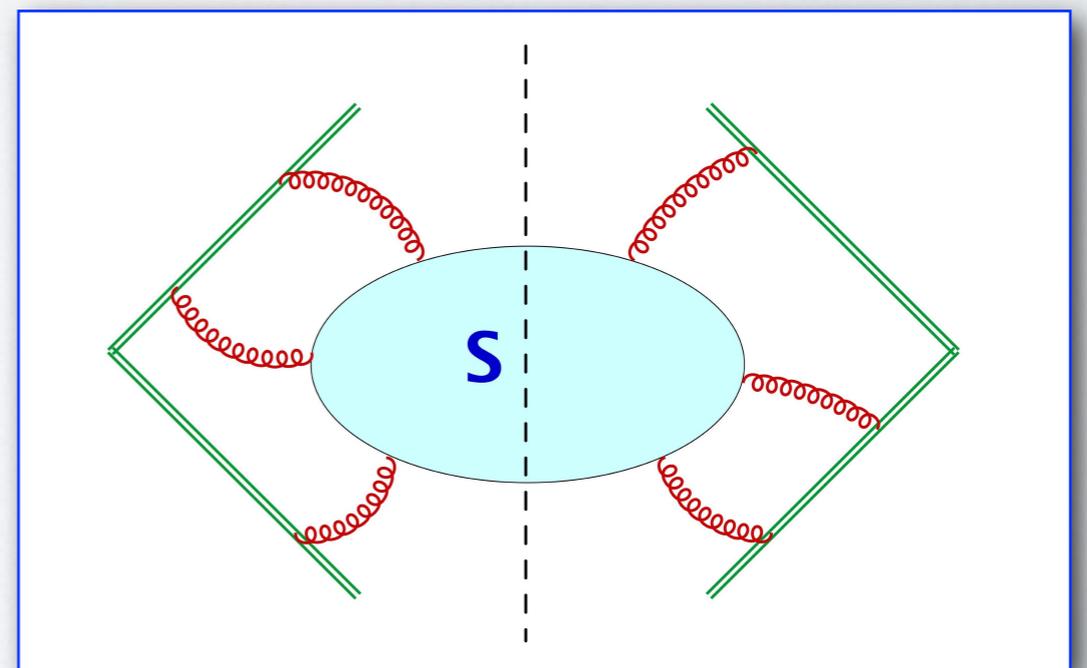
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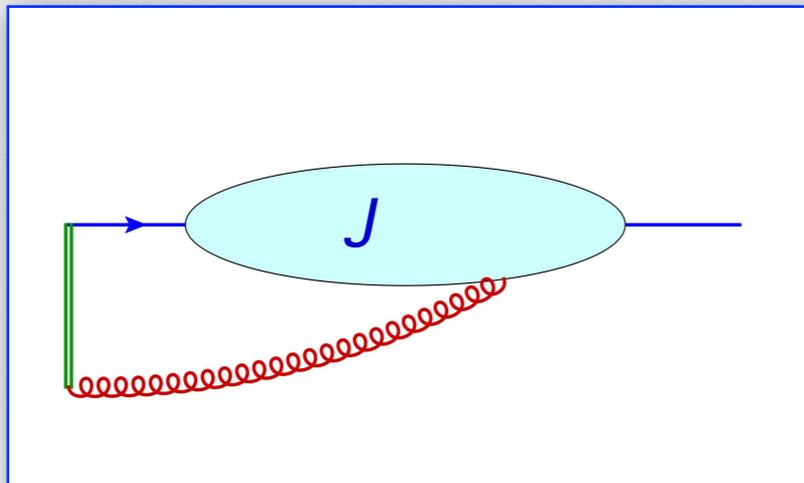
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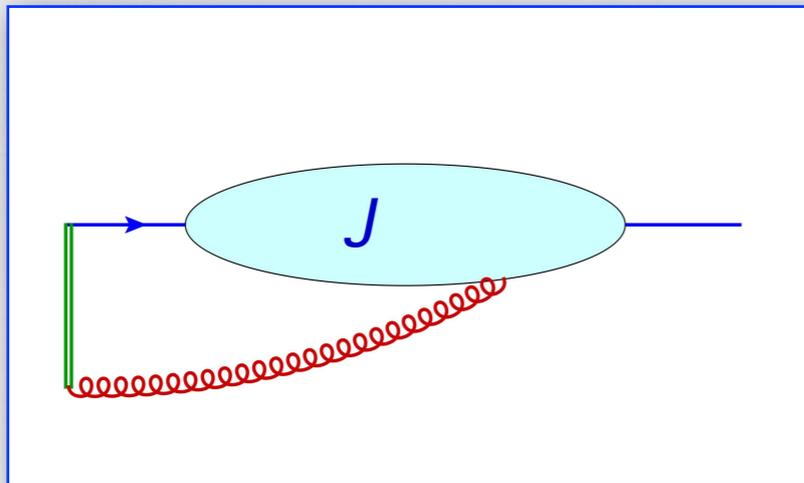
Soft-collinear poles can be subtracted
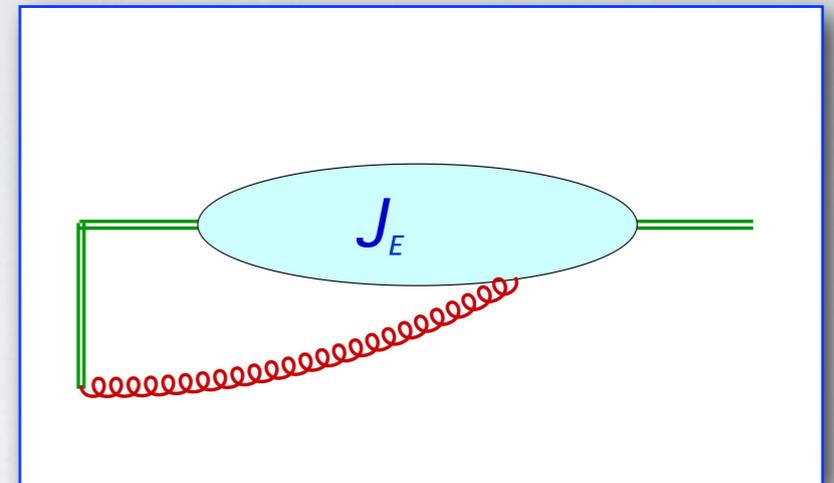
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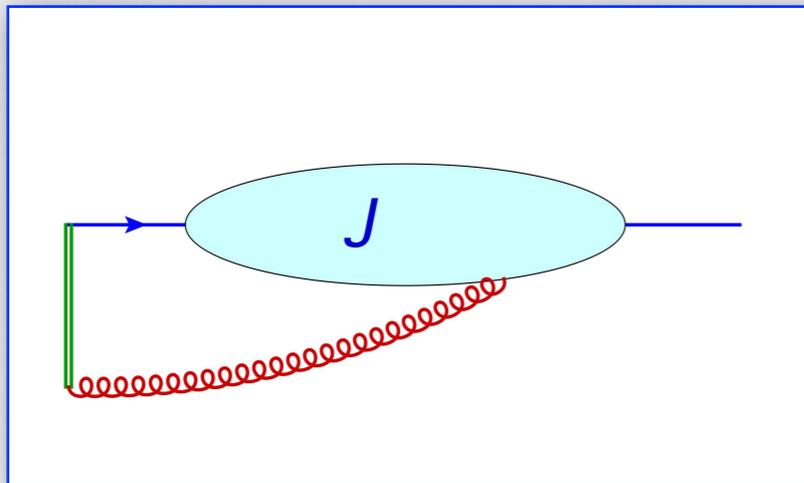
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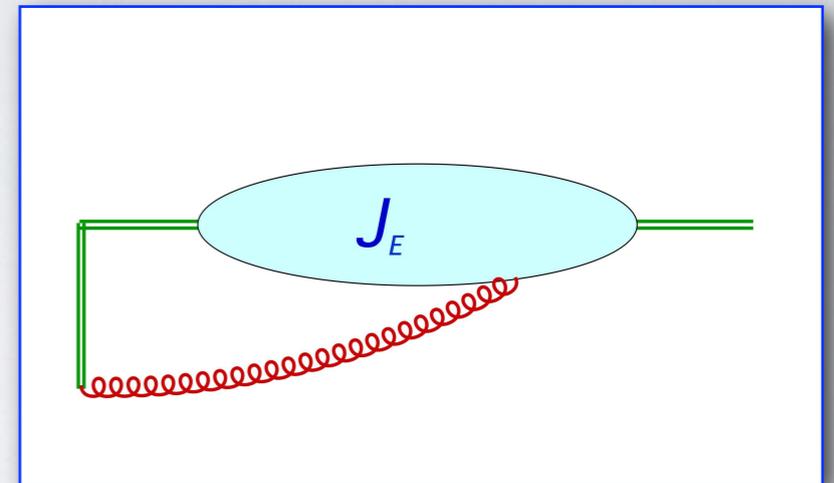
- Inclusive `jet cross sections' are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local collinear counterterms.
- Eikonal jet cross sections subtract the soft-collinear double counting.
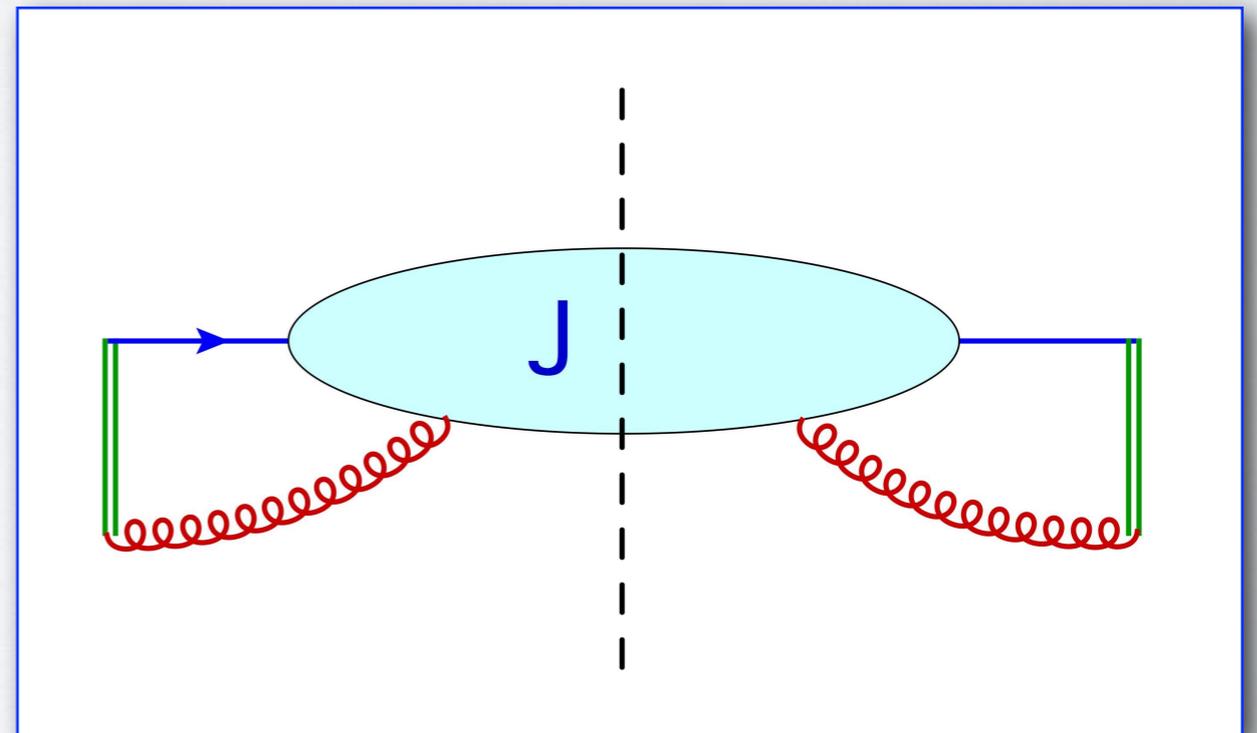
Soft counterterms: all orders

Introduce **eikonal form factors** for the emission of **m soft** partons from **n hard** ones.

$$
\begin{aligned}
\mathcal{S}_{n,m}\left(k_1, \ldots, k_m; \beta_i\right) &\equiv \langle k_1, \lambda_1; \ldots; k_m, \lambda_m | \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0) \, |0\rangle \\
&\equiv \epsilon_{\mu_1}^{*\,(\lambda_1)}(k_1) \ldots \epsilon_{\mu_m}^{*\,(\lambda_m)}(k_m) \, J_{\mathcal{S}}^{\mu_1 \ldots \mu_m}\left(k_1, \ldots, k_m; \beta_i\right) \\
&\equiv \sum_{p=0}^{\infty} \mathcal{S}_{n,m}^{(p)}\left(k_1, \ldots, k_m; \beta_i\right)
\end{aligned}
$$

These matrix elements **define** soft gluon **multiple emission currents**. They are **gauge invariant** and they contain **loop corrections** to all orders.

Existing finite order **calculations** and all-order **arguments** are **consistent** with the **factorisation**

$$
\mathcal{A}_{n,m}\left(k_1, \ldots, k_m; p_i\right) = \mathcal{S}_{n,m}\left(k_1, \ldots, k_m; \beta_i\right) \mathcal{H}_n(p_i) + \mathcal{R}_{n,m}\left(k_1, \ldots, k_m; p_i\right)
$$

with **corrections** that are **finite** in dimensional regularisation, and **integrable** in the soft gluon phase space. It is a **working assumption**: a formal all-order proof is still **lacking**.

Soft counterterms: all orders

The factorisation is reflected at cross-section level, for fixed final state quantum numbers.

$$\sum_{\lambda_i} |\mathcal{A}_{n,m}(k_1,\ldots,k_m;p_i)|^2 \simeq \mathcal{H}_n^\dagger(p_i)\, S_{n,m}(k_1,\ldots,k_m;\beta_i)\, \mathcal{H}_n(p_i)$$

The *cross-section level* "radiative soft functions" are Wilson-line squared matrix elements

$$S_{n,m}(k_1,\ldots,k_m;\beta_i) \equiv \sum_{p=0}^{\infty} S_{n,m}^{(p)}(k_1,\ldots,k_m;\beta_i)$$

$$\equiv \sum_{\lambda_i} \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}(0,\infty) |k_1,\lambda_1;\ldots;k_m,\lambda_m\rangle \langle k_1,\lambda_1;\ldots;k_m,\lambda_m| \prod_{i=1}^{n} \Phi_{\beta_i}(\infty,0) |0\rangle \, .$$

These functions provide a complete list of local soft subtraction counterterms, to all orders. Indeed, summing over particle numbers and integrating over the soft phase space one finds

$$\sum_{m=0}^{\infty} \int d\Phi_m\, S_{n,m}(k_1,\ldots,k_m;\beta_i) = \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}(0,\infty) \prod_{i=1}^{n} \Phi_{\beta_i}(\infty,0) |0\rangle$$

This is a finite fully inclusive soft cross section, order by order in perturbation theory.

Soft current at tree level

At NLO, only the tree-level single-emission current is required, simply defined by

$$\epsilon^{*\,(\lambda)}(k) \cdot J_{\mathcal{S}}^{(0)}(k, \beta_i) = \mathcal{S}_{n,1}^{(0)}(k; \beta_i) = \langle k, \lambda | \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0) \, |0\rangle \Big|_{\text{tree}}$$

One obviously recovers all the well-known results for the leading-order soft gluon current

$$\mathcal{A}_{n,1}^{(0)}(k, p_i) = \epsilon^{*\,(\lambda)}(k) \cdot J_{\mathcal{S}}^{(0)}(k, \beta_i)\, \mathcal{H}_n^{(0)}(p_i) + \mathcal{O}(k^0)$$

$$J_{\mathcal{S}}^{\mu\,(0)}(k, \beta_i) = g \sum_{i=1}^{n} \frac{\beta_i^{\mu}}{\beta_i \cdot k}\, \mathbf{T}_i \cdot$$

For the cross-section, the tree-level single-radiation soft function acts as a local counterterm.

$$\sum_{\lambda} \left| \mathcal{A}_{n,1}^{(0)}(k, p_i) \right|^2 \simeq \mathcal{H}^{(0)\,\dagger}(p_i)\, S_{n,1}^{(0)}(k; \beta_i)\, \mathcal{H}_n^{(0)}(p_i)$$

$$= -4\pi\alpha_s \sum_{i,j=1}^{n} \frac{\beta_i \cdot \beta_j}{\beta_i \cdot k\, \beta_j \cdot k}\, \mathcal{A}_n^{(0)\dagger}(p_i)\, \mathbf{T}_i \cdot \mathbf{T}_j\, \mathcal{A}_n^{(0)}(p_i)$$

- The single-radiative soft function acts as a color operator on the color-correlated Born.
- Beyond NLO, tree-level multiple gluon emission currents also follow from this definition.

At one loop, for single radiation, our definition of the soft currents gives

$$\mathcal{A}_{n,1}(k; p_i) \simeq \mathcal{S}_{n,1}(k; \beta_i)\, \mathcal{H}_n(p_i)$$
$$= \mathcal{S}_{n,1}^{(0)}(k; \beta_i)\, \mathcal{H}_n^{(1)}(p_i) + \mathcal{S}_{n,1}^{(1)}(k; \beta_i)\, \mathcal{H}_n^{(0)}(p_i)$$

The factorisation proposed in the classic work by Catani-Grazzini appears different

$$\mathcal{A}_{n,1}(k; p_i) \simeq \epsilon^{*\,(\lambda)}(k) \cdot J_{\mathrm{CG}}(k, \beta_i)\, \mathcal{A}_n(p_i)$$

but it is easily matched using the factorisation of the non-radiative amplitude

$$\mathcal{A}_n(p_i) \simeq \mathcal{S}_n(\beta_i)\, \mathcal{H}_n(p_i) \quad \longrightarrow \quad \mathcal{H}_n^{(1)}(p_i) = \mathcal{A}_n^{(1)}(p_i) - \mathcal{S}_n^{(1)}(\beta_i)\, \mathcal{A}_n^{(0)}(p_i)$$

Recombining, we get an explicit eikonal expression for the CG one-loop soft current

$$\epsilon^{*\,(\lambda)}(k) \cdot J_{\mathrm{CG}}^{(1)}(k, \beta_i) = \mathcal{S}_{n,1}^{(1)}(k; \beta_i) - \mathcal{S}_{n,1}^{(0)}(k; \beta_i)\, \mathcal{S}_n^{(1)}(\beta_i)$$

The two calculations are easily matched: same diagrammatic content, cancellations and result.

Collinear counterterms: all orders

For collinear poles, introduce jet matrix elements for the emission of m partons. For quarks

$$\overline{u}_s(p)\, \mathcal{J}_{q,m}(k_1, \ldots, k_m; p, n) \equiv \langle p, s; k_1, \lambda_1; \ldots; k_m, \lambda_m|\, \overline{\psi}(0)\, \Phi_n(0, \infty)\, |0\rangle$$

At *cross-section level*, "radiative jet functions" can be defined as Fourier transforms of squared matrix elements, to account for the non-trivial momentum flow. We propose

$$J_{q,m}(k_1, \ldots k_m; l, p, n) \equiv \sum_{p=0}^{\infty} J_{q,m}^{(p)}(k_1, \ldots k_m; l, p, n)$$

$$\equiv \int d^d x\, \mathrm{e}^{il\cdot x} \sum_{\{\lambda_j\}} \langle 0|\, \Phi_n(\infty, x)\, \psi(x)\, |p, s; k_j, \lambda_j\rangle\, \langle p, s; k_j, \lambda_j|\, \overline{\psi}(0)\, \Phi_n(0, \infty)\, |0\rangle$$

These functions provide a complete list of local collinear counterterms, to all orders. Summing over particle numbers and integrating over the collinear phase space one finds

$$\sum_{m=0}^{\infty} \int d\Phi_{m+1}\, J_{q,m}(k_1, \ldots, k_m; l, p, n) = \mathrm{Disc}\left[\int d^d x\, \mathrm{e}^{il\cdot x}\, \langle 0|\, \Phi_n(\infty, x)\psi(x)\overline{\psi}(0)\Phi_n(0, \infty)\, |0\rangle\right]$$

A "two-point function", finite order by order in perturbation theory. Note however

- The collinear limit must still be taken (as $l^2 \to 0$), unlike the case of radiative soft functions.
- Working with $n^2 \neq 0$ eliminates spurious collinear poles, but is cumbersome in practice.

Collinear counterterms: NLO

At NLO, only tree-level single-emission contributes, resulting (for quarks) in three diagrams
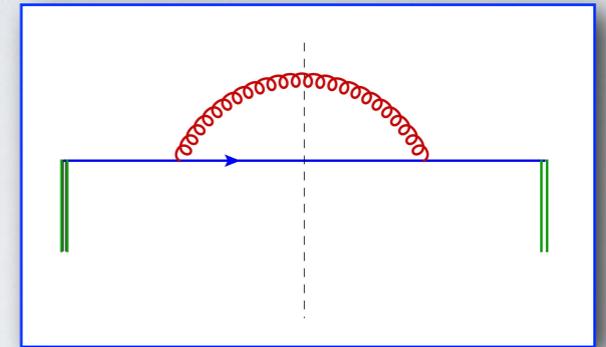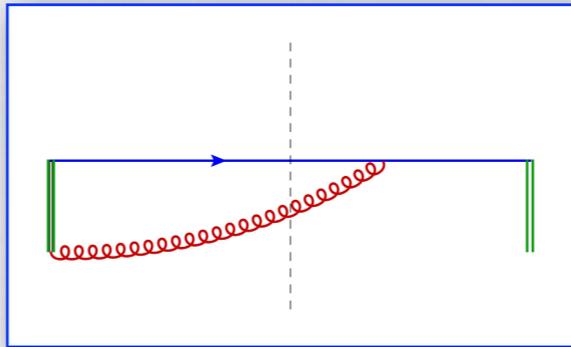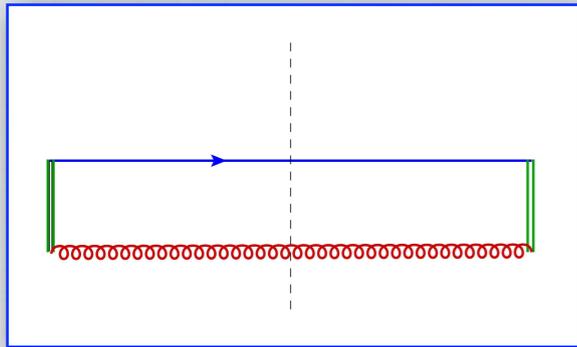
Summing over helicities, and taking the $n^2 \to 0$ limit, one finds a spin-dependent kernel

$$\sum_s J_{q,1}(k;l,p,n) = \frac{4\pi\alpha_s C_F}{l^2}(2\pi)^d \delta^d(l-p-k)\left[-\not{l}\gamma_\mu\not{p}\gamma^\mu\not{l} + \frac{1}{k\cdot n}(\not{l}\not{n}\not{p}+\not{p}\not{n}\not{l})\right]$$

With a Sudakov decomposition

$$p^\mu = z l^\mu + \mathcal{O}(l_\perp), \qquad k^\mu = (1-z)l^\mu + \mathcal{O}(l_\perp), \qquad n^2 = 0$$

and taking $l_\perp \to 0$, one recovers the full unpolarised DGLAP LO splitting kernel.

$$\sum_s J_{q,1}(k;l,p,n) = \frac{8\pi\alpha_s C_F}{l^2}(2\pi)^d \delta^d(l-p-k)\left[\frac{1+z^2}{1-z} - \epsilon(1-z) + \mathcal{O}(l_\perp)\right]$$

- The three diagrams map precisely to the axial gauge calculation by Catani, Grazzini.
- All LO DGLAP kernels are easily reproduced, triple collinear limits are under way.

The outlines of a subtraction procedure emerge. Begin by expanding the virtual matrix element

$$
\mathcal{A}_n(p_i) = \left[\mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) \right.
$$
$$
\left. + \sum_{i=1}^{n} \left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},i}^{(1)}(\beta_i) \right) \mathcal{S}_n^{(0)}(\beta_i) \, \mathcal{H}_n^{(0)}(p_i) \right] \left(1 + \mathcal{O}\left(\alpha_s^2\right) \right)
$$

From the master formula, get the virtual poles of the cross section in terms of virtual kernels

$$
V_n \equiv 2\,\mathbf{Re}\left[\mathcal{A}_n^{(0)*} \mathcal{A}_n^{(1)} \right] \simeq \mathcal{H}_n^{(0)\,\dagger}(p_i) S_{n,0}^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \sum_i \left(J_{i,0}^{(1)}(p_i) - J_{\mathrm{E},i,0}^{(1)}(\beta_i) \right) \left| \mathcal{A}_n^{(0)}(p_i) \right|^2
$$

Go through the list of proposed soft and collinear counterterms to collect the relevant ones

$$
S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1 \, S_{n,1}^{(0)}(k,\beta_i) = \text{finite}
$$

$$
J_{i,0}^{(1)}(l,p,n) + \int d\Phi_1 \, J_{i,1}^{(0)}(k;l,p,n) = \text{finite}
$$

Construct the appropriate local functions.

$$
K_{n+1}^{\mathrm{NLO,\,s}} = \mathcal{H}_n^{(0)\,\dagger}(p_i) \, S_{n,1}^{(0)}(k,\beta_i) \, \mathcal{H}_n^{(0)}(p_i)
$$

$$
K_{n+1}^{\mathrm{NLO,\,c}} = \sum_{i=1}^{n} J_{i,1}^{(0)}(k_i;l,p_i,n_i) \left| \mathcal{A}_n^{(0)}(p_1,\ldots,p_{i-1},l,p_{i+1},\ldots,p_n) \right|^2
$$

with a similar expression for the anti-subtraction of the soft-collinear region in terms of $J_{\mathcal{E}}$.

Let us follow the **same procedure** at NNLO. Collect the poles of the **virtual** amplitude

$$
\begin{aligned}
\mathcal{A}_n^{(2)}(p_i) &= \mathcal{S}_n^{(2)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(2)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) \\
&+ \sum_{i=1}^{n}\left[\mathcal{J}_i^{(2)}(p_i) - \mathcal{J}_{\mathrm{E},i}^{(2)}(\beta_i) - \mathcal{J}_{\mathrm{E},i}^{(1)}(\beta_i)\left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},i}^{(1)}(\beta_i) \right) \right] \mathcal{A}_n^{(0)}(p_i) \\
&+ \sum_{i<j=1}^{n}\left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},i}^{(1)}(\beta_i) \right)\left(\mathcal{J}_j^{(1)}(p_j) - \mathcal{J}_{\mathrm{E},j}^{(1)}(\beta_j) \right) \mathcal{A}_n^{(0)}(p_i) \\
&+ \sum_{i=1}^{n}\left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},i}^{(1)}(\beta_i) \right)\left[\mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) \right]
\end{aligned}
$$

Cross-section level soft and jet functions have **non-trivial structure** starting at NNLO

$$
S_n^{(2)} = \mathcal{S}_n^{(0)\,\dagger}\mathcal{S}_n^{(2)} + \mathcal{S}_n^{(2)\,\dagger}\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(1)\,\dagger}\mathcal{S}_n^{(1)}
$$

$$
J_{q,m}^{(2)} = \int d^d x\, e^{il\cdot x} \sum_{\{\lambda_j\}}\left[\mathcal{J}_{q,m}^{(1)\,\dagger}(x)\,\slashed{p}\,\mathcal{J}_{q,m}^{(1)}(0) + \mathcal{J}_{q,m}^{(0)\,\dagger}(x)\,\slashed{p}\,\mathcal{J}_{q,m}^{(2)}(0) + \mathcal{J}_{q,m}^{(0)}(x)\,\slashed{p}\,\mathcal{J}_{q,m}^{(2)\,\dagger}(0) \right]
$$

All poles of the **squared virtual** amplitude can nonetheless be **expressed** in terms of **squared jets** and **eikonal** correlators, which leads to the **identification** of local NNLO counterterms.

Cross-section level **double-virtual** poles originate from **a number** of **different** configurations

$$(VV)_n \equiv (VV)_n^{(2s)} + (VV)_n^{(1s)} + \sum_{i=1}^{n} (VV)_{n,i}^{(2hc)} + \sum_{i<j=1}^{n} (VV)_{n,ij}^{(2hc)} + \sum_{i=1}^{n} (VV)_{n,i}^{(1hc,1s)} + \sum_{i=1}^{n} (VV)_{n,i}^{(1hc)}$$

Focus on **double collinear** radiation along the direction of a **selected hard particle**. One finds

$$(VV)_{n,i}^{(2hc)} = \left[J_{i,0}^{(2)} - J_{E,i,0}^{(2)} - J_{E,i,0}^{(1)} \left(J_{i,0}^{(1)} - J_{E,i,0}^{(1)} \right) \right] \left| \mathcal{A}_n^{(0)} \right|^2$$

It is easy to **identify finite combinations** of **virtual** and **real** (hard) collinear radiation

$$J_{i,0}^{(2)} + \int d\Phi_1 \, J_{i,1}^{(1)} + \int d\Phi_2 \, J_{i,2}^{(0)} = \text{finite}$$

$$\left[J_{E,i,0}^{(1)} + \int d\Phi_1 \, J_{E,i,1}^{(0)} \right] \left[J_{i,0}^{(1)} - J_{E,i,0}^{(1)} + \int d\Phi_1' \left(J_{i,1}^{(0)} - J_{E,i,1}^{(0)} \right) \right] = \text{finite}$$

Real radiation **naturally organises** into **single** and **double** unresolved, and **real-virtual** terms

$$K_{n+2,i}^{\text{NNLO},(\mathbf{2},\text{hc})} = \left[J_{i,2}^{(0)} - J_{E,i,2}^{(0)} - J_{E,i,1}^{(0)} \left(J_{i,1}^{(0)} - \mathcal{J}_{E,i,1}^{(0)} \right) \right] \left| \mathcal{A}_n^{(0)} \right|^2$$

$$K_{n+2,i}^{\text{NNLO},(\mathbf{1},\text{hc})} = \left(J_{i,1}^{(0)} - \mathcal{J}_{E,i,1}^{(0)} \right) \left| \mathcal{A}_{n+1}^{(0)} \right|^2$$

$$K_{n+1,i}^{\text{NNLO},(\mathbf{RV},\text{hc})} = \left[J_{i,1}^{(1)} - \mathcal{J}_{E,i,1}^{(1)} - J_{i,0}^{(1)} J_{E,i,1}^{(0)} - J_{E,i,0}^{(1)} J_{i,1}^{(0)} + 2 J_{E,i,0}^{(1)} J_{E,i,1}^{(0)} \right] \left| \mathcal{A}_n^{(0)} \right|^2 .$$

Cross-section level **double-virtual** poles originate from **a number** of **different** configurations

$$(VV)_n \equiv (VV)_n^{(2\mathrm{s})} + (VV)_n^{(1\mathrm{s})} + \sum_{i=1}^{n}(VV)_{n,i}^{(2\mathrm{hc})} + \sum_{i<j=1}^{n}(VV)_{n,ij}^{(2\mathrm{hc})} + \sum_{i=1}^{n}(VV)_{n,i}^{(1\mathrm{hc},1\mathrm{s})} + \sum_{i=1}^{n}(VV)_{n,i}^{(1\mathrm{hc})}$$

Focus on **double soft** and **single soft** radiation. One finds

$$(VV)_n^{(2\mathrm{s})} = \mathcal{H}_n^{(0)\,\dagger}\, S_{n,0}^{(2)}\, \mathcal{H}_n^{(0)}$$

$$(VV)_n^{(1\mathrm{s})} = \mathcal{H}_n^{(0)\,\dagger} S_{n,0}^{(1)} \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger} S_{n,0}^{(1)} \mathcal{H}_n^{(0)}$$

It is easy to **identify finite combinations** of **virtual**, **real-virtual** and **double real** soft radiation

$$S_{n,0}^{(2)}(\beta_i) + \int d\Phi_1\, S_{n,1}^{(1)}(k,\beta_i) + \int d\Phi_2\, S_{n,2}^{(0)}(k_1,k_2,\beta_i) = \text{finite}\,.$$

Real radiation **naturally organises** into **single** and **double** unresolved, and **real-virtual** terms

$$K_{n+2}^{\mathrm{NNLO},\,(\mathbf{2},\mathrm{s})} = \mathcal{H}_n^{(0)\,\dagger}\, S_{n,2}^{(0)}\, \mathcal{H}_n^{(0)}$$

$$K_{n+2}^{\mathrm{NNLO},\,(\mathbf{1},\mathrm{s})} = \mathcal{H}_{n+1}^{(0)\,\dagger}\, S_{n+1,1}^{(0)}\, \mathcal{H}_{n+1}^{(0)}$$

$$K_{n+1}^{\mathrm{NNLO},\,(\mathbf{RV},\mathrm{s})} = \mathcal{H}_n^{(0)\,\dagger}\, S_{n,1}^{(0)}\, \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,1}^{(0)}\, \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(0)\,\dagger}\, S_{n,1}^{(1)}\, \mathcal{H}_n^{(0)}$$

OUTLOOK

Outlook

- A number of successful NNLO subtraction algorithms are available.

- They are computationally expensive, either analytically, or numerically, or both.

- Extensions to multi-leg processes or higher orders is expected to be useful but hard.

- Work on refining existing tools to find the `minimal toolbox' is necessary and under way.

- The factorisation of soft and collinear virtual amplitudes contains important information.

- A general all-order definition of soft and/or collinear counterterms has been proposed.

- Existing results at NLO and beyond are reproduced and systematised.

- Tracing the real emission counterterms starting from virtual poles is a useful strategy.

- A parallel effort to construct a detailed analytic subtraction algorithm is under way.

- What we have is promising preliminary evidence: a lot of work remains to be done.

THANK YOU

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