Local analytic sector subtraction: the Torino scheme

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in collaboration with:
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based on:
Magnea et al., arXiv:1806.09570, arXiv:1809.05444

Motivations

- **Small deviations** from Standard Model predictions can provide important tests for New Physics models.
- Hunting for such deviations requires **high precision predictions** to compare with high precision experiments.
- **Next-to-next-to-leading** (NNLO) in **QCD** is the current accuracy standard.
- The automation of QCD computations needs a **fully general** and efficient **treatment of the IR singularities**.

Schemes and tricks to deal with the IR

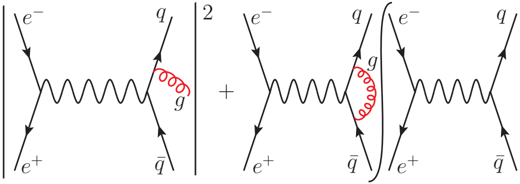
Few scheme available at NLO:

- Slicing: *[Giele, Glover]*
- Subtraction: dipole*[Catani, Seymour 9602277]*, FKS *[Frixione et al. 9512328]*, NS *[Nagy, Soper 0308127]*

Many schemes available at NNLO:

- Slicing: $q_\perp$ *[Catani, Grazzini 0703012]*, N-Jettiness *[Boughezal et al. 1505.03893, Gaunt et al. 1505.04794]*
- Subtraction: Antenna *[Gehrmann-DeRidder et al. 0505111]*, ColorfullNNLO *[Del Duca et al. 1603.08927]*, Nested soft-collinear *[Caola et al. 1702.01352]*, Geometric IR subtraction *[Herzog 1804.07949]*, ϵ-prescription *[Frixione, Grazzini 0411399]*, Sector decomposition *[Bonoth et al. 0402265, Anastasiou et al. 0311311]*, residue subtraction *[Czakon 1005.0274]*
- New stategies: Unsubtraction *[Sborlini et al. 1608.01584]*, FDR *[Pittau 1208.5457]*

$\rightarrow$ Many options, but still there is room for improvement!!!



Real contribution

complicated phase space integration

↓

Subtraction

↓

counterterm

Virtual contribution

divide singularities according to their nature

↓

Factorisation

↓

completeness

The procedure is implemented at NLO and NNLO

Torino Subtraction scheme at NLO

Subtraction pattern

Given a generic amplitude with n **massless** particles in the final state *[partons in the final state only]*

$$\mathcal{A}_n(p_i) = \mathcal{A}_n^{(0)}(p_i) + \mathcal{A}_n^{(1)}(p_i) + \mathcal{A}_n^{(2)}(p_i) + \dots$$

An **IR-safe** *observable X receives contribution at NLO according to*

$$\frac{d\sigma^{\rm NLO}}{dX} = \lim_{d\to 4}\left\{\int d\Phi_n\, V_n\,\delta_n + \int d\Phi_{n+1}\, R_{n+1}\,\delta_{n+1}\right\}$$

where $\delta_i = \delta(X - X_i)$, X_i the i-particle configuration, and

$$V_n = 2\mathbf{Re}\left[\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)}\right] \qquad R_{n+1} = \left|\mathcal{A}_{n+1}^{(0)}\right|^2.$$

Problem

Numerical implementation *requires to handle finite quantities $\to$ radiation IR poles have to be subtracted before performing the phase space integration.*

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- has the same singular limits as R, locally in phase space
- is analytically integrable in d dim

$$\frac{d\sigma_{ct}^{\mathrm{NLO}}}{dX} = \int \Phi_{n+1}\, K_{n+1}\,, \quad I_n = \int d\Phi_{\mathrm{rad}}\, K_{n+1}$$

$$\frac{d\sigma^{\mathrm{NLO}}}{dX} = \underbrace{\int d\Phi_n \left(V_n + I_n\right)\delta_n}_{\substack{\text{finite in } d=4 \\ \text{integrable in } \Phi_n}} + \underbrace{\int d\Phi_{n+1}\left(R_{n+1}\,\delta_{n+1} - K_{n+1}\,\delta_n\right)}_{\substack{\text{finite in } d=4 \\ \text{integrable in } \Phi_{n+1}}}$$

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$$\frac{d\sigma^{\mathrm{NLO}}}{dX} = \int d\Phi_n \left(V - I\right)^{(4)} \delta_n + \int d\Phi^{(4)}_{n+1} \left(R^{(4)} \delta_{n+1} - K^{(4)}\, \delta_n\right)$$

Implementation of the Subtraction method: the main ingredients

Ingredients of our method:

- **Fundamental limits $\mathbf{S}_i$, $\mathbf{C}_{ij}$** selecting the leading behaviour in terms of invariants $s_{ab} = 2k_a \cdot k_b$

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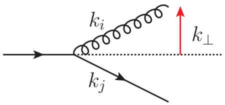

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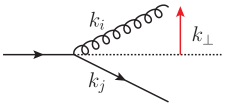

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- Born matrix element

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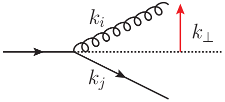

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$$\mathbf{S}_i R(\{k\}) = -\mathcal{N} \sum_{c,d} \delta_{f_i g} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}(\{k\}_{\not i})$$

$$\mathbf{C}_{ij} R(\{k\}) = \mathcal{N} \frac{1}{s_{ij}} P_{ij}^{\mu\nu}(s_{ir}, s_{jr})\, B_{\mu\nu}(\{k\}_{\not i \not j}, k)$$

$$\mathbf{S}_i \mathbf{C}_{ij} R(\{k\}) = 2\mathcal{N}\, C_{f_j}\, \delta_{f_i g} \frac{s_{jr}}{s_{ij}\, s_{ir}} B(\{k\}_{\not i})$$

B_{cd}=color-correlated Born, $B_{\mu\nu}$=spin-correlated Born.

Born kinem.: mass-shell condition and momenta conservation just in the limits.

Implementation of the Subtraction method: the main ingredients

- **partition of the phase space** Φ_{n+1} with sector functions $\mathcal{W}_{ij}$, that satisfy two requirements *[Frixione, Kunszt, Signer 9512328]*:

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$$\mathbf{S}_i \, \mathcal{W}_{ab} = 0 \,, \quad \forall i \neq a \qquad\qquad \mathbf{C}_{ij} \, \mathcal{W}_{ab} = 0 \,, \quad \forall a, b \notin \pi(i,j)$$

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  - explicit form

$$\text{CM } q^\mu = (\sqrt{s}, \vec{0}), \qquad e_i = \frac{s_{qi}}{s} \qquad \omega_{ij} = \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}$$

$$\boxed{\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{k,l\neq k} \sigma_{kl}}\,, \qquad \sigma_{ij} = \frac{1}{e_i\, \omega_{ij}}}$$

$$\mathbf{S}_i \mathcal{W}_{ab} = \delta_{ia} \frac{1/\omega_{ab}}{\sum_{c\neq a} 1/\omega_{ac}} \qquad \mathbf{C}_{ij}\mathcal{W}_{ab} = (\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}) \frac{e_b}{e_a + e_b}$$

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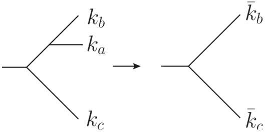
- **momentum mapping**: $\{k_1, \ldots, k_{n+1}\} \rightarrow \{\bar{k}_1, \ldots, \bar{k}_n\}$ *[Catani, Seymour 9605323]*:

  - phase space factorisation $d\Phi_{n+1} = d\bar{\Phi}_n \, d\bar{\Phi}_{\text{rad}}$
  - n on-shell particles conserving momentum.

$$\{\bar{k}\}^{(abc)} = \left\{ \{k\}_{\not a \not b \not c}, \bar{k}_b^{(abc)}, \bar{k}_c^{(abc)} \right\}$$

$$\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$$



Implementation of the Subtraction method: counterterm construction

Definition of the counterterm

Sector: $\mathcal{W}_{ij} \to$ minimal singularity structure $\mathbf{S}_i, \mathbf{C}_{ij}$

Candidate counterterm: $K_{ij} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1 - \mathbf{S}_i)\Big]\, R\mathcal{W}_{ij}$

$\to$ $\mathbf{S}_i, \mathbf{C}_{ij}$ **commute** both on R and on sector function

$\to$ **overlap** between $\mathbf{S}_i, \mathbf{C}_{ij}$ taken into account

Mapping $\{k_{n+1}\} \to \{k_n\}^{(abc)}$: local counterterm in the remapped kinematic

$$\overline{K}_{ij} \equiv (\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij})\, R\, \mathcal{W}_{ij}$$

Barred limits have to fulfil the **consistency relations**

$$\mathbf{S}_i\, \overline{\mathbf{S}}_i\, R = \mathbf{S}_i\, RR$$
$$\mathbf{C}_{ij}\, \overline{\mathbf{C}}_{ij}\, R = \mathbf{C}_{ij}\, RR$$
$$\mathbf{C}_{ij}\, \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R = \mathbf{C}_{ij}\, \overline{\mathbf{S}}_i\, RR \qquad \mathbf{S}_i\, \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R = \mathbf{S}_i\, \overline{\mathbf{C}}_{ij}\, RR$$

Such that

$$\boxed{R\mathcal{W}_{ij} - \overline{K}_{ij} = \text{finite}}$$

Mapping $\{k_{n+1}\} \to \{k_n\}^{(abc)}$ *(abc)* chosen according to the **invariants in the kernels**

$$\overline{\mathbf{S}}_i R(\{k\}) = -\mathcal{N} \sum_{c,d\neq i} \delta_{f_i g} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}(\{\bar{k}\}^{(icd)})$$

$$\overline{\mathbf{C}}_{ij} R(\{k\}) = \mathcal{N} \frac{1}{s_{ij}} P_{ij}^{\mu\nu}(s_{ir}, s_{jr}) B_{\mu\nu}(\{\bar{k}\}^{(ijr)})$$

$$\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} R(\{k\}) = 2\mathcal{N} C_{f_j} \delta_{f_i g} \frac{s_{jr}}{s_{ij}\, s_{ir}} B(\{\bar{k}\}^{(ijr)})$$

$$\begin{aligned} P_{ij}^{\mu\nu}(s_{ir}, s_{jr}) B_{\mu\nu} &= P_{ij}(s_{ir}, s_{jr}) B + Q_{ij}^{\mu\nu}(s_{ir}, s_{jr}) B_{\mu\nu} \\ &\equiv P_{ij}(x_i, x_j) B + Q_{ij}^{\mu\nu}(x_i, x_j) B_{\mu\nu} \end{aligned}$$

$$x_i = \frac{s_{ir}}{s_{ir} + s_{jr}} \qquad x_j = \frac{s_{jr}}{s_{ir} + s_{jr}}$$

- Collinear limit: single mapping $\to$ *dipole=(ijr)*
- Soft limit: different mapping for each contribution to $\mathbf{S}_i R(\{k\}) \to$ *dipole=(icd)*

Implementation of the Subtraction method: counterterm construction

Sector function sum rules $\rightarrow$ summing over sectors $\overline{K}$ becomes independent of $\mathcal{W}_{ij}$

$$\overline{K} = \sum_{i,j\neq i} \overline{K}_{ij} = \sum_i \left(\overline{\mathbf{S}}_i R\right) \Big[\overbrace{\overline{\mathbf{S}}_i \sum_{j\neq i} \mathcal{W}_{ij}}^{=1} \Big] + \sum_{i,j>i} \left(\mathbf{C}_{ij} R\right) \Big[\overbrace{\overline{\mathbf{C}}_{ij} \left(\mathcal{W}_{ij} + \mathcal{W}_{ji}\right)}^{=1} \Big]$$

$$- \sum_{i,j\neq i} \left(\overline{\mathbf{S}}_i \mathbf{C}_{ij} R\right) \Big[\underbrace{\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}}_{=1} \Big]$$

$$= \sum_i \overline{\mathbf{S}}_i R + \sum_{i,j>i} \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j\right) R$$

Remarks

- the integrated counterterm has to **match the poles of** V, which is **not** split into sectors.
- the sector functions would have made the integration much more involved.

$\rightarrow$ this way **analytic integration** is feasible with **standard techniques**.

Implementation of the Subtraction method: counterterm integration

- **Parametrisation of the phase space** *[Catani, Seymour 9605323]*

$$d\Phi_{n+1} = d\Phi_n^{(abc)}\, d\Phi_{\rm rad}^{(abc)} \equiv d\Phi_n^{(abc)} \times d\Phi_{\rm rad}\left(s_{bc}^{(abc)}; y, z, \phi\right)$$

$$d\Phi_n^{(abc)} \propto \left(s_{bc}^{(abc)}\right)^{1-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\,(1-y)\left[(1-y)^2 y(1-z)z\right]^{-\epsilon}$$

$$s_{bc}^{(abc)} = s_{abc}\,, \quad s_{ab} = y s_{bc}^{(abc)}\,, \quad s_{ac} = z(1-y)s_{bc}^{(abc)}\,, \quad s_{bc} = (1-z)(1-y)s_{bc}^{(abc)}$$

- **Integration**
  1. we choose different parametrisation for the soft and the hard-collinear contr.
  2. soft kernel is parametrised differently for each term of the sum.

$$I^s = -\mathcal{N}\frac{\varsigma_{n+1}}{\varsigma_n}\sum_i \delta_{f_i g}\sum_{c,d\neq i}\int d\Phi_{\rm rad}\left(s_{cd}^{(icd)}; y, z, \phi\right)\frac{s_{cd}}{s_{ic}\, s_{id}}\, B_{cd}\left(\{\bar{k}\}^{(icd)}\right)$$

$$= -\mathcal{N}\frac{\varsigma_{n+1}}{\varsigma_n}\sum_i \delta_{f_i g}\sum_{c,d\neq i} B_{cd}\left(\{\bar{k}\}^{(icd)}\right)\left(s_{cd}^{(icd)}\right)^{-\epsilon}\frac{(4\pi)^{\epsilon-2}\,\Gamma(1-\epsilon)\,\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)}$$

Remark:

- freedom to **adapt the parametrisation** to the invariants appearing in the kernels.
- integrated counterterm exact in ϵ.

Subtraction pattern at NNLO

NNLO Subtraction pattern

- more configurations contribute

$$\frac{d\sigma^{\mathrm{NNLO}}}{dX} = \int d\Phi_n \, VV_n \, \delta_n(X) + \int d\Phi_{n+1} \, RV_{n+1} \, \delta_{n+1}(X) + \int d\Phi_{n+2} \, RR_{n+2} \, \delta_{n+2}(X)$$

$$RR_{n+2} = \left| \mathcal{A}^{(0)}_{n+2} \right|^2 \quad VV_n = \left| \mathcal{A}^{(1)}_{n} \right|^2 + 2\mathbf{Re}\left[\mathcal{A}^{(0)\dagger}_{n} \mathcal{A}^{(2)}_{n} \right] \quad RV_{n+1} = 2\mathbf{Re}\left[\mathcal{A}^{(0)\dagger}_{n+1} \mathcal{A}^{(1)}_{n+1} \right]$$

- more counterterms to add and subtract

$$\int d\Phi_{n+2} \, K^{(1)} \, \delta_{n+1} \; : \qquad K^{(1)} \quad \rightarrow \quad \text{same 1-unr. singularities as RR}$$

$$\int d\Phi_{n+2} \left(K^{(2)} - K^{(12)} \right) \delta_n \; : \qquad K^{(2)} - K^{(12)} \quad \rightarrow \quad \text{same 2-unr. singularities as RR.}$$

[1-unr.(2-unr.), pure 2-unr.]

$$\int d\Phi_{n+1} \, K^{(\mathrm{RV})} \, \delta_n \; : \qquad K^{(\mathrm{RV})} \quad \rightarrow \quad \text{same 1-unr. singularities as RV}$$

and integrate in the radiative phase space

$$I^{(i)} = \int d\Phi_{\mathrm{rad},i} \, K^{(i)} \,, \quad I^{(12)} = \int d\Phi_{\mathrm{rad},1} \, K^{(12)} \,, \quad I^{(\mathrm{RV})} = \int d\Phi_{\mathrm{rad}} \, K^{(\mathrm{RV})} \,,$$

Subtraction pattern at NNLO

$$\frac{d\sigma^{\rm NNLO}}{dX} = \int d\Phi_n \Big[\underbrace{VV_n}_{\text{singular in d=4 , finite in } \Phi_n} \Big] \delta_n$$
$$+ \int d\Phi_{n+1} \Big[\underbrace{\big(RV_{n+1}\big)}_{\text{singular in d=4, singular in } \Phi_{n+1}} \delta_{n+1} \Big]$$
$$+ \int d\Phi_{n+2} \Big[\underbrace{RR_{n+2}}_{\text{finite in d=4, singular in } \Phi_{n+2}} \delta_{n+2} \Big]$$

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$$+ \int d\Phi_{n+2} \Big[\underbrace{RR_{n+2}\, \delta_{n+2} - K^{(1)} \delta_{n+1} - \big(K^{(2)} - K^{(12)}\big) \delta_n}_{\text{finite in d=4 and in } \Phi_{n+2}} \Big]$$

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Subtraction algorithm at NNLO: ingredients

Ingredients of our method:

- **new singular configurations of RR**:

$$\mathbf{S}_{ij} \to ij \text{ soft} \qquad \mathbf{C}_{ijk} \to ijk \text{ collinear}$$
$$\mathbf{C}_{ijkl} \to (ij),(kl) \text{ indep. collinear} \qquad \mathbf{SC}_{ijk} \to i \text{ soft, } jk \text{ collinear}$$

- **partition of Φ_{n+2}**:

$$\mathcal{W}_{ijkl} \qquad \begin{cases} i,k & \to \textit{soft} \\ ij,kl & \to \textit{collinear} \end{cases} \qquad \begin{cases} \text{sum rules} \\ \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \mathcal{W}_{ijkl} = 1 \end{cases}$$

different topologies to select the minimum number of singularities:

$$\mathcal{W}_{ijjk} \;:\; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}$$
$$\mathcal{W}_{ijkj} \;:\; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ik} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk} \quad \mathbf{SC}_{kij}$$
$$\mathcal{W}_{ijkl} \;:\; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ik} \quad \mathbf{C}_{ijkl} \quad \mathbf{SC}_{ikl} \quad \mathbf{SC}_{kij}$$

■ single unresolved limits ■ double unresolved limits

factorisation into NLO sector function under single-unresolved limits

$$\mathbf{S}_i \mathcal{W}_{ijkl} = \mathcal{W}_{kl}\, \mathbf{S}_i \tilde{\mathcal{W}}_{ij} \quad \mathbf{C}_{ij} \mathcal{W}_{ijkl} = \mathcal{W}_{kl}\, \mathbf{C}_{ij} \tilde{\mathcal{W}}_{ij} \quad \mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ijkl} = \mathcal{W}_{kl}\, \mathbf{S}_i \mathbf{C}_{ij} \tilde{\mathcal{W}}_{ij}$$

Subtraction algorithm at NNLO: ingredients

- **counterterm identification** [sector $\mathcal{W}_{ijjk}$]

$$\underbrace{(1-\mathbf{S}_i)\,(1-\mathbf{C}_{ij})}_{1-\mathbf{L}^{(1)}_{ij}}\;\underbrace{(1-\mathbf{S}_{ij})\,\left(1-\mathbf{C}_{ijk}\right)\,\left(1-\mathbf{SC}_{ijk}\right)}_{1-\mathbf{L}^{(2)}_{ijjk}}\,RR\,\mathcal{W}_{ijjk} = \text{finite}$$

$$\left(1-\mathbf{L}^{(1)}_{ij}-\mathbf{L}^{(2)}_{ijjk}+\mathbf{L}^{(1)}_{ij}\mathbf{L}^{(2)}_{ijjk}\right)RR\,\mathcal{W}_{ijjk} = \text{finite}$$

according to the number of unresolved partons we define

$$RR\,\mathcal{W}_{ijjk} - K^{(\mathbf{1})}_{ijjk} - K^{(\mathbf{2})}_{ijjk} + K^{(\mathbf{12})}_{ijjk} = \text{finite}$$

$^{(1)}$ = one unres. , $^{(2)}$ = two unres. democratic , $^{(12)}$ = two unres. hierarchical

$$K^{(\mathbf{1})}_{ijjk} = \left[\mathbf{S}_i + \mathbf{C}_{ij}\,(1-\mathbf{S}_i)\right]RR\,\mathcal{W}_{ijjk}$$

$$K^{(\mathbf{2})}_{ijjk} = \left[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ij})\left(1-\mathbf{C}_{ijk}\right)\right]RR\,\mathcal{W}_{ijjk}$$

$$K^{(\mathbf{12})}_{ijjk} = \left\{\left[\mathbf{S}_i + \mathbf{C}_{ij}\,(1-\mathbf{S}_i)\right]\left[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ij})\left(1-\mathbf{C}_{ijk}\right)\right]\right\}RR\,\mathcal{W}_{ijjk}$$

Remarks:

- $\mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{S}_{ij}, \mathbf{C}_{ijk}, \mathbf{SC}_{ijk}$ commute

Subtraction algorithm at NNLO: ingredients

- **Singular structure of RR** under the fundamental limits
  - universal kernel *[Catani, Grazzini 9903516, 9810389]* *[Campbell, Glover 9710255]*
  - Born matrix element

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \Big[\sum_{e,f\neq i,j} \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{ef}^{(j)}\, B_{cdef}\big(\{k\}_{\not{i}\not{j}}\big) + \mathcal{I}_{cd}^{(ij)}\, B_{cd}\big(\{k\}_{\not{i}\not{j}}\big) \Big]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}(s_{ir}, s_{jr}, s_{kr})\, B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{ijk}\big)$$

$$\mathbf{C}_{ijkl}\, RR(\{k\}) \propto \frac{1}{s_{ij}\, s_{kl}}\, P_{ij}^{\mu\nu}(s_{ir}, s_{jr})\, P_{kl}^{\rho\sigma}(s_{kr'}, s_{lr'})\, B_{\mu\nu\rho\sigma}\big(\{k\}_{\not{i}\not{j}\not{k}\not{l}}, k_{ij}, k_{kl}\big)$$

$$\mathbf{SC}_{ijk}\, RR(\{k\}) = \mathbf{CS}_{jki} RR(\{k\}) \propto \frac{1}{s_{jk}} \sum_{c,d\neq i} P_{jk}^{\mu\nu}\, \mathcal{I}_{cd}^{(i)}\, B_{\mu\nu}^{cd}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{jk}\big)$$

$\mathcal{I}_{cd}^{(i)}$ = single eikonal current, $\mathcal{I}_{cd}^{(ij)}$ = double eikonal current.
$P_{ijk}^{\mu\nu}(s_{ir}, s_{jr}, s_{kr})$ = triple splitting function.

Born kinem.:
$K_{ijk}^{(1)}, K_{ijk}^{(12)}, K_{ijk}^{(2)}$ do not satisfy **mass-shell condition** and **momenta conservation**

$\Longrightarrow$ momentum mapping needed!

Subtraction algorithm at NNLO: ingredients

- **double momentum mapping**: $\{k_1, \dots, k_{n+2}\} \to \{\bar{k}_1, \dots, \bar{k}_n\}$.

two kind of mapping to treat different kernels and simplify the integration.

1) two-steps mapping

$$\bar{k}_n^{(acd,bef)} = \bar{k}_n^{(acd)}, \quad n \neq a, b, e, f$$

$$\bar{k}_e^{(acd,bef)} = \bar{k}_b^{(acd)} + \bar{k}_e^{(acd)} - \frac{\bar{s}_{be}^{(acd)}}{\bar{s}_{bf}^{(acd)} + \bar{s}_{ef}^{(acd)}} \bar{k}_f^{(acd)} \qquad \bar{k}_f^{(acd,bef)} = \frac{\bar{s}_{bef}^{(acd)}}{\bar{s}_{bf}^{(acd)} + \bar{s}_{ef}^{(acd)}} \bar{k}_f^{(acd)}$$

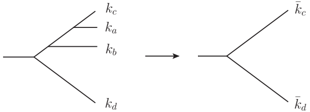
PS fact.: $d\Phi_{n+2} = d\Phi_n^{(acd,bef)} \cdot d\Phi_{\mathrm{rad},1}(\bar{s}_{bef}^{(acd)}; y', z', \phi') \cdot d\Phi_{\mathrm{rad},1}(s_{acd}; y, z, \phi)$

2) one-step mapping

$$\bar{k}_n^{(abcd)} = k_n, \quad n \neq a, b, c, d$$

$$\bar{k}_c^{(abcd)} = k_a + k_b + k_c - \frac{s_{abc}}{s_{ad} + s_{bd} + s_{cd}} k_d$$

$$\bar{k}_d^{(abcd)} = \frac{s_{abcd}}{s_{ad} + s_{bd} + s_{cd}} k_d$$



PS fact.: $d\Phi_{n+2} = d\Phi_n^{(abcd)} \cdot d\Phi_{\mathrm{rad},2}(\bar{s}_{cd}^{(abcd)}; y, z, \phi, y', z', x')$.

From the ingredients to the recipe

Example: **double unresolved counterterm** and its integral

Applying the sum rules to the sector functions we end up with

$$\overline{K}^{(2)} = \sum_i \Big\{ \sum_{j>i} \overline{\mathbf{S}}_{ij} + \sum_{j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) + \sum_{j>i} \sum_{\substack{k>i \\ k\neq j}} \sum_{\substack{l>k \\ l\neq j}} \overline{\mathbf{C}}_{ijkl} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl}\right) + \dots \Big\} RR \,,$$

- **No sector functions left** as needed for matching the VV poles.
- Full freedom in defining the mapped terms.

From the ingredients to the recipe

Example: **double unresolved counterterm** and its integral

Applying the sum rules to the sector functions we end up with

$$\overline{K}^{(2)} = \sum_i \Big\{ \sum_{j>i} \overline{\mathbf{S}}_{ij} + \sum_{j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) + \sum_{j>i} \sum_{\substack{k>i \\ k\neq j}} \sum_{\substack{l>k \\ l\neq j}} \overline{\mathbf{C}}_{ijkl} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl}\right) + \dots \Big\} RR \,,$$

- **No sector functions left** as needed for matching the VV poles.
- Full freedom in defining the mapped terms.

Starting from the limit

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \Big[\sum_{e,f\neq i,j} \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{ef}^{(j)}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + \mathcal{I}_{cd}^{(ij)}\, B_{cd}\big(\{k\}_{\not i \not j}\big) \Big]$$

we are free to map each term separately, adapting the choice to the invariants appearing in the kernel

$$\begin{aligned} \bar{\mathbf{S}}_{ij}\, RR \;\propto\; \sum_{\substack{c\neq i,j\\ d\neq i,j,c}} \Bigg[& \sum_{\substack{e\neq i,j,c,d\\ f\neq i,j,c,d}} \mathcal{I}_{cd}^{(i)}\, \overline{\mathcal{I}}_{ef}^{(j)(icd)} B_{cdef}\Big(\{\bar{k}\}^{(icd,jef)}\Big) \\ & + 4 \sum_{e\neq i,j,c,d} \mathcal{I}_{cd}^{(i)}\, \overline{\mathcal{I}}_{ed}^{(j)(icd)} B_{cded}\Big(\{\bar{k}\}^{(icd,jed)}\Big) \\ & + 2\, \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{cd}^{(j)} B_{cdcd}\Big(\{\bar{k}\}^{(ijcd)}\Big) + \Big(\mathcal{I}_{cd}^{(ij)} - \frac{1}{2}\mathcal{I}_{cc}^{(ij)} - \frac{1}{2}\mathcal{I}_{dd}^{(ij)}\Big) B_{cd}\Big(\{\bar{k}\}^{(ijcd)}\Big) \Bigg] \end{aligned}$$

The PS parametrisation follows the mapping structure to simplify the integral

Starting from the limit

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{ef}^{(j)}\, B_{cdef}\left(\{k\}_{\not i \not j}\right) + \mathcal{I}_{cd}^{(ij)}\, B_{cd}\left(\{k\}_{\not i \not j}\right)\right]$$

we are free to map each term separately, adapting the choice to the invariants appearing in the kernel

$$\begin{aligned}
\bar{\mathbf{S}}_{ij}\, RR \;\propto\; & \sum_{\substack{c\neq i,j\\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d\\ f\neq i,j,c,d}} \mathcal{I}_{cd}^{(i)}\, \overline{\mathcal{I}}_{ef}^{(j)(icd)} B_{cdef}\left(\{\bar{k}\}^{(icd,jef)}\right) \\
& + 4 \sum_{e\neq i,j,c,d} \mathcal{I}_{cd}^{(i)}\, \overline{\mathcal{I}}_{ed}^{(j)(icd)} B_{cded}\left(\{\bar{k}\}^{(icd,jed)}\right) \\
& + 2\, \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{cd}^{(j)} B_{cdcd}\left(\{\bar{k}\}^{(ijcd)}\right) + \left(\mathcal{I}_{cd}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{cc}^{(ij)} - \frac{1}{2}\, \mathcal{I}_{dd}^{(ij)}\right) B_{cd}\left(\{\bar{k}\}^{(ijcd)}\right) \Bigg]
\end{aligned}$$

The PS parametrisation follows the mapping structure to simplify the integral

$$\begin{aligned}
I_{\mathrm{SS},cdef}^{(2)} &= \int d\Phi_{\mathrm{rad},2}\, \mathcal{I}_{cd}^{(i)}\, \overline{\mathcal{I}}_{ef}^{(j),(icd)} = \int d\overline{\Phi}_{\mathrm{rad}}^{(icd,jef)}\, \overline{\mathcal{I}}_{ef}^{(j),(icd)} \int d\Phi_{\mathrm{rad}}^{(icd)}\, \mathcal{I}_{cd}^{(i)} \\
&= \delta_{f_i g}\, \delta_{f_j g} \left[\frac{(4\pi)^{\epsilon-2}}{(\bar{s}_{cd}^{(icd,jef)})^{\epsilon}}\, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\, \Gamma(2-3\epsilon)} \right] \left[\frac{(4\pi)^{\epsilon-2}}{(\bar{s}_{ef}^{(icd,jef)})^{\epsilon}}\, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\, \Gamma(2-3\epsilon)} \right]
\end{aligned}$$

All the contributions to $\overline{K}^{(2)}$ have been integrated

$$I^{(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left[I^{(2)}_{\rm ss} + I^{(2)}_{\rm hcc} + I^{(2)}_{\rm cc4} + I^{(2)}_{\rm sc3} \right]$$

and organised according to the different colour structures

$$\begin{aligned} I^{(2)}_{\rm ss} = & \Bigg[2\Big(\sum_{a,b} C_{f_a} C_{f_b}\Big) I^{\rm ss}_{C_f C_f} + 8\Big(\sum_{a} C^2_{f_a}\Big) I^{\rm ss}_{C_f^2} \\ & -\Big(\sum_a C_{f_a}\Big)\Big(N_f\, T_R\, I^{\rm ss}_{C_f T_R} - \frac{C_A}{2}\, I^{\rm ss}_{C_f C_A}\Big)\Bigg] B(\{\bar{k}\}) \\ & +2\sum_{c,d\neq c}\Bigg[-2\Big(\sum_a C_{f_a}\Big) I^{\rm ss}_{C_f B_{cd}} - 2C_{f_d} I^{\rm ss}_{C_d B_{cd}} + N_f\, T_R\, I^{\rm ss}_{T_R B_{cd}} - \frac{C_A}{2}\, I^{\rm ss}_{C_A B_{cd}}\Bigg] B_{cd}(\{\bar{k}\}) \\ & +2\sum_{c,d\neq c} I^{\rm ss}_{B_{cdcd}}\, B_{cdcd}(\{\bar{k}\}) + 4\sum_{\substack{c,d\neq c\\ e\neq d}} I^{\rm ss}_{B_{cded}}\, B_{cded}(\{\bar{k}\}) \\ & +\sum_{\substack{c,d\neq c\\ e,f\neq e}} I^{\rm ss}_{B_{cdef}}\, B_{cdef}(\{\bar{k}\}) + \mathcal{O}(\epsilon)\,. \end{aligned}$$

Remark: $I^{(2)}_{\rm cc4}, I^{(2)}_{\rm sc3}$ feature a NLO×NLO complexity.

$$I^{ss}_{C_f C_f} = \frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\left(16-\frac{7}{6}\pi^2\right)\frac{1}{\epsilon^2}+\left(60-\frac{14}{3}\pi^2-\frac{50}{3}\zeta(3)\right)\frac{1}{\epsilon}+216-\frac{56}{3}\pi^2-\frac{200}{3}\zeta(3)+\frac{29}{120}\pi^4$$

$$I^{ss}_{C_f^2} = \left(1-\frac{\pi^2}{6}\right)\frac{1}{\epsilon^2}+\left(10-\frac{2}{3}\pi^2-6\zeta(3)\right)\frac{1}{\epsilon}+68-4\pi^2-24\zeta(3)-\frac{7}{72}\pi^4$$

$$I^{ss}_{C_f T_R} = \frac{2}{3}\frac{1}{\epsilon^3}+\frac{34}{9}\frac{1}{\epsilon^2}+\left(\frac{464}{27}-\frac{7}{9}\pi^2\right)\frac{1}{\epsilon}+\frac{5896}{81}-\frac{131}{27}\pi^2-\frac{76}{9}\zeta(3)$$

$$I^{ss}_{C_f C_A} = \frac{2}{\epsilon^4}+\frac{35}{3}\frac{1}{\epsilon^3}+\left(\frac{487}{9}-\frac{8}{3}\pi^2\right)\frac{1}{\epsilon^2}+\left(\frac{6248}{27}-\frac{269}{18}\pi^2-\frac{154}{3}\zeta(3)\right)\frac{1}{\epsilon}+\frac{77404}{81}-\frac{3829}{54}\pi^2-\frac{2050}{9}\zeta(3)-\frac{23}{60}\pi^4$$

$$I^{ss}_{C_f B_{cd}} = \ln\frac{\bar{s}_{cd}}{\mu^2}\Bigg[-\frac{1}{\epsilon^3}-\frac{4}{\epsilon^2}-\left(16-\frac{7}{6}\pi^2\right)\frac{1}{\epsilon}-60+\frac{14}{3}\pi^2+\frac{50}{3}\zeta(3) + \frac{1}{2}\ln\frac{\bar{s}_{cd}}{\mu^2}\left(\frac{1}{\epsilon^2}+\frac{4}{\epsilon}+16-\frac{7}{6}\pi^2\right)-\frac{1}{6}\ln^2\frac{\bar{s}_{cd}}{\mu^2}\left(\frac{1}{\epsilon}+4\right)+\frac{1}{24}\ln^3\frac{\bar{s}_{cd}}{\mu^2}\Bigg]$$

$$I^{ss}_{C_d B_{cd}} = 4\ln\frac{\bar{s}_{cd}}{\mu^2}\left[-\left(1-\frac{\pi^2}{6}\right)\frac{1}{\epsilon}-10+\frac{2}{3}\pi^2+6\zeta(3)+\frac{1}{2}\ln\frac{\bar{s}_{cd}}{\mu^2}\left(1-\frac{\pi^2}{6}\right)\right]$$

$$I^{ss}_{T_R B_{cd}} = \ln\frac{\bar{s}_{cd}}{\mu^2}\left[-\frac{2}{3}\frac{1}{\epsilon^2}-\frac{34}{9}\frac{1}{\epsilon}-\frac{464}{27}+\frac{7}{9}\pi^2+\ln\frac{\bar{s}_{cd}}{\mu^2}\left(\frac{2}{3}\frac{1}{\epsilon}+\frac{34}{9}\right)-\frac{4}{9}\ln^2\frac{\bar{s}_{cd}}{\mu^2}\right]$$

$$I^{ss}_{C_A B_{cd}} = \ln\frac{\bar{s}_{cd}}{\mu^2}\Bigg[-\frac{2}{\epsilon^3}-\frac{35}{3}\frac{1}{\epsilon^2}-\left(\frac{487}{9}-\frac{8}{3}\pi^2\right)\frac{1}{\epsilon}-\frac{6248}{27}+\frac{269}{18}\pi^2+\frac{154}{3}\zeta(3) + \ln\frac{\bar{s}_{cd}}{\mu^2}\left(\frac{2}{\epsilon^2}+\frac{35}{3}\frac{1}{\epsilon}+\frac{487}{9}-\frac{8}{3}\pi^2\right)-\frac{2}{3}\ln^2\frac{\bar{s}_{cd}}{\mu^2}\left(\frac{2}{\epsilon}+\frac{35}{3}\right)+\frac{2}{3}\ln^3\frac{\bar{s}_{cd}}{\mu^2}\Bigg]$$

$$I^{ss}_{B_{cdcd}} = -4(1-\zeta(3))\left(\frac{1}{\epsilon}-2\ln\frac{\bar{s}_{cd}}{\mu^2}\right)-40-\frac{\pi^2}{3}+12\zeta(3)+\frac{13}{36}\pi^4$$

$$I^{ss}_{B_{cded}} = \ln\frac{\bar{s}_{cd}}{\mu^2}\ln\frac{\bar{s}_{ed}}{\mu^2}\left(1-\frac{\pi^2}{6}\right)$$

$$I^{ss}_{B_{cdef}} = \ln\frac{\bar{s}_{cd}}{\mu^2}\ln\frac{\bar{s}_{ef}}{\mu^2}\left[\frac{1}{\epsilon^2}+\frac{4}{\epsilon}+16-\frac{7}{6}\pi^2-\frac{1}{2}\left(\ln\frac{\bar{s}_{cd}}{\mu^2}+\ln\frac{\bar{s}_{ef}}{\mu^2}\right)\left(\frac{1}{\epsilon}+4\right)+\frac{1}{6}\left(\ln^2\frac{\bar{s}_{cd}}{\mu^2}+\ln^2\frac{\bar{s}_{ef}}{\mu^2}\right)+\frac{1}{4}\ln\frac{\bar{s}_{cd}}{\mu^2}\ln\frac{\bar{s}_{ef}}{\mu^2}\right]$$

Outlook

Outlook

Some work is done:

- General structure of a local, analytic sector subtraction has been proposed.
- All the integrals needed for $K^{(2)}$ and $K^{(RV)}$ are done.

Some work is in progress:

- Combining the results to check the cancellation of the IR poles for a generic process.

A lot of work remains to be done:

- Implementation in a differential code.
- Generalisation to initial state radiation.
- Extension to massive particles.

Backup

Example: **one unresolved counterterm** and its integral

$$K^{(1)} = \sum_{i,j\neq i} \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i) \Big] \, RR \sum_{k\neq i,j} \Big(\mathcal{W}_{ijjk} + \mathcal{W}_{ijkj} + \sum_{l\neq i,j,k} \mathcal{W}_{ijkl} \Big)$$

NNLO sectors factorise into NLO sectors and mapping is applied

$$\begin{aligned} \overline{K} \;&=\; \sum_{i,j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \Big[\Big(\mathbf{S}_i \, \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{S}}_i \, RR \big) \overline{\mathcal{W}}_{kl} + \Big(\mathbf{C}_{ij} \, \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{C}}_{ij} \, RR \big) \overline{\mathcal{W}}_{kl} \\ &\qquad - \Big(\mathbf{S}_i \mathbf{C}_{ij} \, \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} \, RR \big) \overline{\mathcal{W}}_{kl} \Big] \\ &=\; \sum_{\substack{k\neq i \\ l\neq i,k}} \underbrace{\overline{\mathcal{W}}_{kl}}_{\text{NLO sector}} \; \underbrace{\Big[\sum_i \overline{\mathbf{S}}_i \, RR + \sum_{i,j>i} \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j) \, RR \Big]}_{\text{1-unresolved structure}} \end{aligned}$$

Kinematic mapping of sector functions allows to factorise the structure of **NLO sectors out of the radiation phase space**, and integrate only single-unresolved kernels.

$$I^{(1)} \propto \sum_{k,l} \overline{\mathcal{W}}_{kl} \Big[\sum_{i,j>i} \int d\Phi_{\mathrm{rad},1}^{(ijr)} \, \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j) \, RR(\{k\}) + \sum_i \int d\Phi_{\mathrm{rad},1} \, \overline{\mathbf{S}}_i \, RR(\{k\}) \Big]$$

The tripoles mistery

$$\int d\Phi_n \underbrace{\left[VV_n + I^{(2)} + I^{(\mathrm{RV})}\right]}_{\text{finite in d=4 and in } \Phi_n} \delta_n$$

VV: Infrared structure of gauge amplitudes

$$\mathcal{A}\left(\frac{p_i}{\mu}, \alpha_s, \epsilon\right) = \mathbf{Z}\left(\frac{p_i}{\mu}, \alpha_s, \epsilon\right) \mathcal{H}\left(\frac{p_i}{\mu}, \alpha_s, \epsilon\right)$$

$\mathcal{H}$ finite for $\epsilon \to 0$, $\mathbf{Z}$ color operator with universal form

$$\mathbf{Z}\left(\frac{p_i}{\mu}, \alpha_s, \epsilon\right) = \mathcal{P}\exp\left[\int_0^\mu \frac{d\lambda}{\lambda}\,\Gamma\left(\frac{p_i}{\lambda}, \alpha_s, \epsilon\right)\right]$$

$\Gamma =$ anomalous dimension matrix $\to$ Dipole formula

$$\Gamma\left(\frac{p_i}{\lambda}, \alpha_s, \epsilon\right) = \frac{1}{2}\hat{\gamma}_K\big(\alpha_s(\lambda, \epsilon)\big) \sum_{i, j>i} \ln\left(\frac{2 p_i \cdot p_j e^{i\pi\sigma_{ij}}}{\lambda^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j - \sum_i \gamma_i\big(\alpha_s(\lambda, \epsilon)\big)$$

RV: Collinear, soft and soft-collinear limits *[Bern et al. 9903516]* *[Catani, Grazzini 0007142]*

$$\mathbf{C}_{ij}\, RV = \frac{1}{s_{ij}}\left[a_c\, P_{ij}^{\mu\nu} V_{\mu\nu} + b_c P_{ij}^{(1)\mu\nu} B_{\mu\nu}\right]$$

$$\mathbf{S}_i\, RV = \sum_{k,\, l}\left[a_s\, \mathcal{I}_{kl}^{(i)} V_{kl} + \left(\frac{b_s}{\epsilon^2}\left(\mathcal{I}_{kl}^{(i)}\right)^{1+\epsilon} + \frac{c_s}{\epsilon}\mathcal{I}_{kl}^{(i)}\right) B_{kl} + \frac{d_s}{\epsilon}\sum_{p\neq k,l} \mathcal{I}_{kl}^{(i)} \left(\mathcal{I}_{lp}^{(i)}\right)^{\epsilon} B_{klp}\right]$$

$$\mathbf{S}_i\mathbf{C}_{ij}\, RV = a_{sc}\, \mathcal{I}_{jr}^{(i)} V - \left(\frac{b_{sc}}{\epsilon^2}\left(\mathcal{I}_{jr}^{(i)}\right)^{1+\epsilon} + \frac{c_{sc}}{\epsilon}\, \mathcal{I}_{jr}^{(i)}\right) B$$

$\{a_i\}, \{b_i\}, \{c_i\}, d_s$ coefficients

$B_{klp} = \sum_{a,\, b,\, c} f_{abc}\, \langle \mathcal{M}_B | T_k^a T_l^b T_p^c | \mathcal{M}_B \rangle \to$ tripole

$$V_{\mu\nu} = \frac{\alpha_s}{\pi}\left[-\frac{1}{2\epsilon^2}\left(\sum_i C_{f_i}\right) B_{\mu\nu} + \frac{1}{\epsilon}\left(\sum_i \gamma_i^{(1)}\right) B_{\mu\nu} - \frac{1}{2\epsilon}\sum_{i,j\neq i} \ln\frac{s_{ij}}{\mu^2}\, B_{\mu\nu,ij} + H_{\mu\nu}\right]$$

Remark: $\mathbf{S}_i\mathbf{C}_{ij} RV$ is independent of tripoles thank to the symmetry properties of B_{klp}.

$$K^{(\mathbf{RV})} = \underbrace{\sum_i \mathbf{S}_i RV}_{\substack{\text{tripoles} \\ \downarrow \\ I_s^{(\mathbf{RV})}:\ \text{tripoles}}} - \underbrace{\sum_{i,j>i} \big(\mathbf{S}_i\, \mathbf{C}_{ij} + \mathbf{S}_j\, \mathbf{C}_{ij}\big) RV}_{\substack{\text{no tripoles} \\ \downarrow \\ I_{sc}^{(\mathbf{RV})}:\ \text{no tripoles}}} + \underbrace{\sum_{i,j>i} \mathbf{C}_{ij} RV}_{\substack{\text{no tripoles} \\ \downarrow \\ I_c^{(\mathbf{RV})}:\ \text{no tripoles}}}$$

Question: Does the mapping procedure modify this structure?

YES!

consistency relations:

$$\mathbf{S}_i\, RV = \mathbf{S}_i\, \overline{\mathbf{S}}_i\, RV\,, \qquad \mathbf{C}_{ij}\, RV = \mathbf{C}_{ij}\, \overline{\mathbf{C}}_{ij}\, RV\,,$$

$$\mathbf{S}_i\, \overline{\mathbf{C}}_{ij}\, RV = \mathbf{S}_i\, \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, RV\,, \qquad \mathbf{C}_{ij}\, \overline{\mathbf{S}}_i\, RV = \mathbf{C}_{ij}\, \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, RV\,.$$

$$\overline{K}^{(\mathbf{RV})} = \underbrace{\underbrace{\sum_i \overline{\mathbf{S}}_i RV}_{\substack{\text{tripoles} \\ \downarrow \\ I_s^{(\mathbf{RV})}:\ \text{tripoles}}} - \underbrace{\sum_{i,j>i} \big(\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij} + \overline{\mathbf{S}}_j\, \overline{\mathbf{C}}_{ij}\big) RV}_{\substack{\text{tripoles} \\ \downarrow \\ I_{sc}^{(\mathbf{RV})}:\ \text{tripoles}}}}_{\substack{\downarrow \\ \text{no tripoles}}} + \underbrace{\sum_{i,j>i} \overline{\mathbf{C}}_{ij} RV}_{\substack{\text{no tripoles} \\ \downarrow \\ I_c^{(\mathbf{RV})}:\ \text{no tripoles}}}$$

Double virtual poles

$$VV\Big|_{1/\epsilon} = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg\{ -\frac{1}{\epsilon^4}\frac{1}{8}\left(\sum_i C_{f_i}\right)^2 B$$

$$+\frac{1}{\epsilon^3}\frac{1}{4}\left(\sum_i C_{f_i}\right)\left[\left(\frac{3}{8}\,b_0 + 2\sum_i \gamma_i^{(1)}\right) B - \sum_{i,j\neq i} \ln\frac{s_{ij}}{\mu^2}\,B_{ij}\right]$$

$$+\frac{1}{\epsilon^2}\frac{1}{4}\Bigg[\left(-\frac{b_0}{2}\sum_i \gamma_i^{(1)} - \frac{\widehat{\gamma}_K^{(2)}}{4}\sum_i C_{f_i} - 2\left(\sum_i \gamma_i^{(1)}\right)^2\right) B$$

$$+\left(\frac{b_0}{4} + 2\sum_i \gamma_i^{(1)}\right)\sum_{i,j\neq i}\ln\frac{s_{ij}}{\mu^2}\,B_{ij} - \frac{1}{4}\sum_{\substack{i,j\neq i\\ k,l\neq k}} \ln\frac{s_{ij}}{\mu^2}\,\ln\frac{s_{kl}}{\mu^2}\,B_{ijkl}\Bigg]$$

$$+\frac{1}{\epsilon}\frac{1}{8}\left[4\sum_i \gamma_i^{(2)}\,B - \widehat{\gamma}_K^{(2)}\sum_{i,j\neq i}\ln\frac{s_{ij}}{\mu^2}\,B_{ij}\right]\Bigg\}$$

$$+\left(\frac{\alpha_s}{\pi}\right)\left\{-\frac{1}{\epsilon^2}\frac{1}{2}\left(\sum_i C_{f_i}\right)V + \frac{1}{\epsilon}\left(\sum_i \gamma_i^{(1)}\right)V - \frac{1}{\epsilon}\frac{1}{2}\sum_{i,j\neq i}\ln\frac{s_{ij}}{\mu^2}\,V_{ij}\right\}.$$

$b_0 = \frac{11C_A - 4T_R N_f}{3}$, $\hat{\gamma}_K^{(1)} = 2$, $\gamma_q^{(1)} = -\frac{3}{4}C_F$, $\gamma_g^{(1)} = -\frac{1}{4}b_0$, $\hat{\gamma}_K^{(2)} = \left(\frac{67}{18} - \zeta(2)\right)C_A - \frac{5}{9}N_f$

$$\gamma_q^{(2)} = \left(-\frac{3}{32} + \frac{3}{4}\zeta(2) - \frac{3}{2}\zeta(3)\right)C_F^2 + \left(-\frac{961}{864} - \frac{11}{16}\zeta(2) + \frac{13}{8}\zeta(3)\right)C_A C_F + \left(\frac{65}{432} + \frac{1}{8}\zeta(2)\right)N_f C_F$$

$$\gamma_g^{(2)} = \left(-\frac{173}{108} + \frac{11}{48}\zeta(2) + \frac{1}{8}\zeta(3)\right)C_A^2 + \left(\frac{8}{27} - \frac{1}{24}\zeta(2)\right)N_f C_A + \frac{1}{8}N_f C_F$$

Cancellation of poles proportional to V

$$VV\Big|_{1/\epsilon}^{V} = -\left(\frac{\alpha_s}{\pi}\right)\left\{\frac{1}{2\epsilon^2}\left(\sum_i C_{f_i}\right)V + \frac{1}{\epsilon}\sum_i\left[\delta_{f_i\{q,\bar{q}\}}\frac{3}{4}C_F + \delta_{f_i g}\frac{11C_A - 4\,T_R\,N_f}{12}\right]V + \frac{1}{2\epsilon}\sum_{i,j\neq i}\ln\frac{s_{ij}}{\mu^2}\,V_{ij}\right\}.$$

The hard-collinear and the soft contributions to $I^{(\mathbf{RV})}$ are

$$I_{\mathrm{HC}}^{(\mathbf{RV})}\Big|_{1/\epsilon}^{V} = \left[I_{\mathrm{C}}^{(\mathbf{RV})} - I_{\mathrm{SC}}^{(\mathbf{RV})}\right]\Big|_{1/\epsilon}^{V} = -\left(\frac{\alpha_s}{\pi}\right)\sum_p\left\{\delta_{f_p g}\frac{C_A + 4\,T_R\,N_f}{12}\,\frac{1}{\epsilon} + \delta_{f_p\{q,\bar{q}\}}\frac{C_F}{4}\,\frac{1}{\epsilon}\right\}V$$

$$I_{\mathrm{S}}^{(\mathbf{RV})}\Big|_{1/\epsilon}^{V} = \left(\frac{\alpha_s}{\pi}\right)\left[\left(\frac{1}{2\epsilon^2}+\frac{1}{\epsilon}\right)\sum_p\left(\delta_{f_p\{q,\bar{q}\}}\,C_F + \delta_{f_p g}\,C_A\right)V + \frac{1}{2\epsilon}\sum_{k,\,l\neq k}\log\frac{s_{kl}}{\mu^2}V_{kl}\right]$$

The contribution $\left[I_{\mathrm{HC}}^{(\mathbf{RV})} - I_{\mathrm{S}}^{(\mathbf{RV})}\right]\Big|_{1/\epsilon}^{V}$ cancels all the poles of VV proportional to V.

$\rightarrow VV + I^{(\mathbf{RV})}$: only "*finite* $\times V$" coming from the finite part of $I^{(\mathbf{RV})}$.