Wilson line geometries in amplitude and PDF factorisation

G. Falcioni

JHEP 1911 (2019) 100, **arxiv:1909.00697**

with

E. Gardi and **C. Milloy**

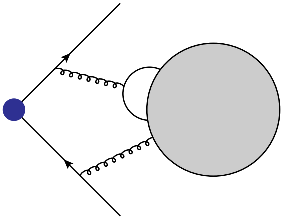
THE UNIVERSITY *of* EDINBURGH

Consider the two-point amplitudes with an off-shell current.
e.g. Massless quark form factor

$$\int dx\, e^{-iq\cdot x} \langle p_2|\overline{\psi}(x)\gamma^\mu\psi(x)|p_1\rangle = \overline{u}(p_2)\gamma^\mu u(p_1)\, F_{\text{quark}}(q^2)$$

General properties

- Single kinematic scale:
  $p_1^2 = p_2^2 = 0$,
  $q^2 = (p_2 - p_1)^2$
- **IR sensitive**: dimensionally regulated $d = 4 - 2\epsilon$, **singular for** $\epsilon \to 0$.



All-order expression

All-order representation (Magnea, Sterman 1990)

$$\log F_i(Q^2) = \int_0^{Q^2} \frac{d\lambda^2}{2\lambda^2}\left[G_i\left(1, \alpha_s(\lambda^2,\epsilon),\epsilon\right) - \gamma_i^{\text{cusp}}\left(\alpha_s(\lambda^2,\epsilon)\right)\log\frac{Q^2}{\lambda^2}\right]$$

i=quark, gluon.

- $\alpha_s(\mu^2,\epsilon)$ $d-$dimensional coupling constant

$$\mu^2\frac{d}{d\mu^2}\alpha_s(\mu^2,\epsilon) = -\epsilon\alpha_s(\mu^2,\epsilon) - b_0\alpha_s^2(\mu^2,\epsilon) - \dots$$

- **IR poles** arise after integration over the scale of α_s

$$\int_0^{Q^2}\frac{d\lambda^2}{\lambda^2}\alpha_s(\lambda^2,\epsilon) = \int_0^{\alpha_s(Q^2,\epsilon)}\frac{d\alpha}{-\epsilon - b_0\alpha\dots} = -\frac{\alpha_s(Q^2,\epsilon)}{\epsilon} + \mathcal{O}\left(\alpha_s^2\right)$$

Singularities of the form factor

Two anomalous dimensions control all the long-distance poles

- Cusp anomalous dimension $\gamma_i^{\text{cusp}} = \sum_{n=1}^{\infty} \gamma_i^{\text{cusp},(n)} \left(\frac{\alpha_s}{\pi}\right)^n$
- γ_{G_i} defined in terms of $G_i = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} G_{n,k}\, \epsilon^k \left(\frac{\alpha_s}{\pi}\right)^n$ such that

$$\int_0^{Q^2} \frac{d\lambda^2}{\lambda^2} G_i(\alpha_s(\lambda^2,\epsilon),\epsilon) = \int_0^{Q^2} \frac{d\lambda^2}{\lambda^2} \gamma_{G_i}(\alpha_s(\lambda^2,\epsilon),\epsilon) + \mathcal{O}\left(\epsilon^0\right)$$

  - Order-by-order expansion of γ_{G_i}

$$\gamma_{G_i} = G_{1,0}\left(\frac{\alpha_s}{\pi}\right) + \left[G_{2,0} - b_0\, G_{1,1}\right]\left(\frac{\alpha_s}{\pi}\right)^2 + \dots$$

Infrared factorisation

Singularities have soft and collinear origin. They **decouple** from the hard process and factorise (Collins 1980, Sen 1981)

$$F(q^2) = H\left(\frac{q^2}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2)\right) \prod_{i=1}^{2} J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$
$$\times \left(\frac{\mathcal{S}(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon)}{\prod_{i=1}^{2} \mathcal{J}_i\left(\frac{(2\beta_i \cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)}\right)$$

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- $J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)$: emissions **collinear** to p_i.
- $\mathcal{S}(\beta_1\cdot\beta_2, \alpha_s(\mu^2), \epsilon)$: **soft** particle exchanges.
- $\mathcal{J}_i\left(\frac{(2\beta_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)$: emissions that are **soft and collinear**.

Building blocks of the factorisation

$\mathcal{S}$, $\mathcal{J}$ and J_i are **defined** as **Wilson-line correlators**.
A Wilson line describes a **hard particle** emitting **soft gluons**

$$W_v(y,x) = \mathbf{P}\exp\left(ig_s\int_x^y d\lambda\, v_\mu\, A^\mu\,(\lambda v_\mu)\right)$$

- v_μ velocity of the hard particle (**no recoil**)
- A^μ in the same **colour representation** of the hard particle.

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Soft and **eikonal jet** functions

$$\mathcal{S}(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon) = \langle 0 | \, T\left[W_{\beta_1}(\infty, 0) W_{\beta_2}(0, \infty)\right] |0\rangle$$

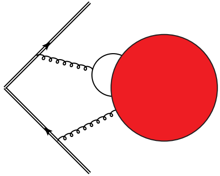
$$\mathcal{J}_i\left(\frac{(2\beta_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) = \langle 0 | \, T\left[W_{n_i}(\infty, 0) W_{\beta_i}(0, \infty)\right] |0\rangle$$

β_i velocity of the particle with momentum p_i, n_i auxiliary vector.

The soft function

$$\mathcal{S} = \langle 0 | T\left[W_{\beta_1}(\infty, 0) W_{\beta_2}(0, \infty)\right] |0\rangle$$

- *Eikonal* form factor describing small momentum exchanges.



- All-order formula as **form factors** (Dixon, Magnea, Sterman 2008)

$$\log \mathcal{S} = -\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\left[\Gamma_\wedge + \gamma^{\text{cusp}} \log\left(\frac{\beta_1 \cdot \beta_2 \mu^2}{\lambda^2}\right)\right]$$

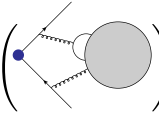
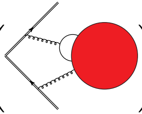
$\Gamma_\wedge$, γ^{cusp} obey **Casimir scaling** up to three loops *e.g.*

$$\frac{\gamma^{\text{cusp}}_{\text{quark}}}{C_F} = \frac{\gamma^{\text{cusp}}_{\text{gluon}}}{C_A}$$

$\log \mathcal{S}$ computed up to 2 loops (Erdoğan, Sterman 2015).

Purely collinear singularities

Subtracting the soft function from the form factor one
isolates purely collinear poles

$\log\left(\frac{J_i|_{\text{pole}}}{\mathcal{J}_i}\right) = \log\Bigg($  $\Bigg) - \log\Bigg($  $\Bigg)$

$$= \frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left[\gamma_{J_i/\mathcal{J}_i} - \frac{\gamma_i^{\text{cusp}}}{2}\log\left(\frac{2(p_i\cdot n_i)^2}{(\beta_i\cdot n_i)^2\mu^2}\right)\right]$$

where $\gamma_{J_i/\mathcal{J}_i}$ is **defined** as the difference

$$\boxed{2\gamma_{J_i/\mathcal{J}_i} = \gamma_{G_i} - \Gamma_{\wedge}}$$

- $\gamma_{J_i/\mathcal{J}_i}$ is **independent** on **process kinematics** $\longrightarrow$ **universal**.

Relations to different processes

Splitting functions[1] describe **collinear** *d.o.f.* in a **process**.

Virtual amplitude (form factors)

$$2\gamma_{J_i/\mathcal{J}_i} = \gamma_{G_i} - \Gamma_\wedge$$

Complete process (Drell-Yan)

$$2B_\delta = \gamma_{G_i} - \frac{\Gamma_{\rm DY}}{2}$$

(Korchemsky, Marchesini 1993, . . .)

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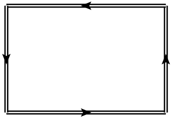
If we restrict B_δ **only to the virtual diagrams** one gets (Dixon, Magnea, Sterman 2008)

$$B_\delta^{\text{virt}} = \gamma_{J/\mathcal{J}} \longrightarrow \Gamma_{\text{DY}} = 2\Gamma_\wedge$$

The **real radiation** modifies this relation and, up to 2 loops, a **new Wilson-line contour** is relevant

$$\Gamma_{\text{DY}} = \frac{\Gamma_\square}{2}, \qquad \text{where} \qquad \Gamma_\square \sim$$



[1]The splitting functions include a contribution $B_\delta\, \delta(1-x)$.

Questions

- What is the relation between the collinear singularities in the jet functions and those in the splitting functions?

There is growing evidence that the difference $\gamma_G - 2B_\delta$ corresponds to a Wilson-line correlator.

i) Casimir scaling checked to three loops (Ravindran, Smith, van Neerven 2004; Moch, Vermaseren, Vogt 2005)

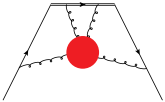
ii) Agreement between Γ_{DY} and $\Gamma_{\Box}$ at 2 loops.

iii) Γ_{DY} does **not** match $\Gamma_{\wedge}$.

- Is there a Wilson-line geometry relating B_δ and γ_G?

PDFs are the probabilities of extracting a parton of momentum xp from a proton P with momentum p *e.g*

$$f_q(x) = \frac{1}{2}\int \frac{dy}{2\pi}\, e^{-iy\, xp\cdot u}\langle P|\overline{\psi}_q(yu)\gamma\cdot u W_u(y,0)\psi_q(0)|P\rangle =$$



They renormalise with the DGLAP equations

$$\frac{df_i}{d\log\mu} = 2\int_x^1 \frac{dz}{z} P_{ij}(z,\alpha_s) f_j\left(\frac{x}{z},\mu\right)$$

In the limit $x \to 1$ the kernel takes the form

(Korchemsky 1989, Berger 2001)

$$P_{ii} = \frac{\gamma^{\text{cusp}}}{(1-x)_+} + B_\delta\,\delta(1-x) + \mathcal{O}(\log(1-x))$$

PDF factorisation

For $x \to 1$ **soft** and **collinear** contributions **factorise** (Korchemsky 1989, Korchemsky, Marchesini 1992). In Mellin space

$$\tilde{f}_i(N) = \int_0^1 dx\, x^{N-1} f_i(x)$$

Factorisation formula

$$\tilde{f}_i(N,\mu) = \left(\prod_{i=1}^{2} \frac{J_i\left(\frac{2(p_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s, \epsilon\right)\Big|_{\text{pole}}}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2}, \alpha_s, \epsilon\right)} \right) \tilde{\mathcal{S}}_{\sqcap}\left(N, \frac{\beta\cdot u\mu}{p\cdot u}, \alpha_s, \epsilon\right)$$

Same collinear singularities as the **form factors**.

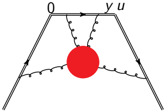
Evolution equation

$$P_{ii} = 2\gamma_{J_i/\mathcal{J}_i}\delta(1-x) + \frac{d\log \mathcal{S}_{\sqcap}}{d\log\mu^2}$$

The PDF soft function

$\mathcal{S}_{\sqcap}$ is a Wilson-line correlator (Korchemsky, Marchesini 1992)

$$\mathcal{S}_{\sqcap} = (p \cdot u) \int \frac{dy}{2\pi} e^{iy(1-x)\, p\cdot u}\, W_{\sqcap} \qquad \text{where } W_{\sqcap} =$$



All-order reperesentation

$$\log W_{\sqcap} = -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \left[2\gamma^{\text{cusp}} \log\left(\frac{i\,(y\, u\cdot\beta - i0)\,\mu}{\sqrt{2}} \right) + \Gamma_{\sqcap} \right]$$

Taking the log-derivative it leads to

$$\boxed{P_{ii} = \frac{\gamma^{\text{cusp}}}{(1-x)_+} + \left(\gamma_{J_i/\mathcal{J}_i} - \frac{\Gamma_{\sqcap}}{2} \right) \delta(1-x)}$$

Discussion

- Separation of **soft** and **purely collinear** contributions in B_δ

$$2B_\delta = 2\gamma_{J/\mathcal{J}} - \Gamma_{\sqcap}$$

- Comparison with the form factor singularities

$$\gamma_G = 2\gamma_{J/\mathcal{J}} - \Gamma_{\wedge}$$

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Conclusion

Relation between form factor singularities and splitting functions

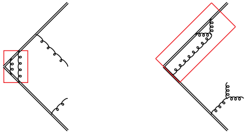
$$\boxed{\gamma_G - 2B_\delta = \Gamma_\sqcap - \Gamma_\wedge}$$

Checked to two loops via direct computation of $\Gamma_\sqcap$ and $\Gamma_\wedge$.

Effects of the Wilson-line geometry

$\Gamma_\wedge$, $\Gamma_\sqcap$ and $\Gamma_\square$ arise from the **renormalisation** of the **UV singularities** of the **different Wilson-line contours**.

- In coordinate space, divergences have two **origins** (Erdoğan, Sterman 2015)



- Cusp configurations
- Collinear to lightlike direction

- **Independent on the rest of the process**.
- **Sensitive** to **infinite/finite** lightlike lines.
  - $\Gamma_\wedge$ involves 2 infinite lines
  - $\Gamma_\sqcap$ has 2 infinite lines and a finite one.

$$\boxed{\Gamma_\sqcap - \Gamma_\wedge \equiv \Gamma_{\text{finite}} \equiv \frac{\Gamma_\square}{4}}$$

Answers

- What is the relation between the collinear singularities in the jet functions and those in the splitting functions?
  - They differ by a Wilson-line geometry that takes into account the **real radiation**.

$$B_\delta = \gamma_{J/\mathcal{J}} - \frac{\Gamma_\sqcap}{2}.$$

- Is there a Wilson-line geometry relating B_δ and γ_G?
  - We find the relation

$$\gamma_G - 2B_\delta = \Gamma_\sqcap - \Gamma_\wedge = \frac{\Gamma_\square}{4}$$

identifying the contribution of a **finite Wilson line**.

Outlook

- Test the relations between finite and infinite line anomalous dimensions on different contours.
- Test the agreement between $\Gamma_{\Box}$ and Γ_{DY}, using the known 3-loop results for the latter.
- Can we extend the relations beyond the singularities of the Wilson loops?
- $\Gamma_{\wedge}$ gives the finite parts of the **gluon Regge trajectory**. Can we explain this agreement?

Thank you

Choice of variable $\rho = i(y\beta \cdot u - i0)$ based on two observations (Korchemsky, Marchesini 1992).

- Reality of $\mathcal{S} \longrightarrow W_\sqcap(-y) = W_\sqcap^*(y)$
- Support of $\mathcal{S}$ in $x \leq 1 \longrightarrow W_\sqcap = W_\sqcap(y\beta \cdot u - i0)$

Starting at one loop

$$\log W_\sqcap^{\text{bare}} = -C_i \int_0^\infty \frac{d\lambda}{\lambda} \int_0^{\frac{\rho}{\sqrt{2}}} \frac{d\sigma}{\sigma} w_1(\epsilon)\, \alpha_s\left(\frac{1}{\lambda\sigma}\right)$$

This representation generalises to higher-orders (Erdoğan, Sterman 2015)

- $\log W_\sqcap$ has only a **single IR pole** and a **single collinear pole** (Frenkel, Gatheral, Taylor 1984)
  - λ, σ are the **largest** parameters
- the integrand **doesn't depend** on **any scale**

The bare expression vanishes **before renormalisation**.
UV singularities arise from integration with $\lambda, \sigma \to 0$.
The renormalisation scale will cut off this region of integration
(Erdoğan, Sterman 2015)

$$\log W_{\sqcap}^{\text{ren}} = -\int_{\frac{1}{\mu}}^{\infty} \frac{d\lambda}{\lambda} \int_{\frac{1}{\mu}}^{\frac{\rho}{\sqrt{2}}} \frac{d\sigma}{\sigma} \gamma^{\text{cusp}} \left(\alpha_s \left(\frac{1}{\lambda\sigma} \right) \right) - \int_{\frac{1}{\mu}}^{\infty} \frac{d\lambda}{\lambda} \Gamma_{\sqcap} \left(\alpha_s \left(\frac{1}{\lambda^2} \right) \right)$$

Which gives the result.