Leading and subleading jet functions

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Precision QCD@LHC

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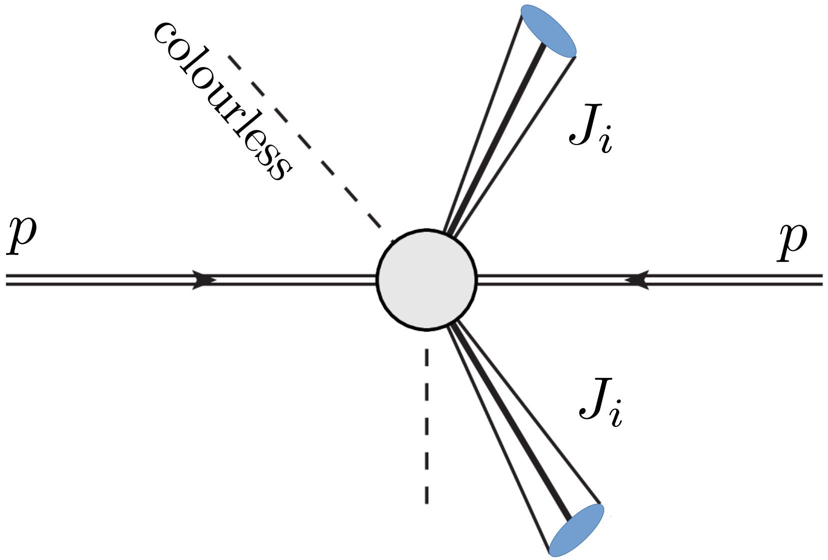
Jets & Jet functions

Jet functions J_i:

- Enter in factorization theorems with final state partons
- Describe collinear radiation of final state partons
- Various types already studied: Exclusive, semi-inclusive, ….

[Ellis, Vermilion, Walsh, Hornig, Lee: JHEP 1011 (2010) 101]
[Kang, Ringer, Vitev: JHEP 1610 (2016) 125]



Jets & Jet functions

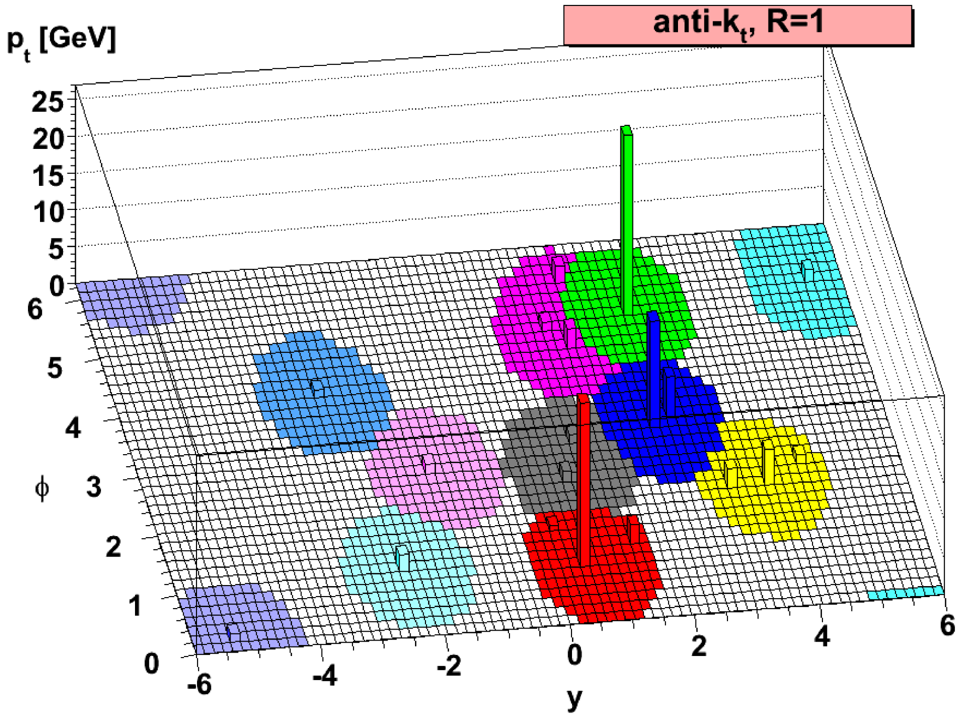
Jets require an algorithm to define
→ *How to cluster final state particles?*

Here we use the anti-k_T algorithm.

[Cacciari, Salam, Soyez: JHEP 0804 (2008) 063]

$$d_{ij} = \min\left(\frac{1}{p_{T,i}^2}, \frac{1}{p_{T,j}^2}\right)\frac{\Delta R_{ij}^2}{R^2}$$

$$d_{iB} = \frac{1}{p_{T,i}^2}$$



Jets & Jet functions

Consider $R \ll 1$ with no other scale hierarchies.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}p_{T,J}} = \sum_{i \in \mathrm{partons}} \int \mathrm{d}z\, \mathrm{d}p_{T,i}\, \frac{\mathrm{d}\hat{\sigma}_{pp \to X+i}}{\mathrm{d}p_{T,i}}\, J_i(z) \delta(p_{T,J} - z p_{T,i}) + \mathcal{O}(R^2)$$

Energy fraction $z = p_{T,J}/p_{T,i}$

Specifically

Leading-jet functions

Track energy fraction of leading jet

Subleading-jet functions

Track energy fraction of leading and subleading jets

Use the jet functions to resum $(\alpha_s \ln R)^n$.

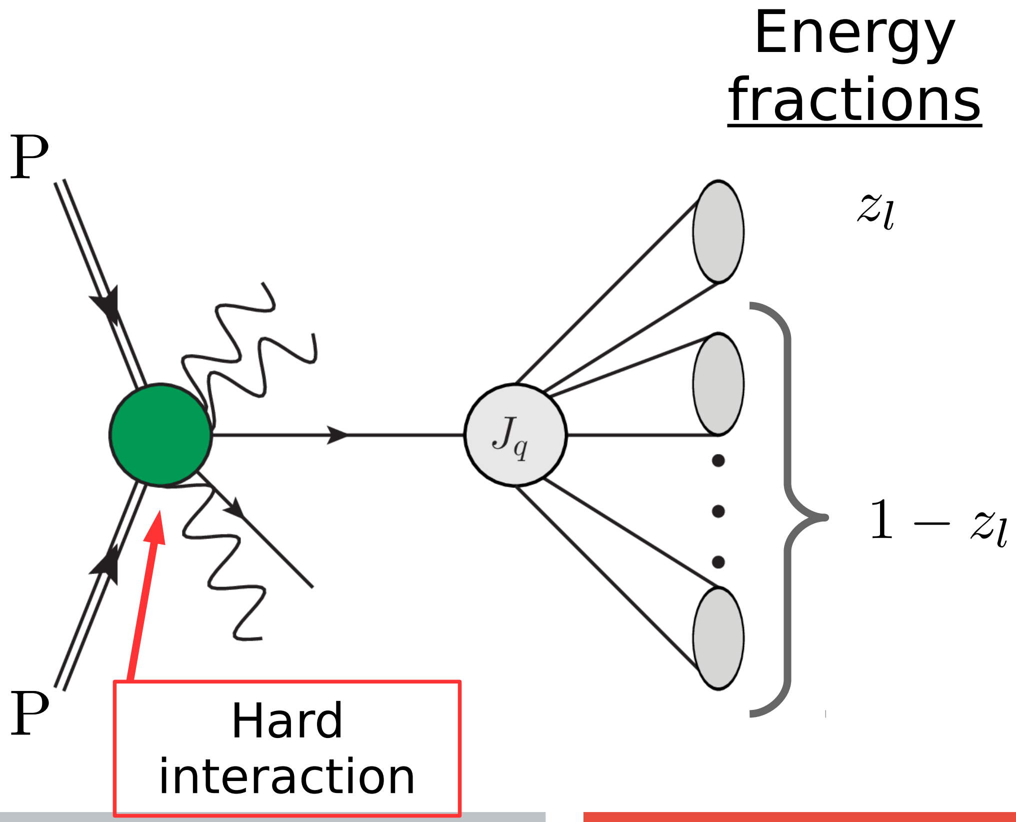
Leading-jet function

Final state partons fragment into jets.

Leading-jet function: probability that the ***hardest*** jet has energy fraction z_l.

$$J_{l,i}(z_l,\mu) = \sum_{n=0} \left(\frac{\alpha_s}{\pi}\right)^n J_{l,i}^{(n)}(z_l,\mu)$$

$$J_{l,q}^{(0)}(z_l,\mu) = J_{l,g}^{(0)}(z_l,\mu) = \delta(1-z_l)$$



NLO calculation: quark

Treat radiation as collinear
Definition in terms of Soft Collinear Effective Theory (SCET) fields

Light cone components: $n^\mu = (1,0,0,1)$ — Collinear to parton; $\bar{n}^\mu = (1,0,0,-1)$ — Anti-collinear to parton

$$p^\mu = \bar{n}\cdot p\,\frac{n^\mu}{2} + n\cdot p\,\frac{\bar{n}^\mu}{2} + p_\perp^\mu$$

$$J_{l,q}(z_l, p_T R, \mu) = \frac{16\pi^3}{2N_c}\sum_X \mathrm{Tr}\Big[\frac{\not{\bar{n}}}{2}\langle 0|\delta(2p_T - \bar{n}\cdot\mathcal{P})\delta^2(\mathcal{P}_\perp)\chi_n(0)|X\rangle\langle X|\bar{\chi}_n(0)|0\rangle\Big]$$
$$\times\,\delta\left(z_l - \frac{\max_{J\in X} p_{T,J}}{p_T}\right)$$

NLO calculation: quark

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$$\times\,\delta\left(z_l - \frac{\max_{J\in X} p_{T,J}}{p_T}\right)$$

χ_n – Collinear quark field

NLO calculation: quark

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$$\times\,\delta\left(z_l - \frac{\max_{J\in X} p_{T,J}}{p_T}\right)$$

Projects out correct momentum for incoming quark

NLO calculation: quark

Treat radiation as collinear
Definition in terms of Soft Collinear Effective Theory (SCET) fields

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$$\times\,\delta\left(z_l - \frac{\max_{J\in X} p_{T,J}}{p_T}\right)$$

Definition of z_l

NLO calculation: quark

At NLO we find

$$J_{l,q}^{(1)} = \Theta\left(z_l - \frac{1}{2}\right)\left\{\left(\frac{1}{2\epsilon} + \ln\frac{\mu}{p_T R}\right)\left[P_{qq}(z_l) + P_{gq}(z_l)\right]\right.$$
$$+ C_F\left[-2\left[\frac{\ln(1-z_l)}{1-z_l}\right]_+ + \left(\frac{13}{4} - \frac{\pi^2}{3}\right)\delta(1-z_l)\right.$$
$$\left.\left. - 2\frac{\ln z_l}{1-z_l} + \left(3 - \frac{2}{z_l}\right)\ln[z_l(1-z_l)] - \frac{1}{2}\right]\right\}$$

Features:

- Non-zero only for $z_l > 0.5$
- Single $\ln R$ term proportional to splitting functions

NLO calculation: quark

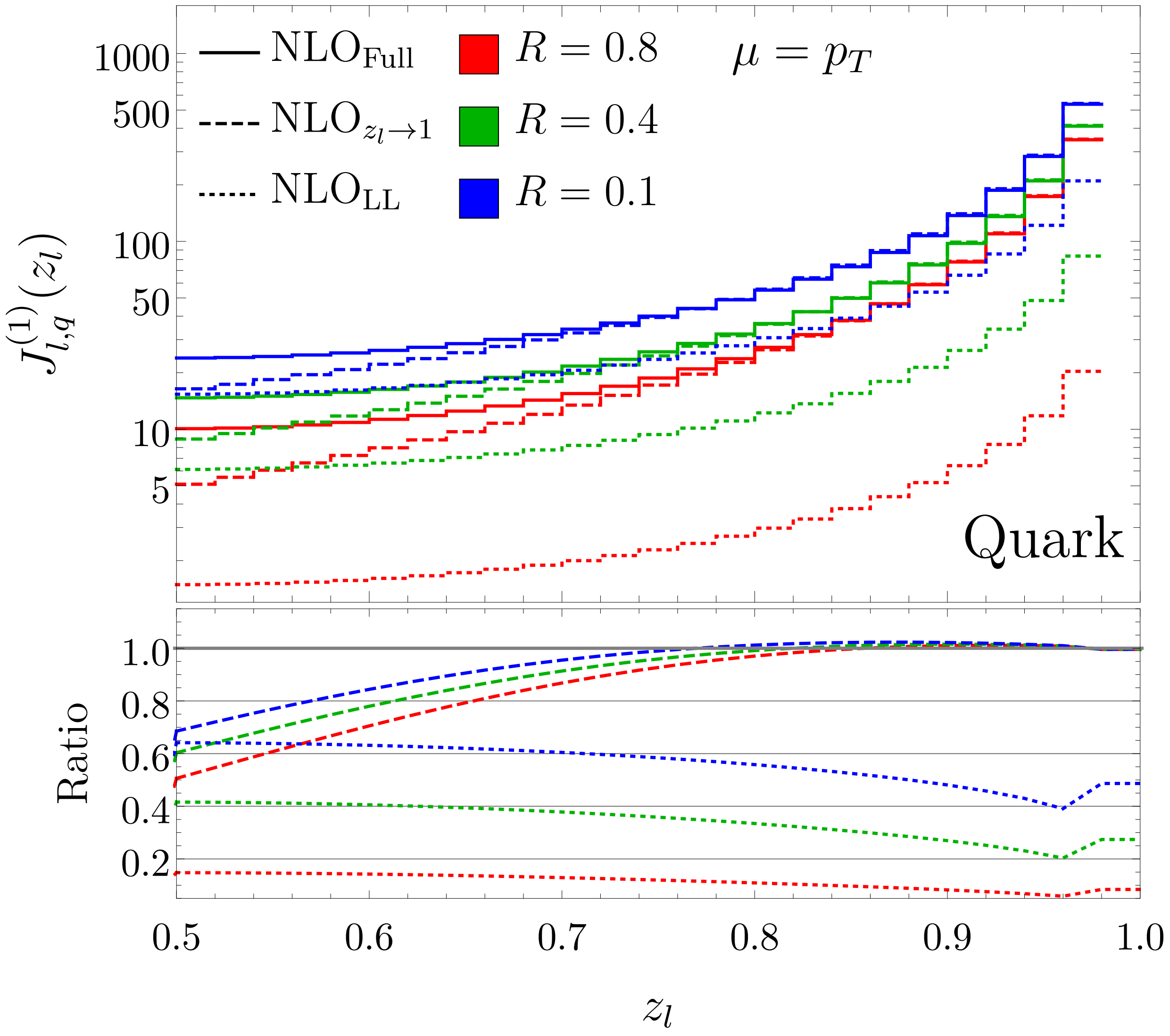
At NLO we find: SOFT LIMIT

$$J_{l,q}^{(1)}\Big|_{z_l\to 1} = 2C_F\left(\frac{1}{[1-z_l]_+}\ln\frac{\mu}{p_T R} - \left[\frac{\ln(1-z_l)}{1-z_l}\right]_+\right) + C_F\delta(1-z_l)\left(\frac{3}{2}\ln\frac{\mu}{p_T R} + \frac{13}{4} - \frac{\pi^2}{3}\right)$$

Features: SOFT LIMIT

- First line same for gluon up to $C_F \to C_A$
- Contains *some* $\ln R$ terms

NLO calculation: quark

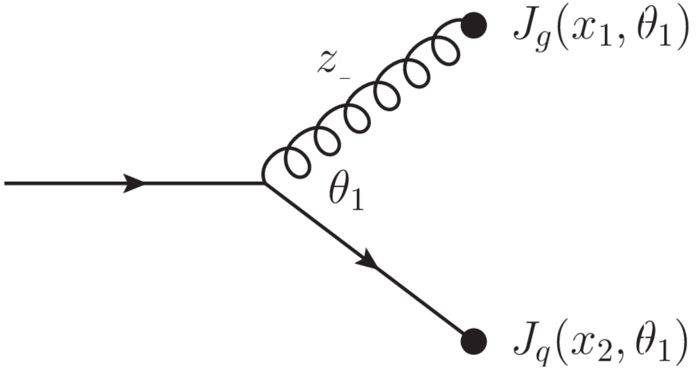


<u>Plot keeping full, Leading log (LL), and soft terms</u>

- LL: poor approximation to full jet function
- Soft approximation works well
- Similar results for gluon

Evolution: leading-jet function

$$
\begin{aligned}
J_{l,q}(z_l, \theta^{\max}) = \delta(1 - z_l) \\
&+ \frac{\alpha_s}{\pi} \int_0^1 dz\, P_{gq}(z) \int_R^{\theta^{\max}} \frac{d\theta_1}{\theta_1} \\
&\times \int_0^1 dx_1\, J_{l,g}(x_1, \theta_1) \int_0^1 dx_2\, J_{l,q}(x_2, \theta_1) \\
&\times \delta\Big(z_l - \max[z x_1, (1 - z) x_2]\Big)
\end{aligned}
$$

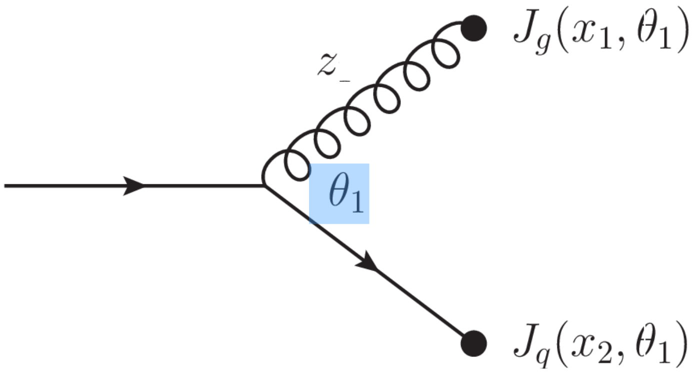


Using a parton shower picture, one can write down a recursive definition at LL.

[Elder, Procura, Thaler, Waalewijn, Zhou: JHEP 1706 (2017) 085]
[Waalewijn: Phys.Rev. D86 (2012) 094030]

Evolution: leading-jet function

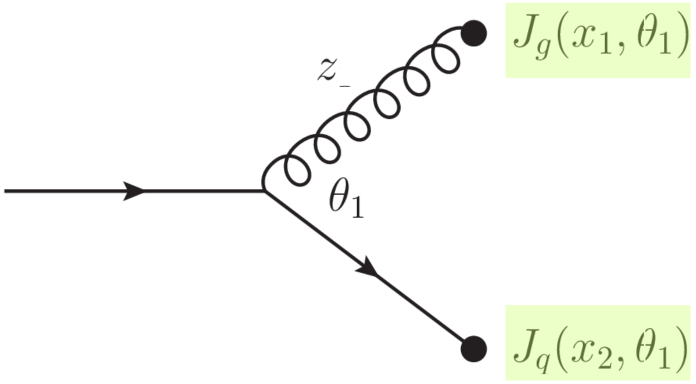
$$J_{l,q}(z_l, \theta^{\max}) = \delta(1 - z_l) + \frac{\alpha_s}{\pi} \int_0^1 dz\, P_{gq}(z) \int_R^{\theta^{\max}} \frac{d\theta_1}{\theta_1} \times \int_0^1 dx_1\, J_{l,g}(x_1, \theta_1) \int_0^1 dx_2\, J_{l,q}(x_2, \theta_1) \times \delta\Big(z_l - \max[z x_1, (1-z) x_2]\Big)$$



Usual angular integration. Emissions smaller than R finish in the same jet, so they do not affect z_l.

Evolution: leading-jet function

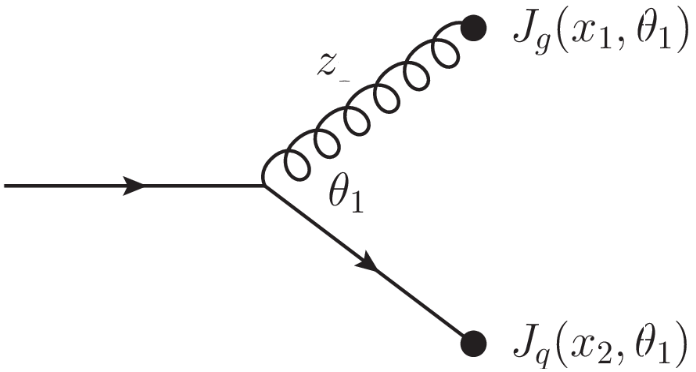
$$J_{l,q}(z_l, \theta^{\max}) = \delta(1 - z_l) + \frac{\alpha_s}{\pi} \int_0^1 dz\, P_{gq}(z) \int_R^{\theta^{\max}} \frac{d\theta_1}{\theta_1} \times \int_0^1 dx_1\, J_{l,g}(x_1, \theta_1) \int_0^1 dx_2\, J_{l,q}(x_2, \theta_1) \times \delta\Big(z_l - \max[z x_1, (1-z) x_2]\Big)$$



- Integrate over possible energy fractions of new 'leading' jets.
- At LL emissions are strongly angular ordered: θ_1 is new $\theta^{\max}$ for further emissions.

Evolution: leading-jet function

$$J_{l,q}(z_l, \theta^{\max}) = \delta(1 - z_l) + \frac{\alpha_s}{\pi} \int_0^1 dz\, P_{gq}(z) \int_R^{\theta^{\max}} \frac{d\theta_1}{\theta_1} \times \int_0^1 dx_1\, J_{l,g}(x_1, \theta_1) \int_0^1 dx_2\, J_{l,q}(x_2, \theta_1) \times \delta\Big(z_l - \max[z x_1, (1 - z) x_2]\Big)$$



Ensures z_l corresponds to the jet with the highest energy fraction.

Evolution: leading-jet function

- Recursive definition allows us to obtain the LL RGE
- Jet function scale is set by $p_T R$, evolve to hard scale p_T
- Convenient to re-write angular integral in terms of p_T
- Dependence on the scale μ explicit (lower bound on θ integral)

Derivative w.r.t. μ gives

$$\mu \frac{d}{d\mu} J_{l,q}(z_l, \mu) = \frac{\alpha_s}{\pi} \int_0^1 dz\, P_{gq}(z) \int_0^1 dx_1\, J_{l,g}(x_1, \mu) \int_0^1 dx_2\, J_{l,q}(x_2, \mu) \\ \times \delta\Big(z_l - \max[z x_1, (1-z) x_2]\Big)$$

Difficult to solve analytically.
Can generate LL solutions to higher order for the jet function.

LL terms at higher order

Using lower order solutions as input to the RGE:

$$J_{l,q}^{(0)}(z_l,\mu) = \delta(1-z_l)$$

$$J_{l,q}^{(1)}(z_l,\mu) = \ln\frac{\mu}{\mu_0}\Big[P_{qq}(z_l) + P_{gq}(z_l)\Big]\Theta\left(z_l - \frac{1}{2}\right)$$

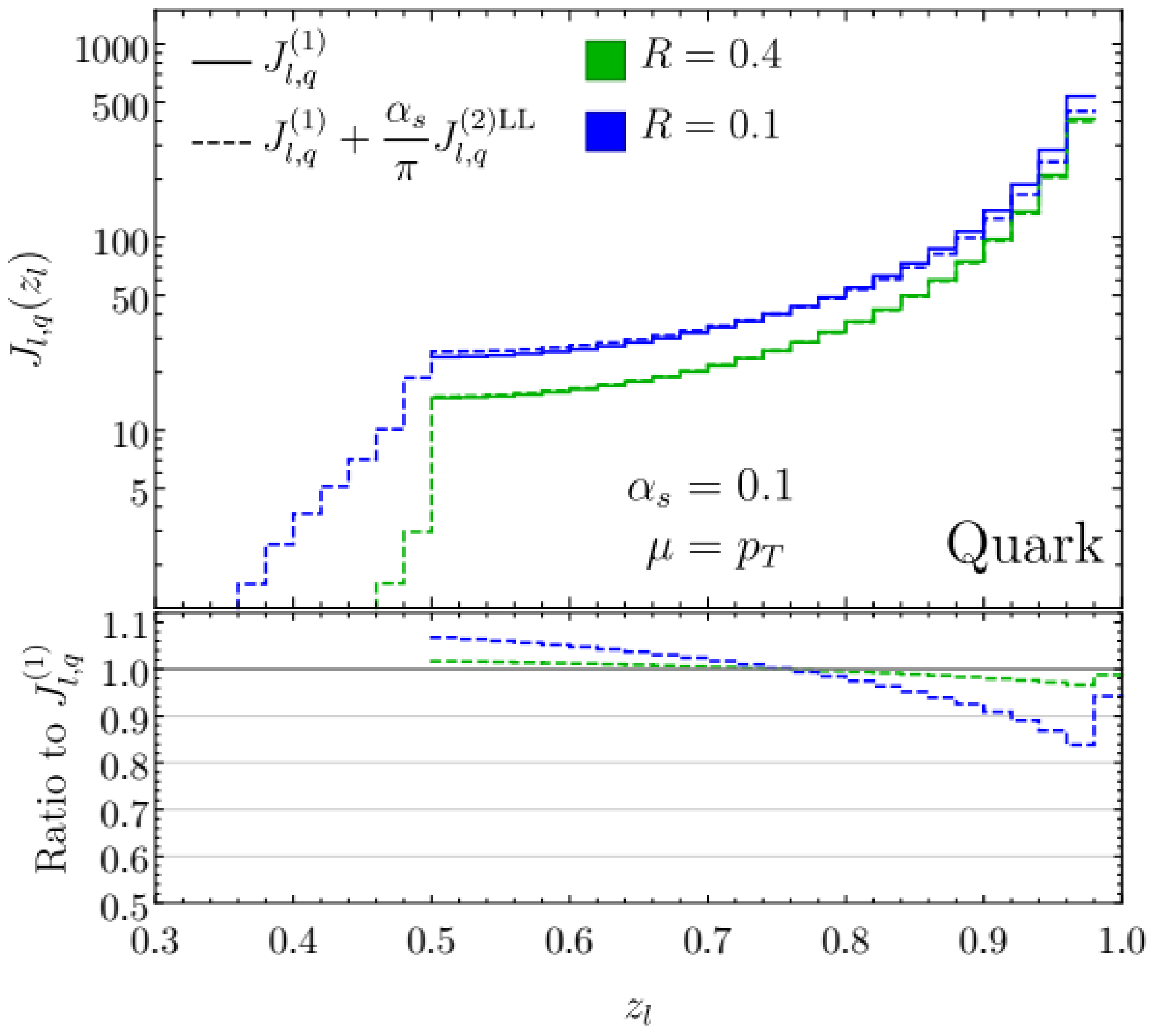
$$J_{l,q}^{(2)}(z_l,\mu) = \frac{1}{2}\ln^2\frac{\mu}{\mu_0}\Bigg\{\Theta\left(z_l-\frac{1}{2}\right)\left[\frac{\beta_0}{2}\left(P_{qq}(z_l)+P_{gq}(z_l)\right) + \int_z^1 \frac{dx}{x}F_q\left(x,\frac{z_l}{x}\right)\right] + \Theta\left(\frac{1}{2}-z_l\right)\Theta\left(z_l-\frac{1}{3}\right)\int_{\frac{1}{2}}^{\frac{z_l}{1-z_l}} dx\left[\frac{1}{x}F_q\left(x,\frac{z_l}{x}\right) + F_q(x,1-z_l)\right]\Bigg\}$$

$F_q(x,y)$ - products of splitting functions.

At $\mathcal{O}(\alpha_s)$ only one splitting – leading parton must have $z_l > 0.5$.

At $\mathcal{O}(\alpha_s^2)$ we have 3 partons. Implies $z_l > 1/3$.

LL terms at higher order



Subleading-jet function

Extend previous framework to also track 2nd hardest jet $J_{s,i}(z_l, z_s, \mu)$
Can be used to:

- Place (loose) jet veto → exclusive production
- Double differential analyses

At (N)LO z_s dependence is trivial

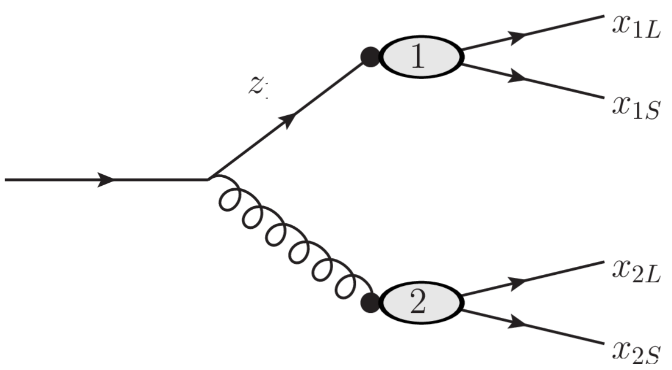
$$J_{s,i}^{(0)}(z_l, z_s, \mu) = J_{s,i}^{(0)}(z_l, \mu)\delta(z_s)$$
$$J_{s,i}^{(1)}(z_l, z_s, \mu) = J_{s,i}^{(1)}(z_l, \mu)\delta(z_s - (1 - z_l))$$

What about LL at NNLO in this case?

Evolution: subleading-jet function

As before we derive an RG equation for our subleading jet function:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} J_{s,i}(z_l, z_s, \mu) = \frac{\alpha_s(\mu)}{\pi} \int_0^1 \mathrm{d}z\, \mathrm{d}x_{1l}\, \mathrm{d}x_{2l}\, \mathrm{d}x_{1s}\, \mathrm{d}x_{2s}$$
$$\times K_{s,i}(x_{1l}, x_{2l}, x_{1s}, x_{2s}, z; \mu)$$
$$\times \Big\{ \Theta\big(zx_{1l} - (1-z)x_{2l}\big)\delta\big(z_l - zx_{1l}\big)\delta\big(z_s - \max[zx_{1s}, (1-z)x_{2l}]\big)$$
$$+ \Theta\big((1-z)x_{2l} - zx_{1l}\big)\delta\big(z_l - (1-z)x_{2l}\big)\delta\big(z_s - \max[zx_{1l}, (1-z)x_{2s}]\big)\Big\}$$



Evolution: subleading-jet function

As before we derive an RG equation for our subleading jet function:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}J_{s,i}(z_l,z_s,\mu)=\frac{\alpha_s(\mu)}{\pi}\int_0^1 \mathrm{d}z\,\mathrm{d}x_{1l}\,\mathrm{d}x_{2l}\,\mathrm{d}x_{1s}\,\mathrm{d}x_{2s}$$

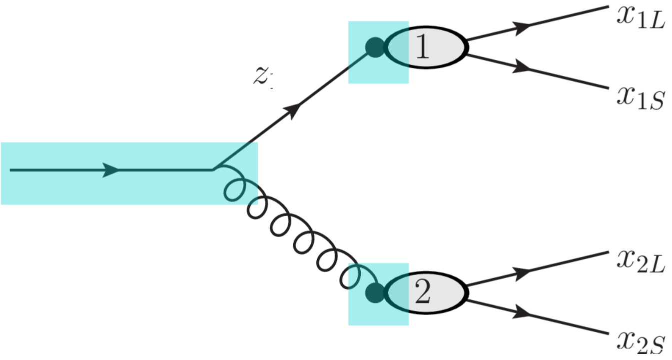
$$\times K_{s,i}(x_{1l},x_{2l},x_{1s},x_{2s},z;\mu)$$

$$\times\Big\{\Theta\big(zx_{1l}-(1-z)x_{2l}\big)\delta\big(z_l-zx_{1l}\big)\delta\big(z_s-\max\left[zx_{1s},(1-z)x_{2l}\right]\big)$$

$$+\,\Theta\big((1-z)x_{2l}-zx_{1l}\big)\delta\big(z_l-(1-z)x_{2l}\big)\delta\big(z_s-\max\left[zx_{1l},(1-z)x_{2s}\right]\big)\Big\}$$

Splitting & jet functions. E.g.

$$K_{s,q}=P_{qq}(z_1)J_{s,q}(x_{1L},x_{1S})J_{s,g}(x_{2L},x_{2S})$$



Evolution: subleading-jet function

As before we derive an RG equation for our subleading jet function:

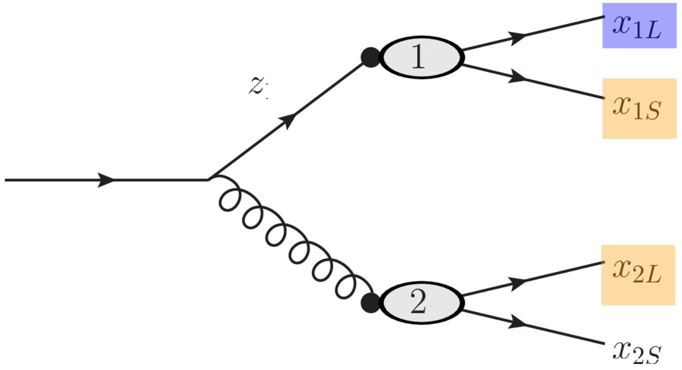
$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} J_{s,i}(z_l, z_s, \mu) = \frac{\alpha_s(\mu)}{\pi} \int_0^1 \mathrm{d}z\, \mathrm{d}x_{1l}\, \mathrm{d}x_{2l}\, \mathrm{d}x_{1s}\, \mathrm{d}x_{2s}$$

$$\times K_{s,i}(x_{1l}, x_{2l}, x_{1s}, x_{2s}, z; \mu)$$

$$\times \Big\{ \Theta\big(zx_{1l} - (1-z)x_{2l}\big)\delta\big(z_l - zx_{1l}\big)\delta\big(z_s - \max[zx_{1s}, (1-z)x_{2l}]\big)$$

$$+ \Theta\big((1-z)x_{2l} - zx_{1l}\big)\delta\big(z_l - (1-z)x_{2l}\big)\delta\big(z_s - \max[zx_{1l}, (1-z)x_{2s}]\big) \Big\}$$

Accounting for the subleading jet.
Depends on where the leading jet is.



Subleading-jet LL terms at NNLO

We can again derive LL results to NNLO (and higher orders) in α_s:

$$J_{s,i}^{(2)\mathrm{LL}}(z_l, z_s, \mu) = \frac{\beta_0}{4}\ln\Big(\frac{\mu}{p_T R}\Big) J_{s,i}^{(1)\mathrm{LL}}(z_l, z_s, \mu)$$
$$+ \frac{1}{2}\ln^2\Big(\frac{\mu}{p_T R}\Big)\Theta\big(z_l - z_s\big)\Theta\big(z_l + 2z_s - 1\big)\Theta\big(1 - z_l - z_s\big)$$
$$\times \Big[F_i(z_l, z_s) + F_i(z_s, z_l) + F_i(1 - z_l - z_s, z_l)\Big]\Big\}$$

where for example

$$F_q(a,b) = \frac{1}{1-a}\Big\{P_{qq}(a)\Big[P_{gg}\Big(\frac{b}{1-a}\Big) + 2n_f P_{qg}\Big(\frac{b}{1-a}\Big)\Big]$$
$$+ P_{qq}(1-a)\Big[P_{qq}\Big(\frac{b}{1-a}\Big) + P_{gq}\Big(\frac{b}{1-a}\Big)\Big]\Big\}$$

Phenomenology: setup

Study the $p_{T,J}$ spectrum in exclusive H+1 jet production.
At LO:

$$\frac{\mathrm{d}\sigma^{\mathrm{LO}}_{pp\to HJ}}{\mathrm{d}p_{T,J}} = \sum_i \int \mathrm{d}p_{T,i} \, \frac{\mathrm{d}\tilde{\sigma}^{(0)}_{pp\to Hi}}{\mathrm{d}p_{T,i}} \int \mathrm{d}z_l \mathrm{d}z_s \, J_{s,i}(z_l, z_s, \mu) \, \delta(p_{T,J} - z_l p_{T,i}) + \mathcal{O}(R^2)$$

$$= \sum_i \frac{\mathrm{d}\tilde{\sigma}^{(0)}_{pp\to Hi}}{\mathrm{d}p_{T,i}} \otimes J_{s,i} + \mathcal{O}(R^2)$$

\`Singular'

Phenomenology: setup

Study the $p_{T,J}$ spectrum in exclusive H+1 jet production.
At LO:

$$\frac{\mathrm{d}\sigma^{\mathrm{LO}}_{pp\to HJ}}{\mathrm{d}p_{T,J}} = \sum_i \int \mathrm{d}p_{T,i}\, \frac{\mathrm{d}\tilde{\sigma}^{(0)}_{pp\to Hi}}{\mathrm{d}p_{T,i}} \int \mathrm{d}z_l \mathrm{d}z_s\, J_{s,i}(z_l, z_s, \mu)\, \delta(p_{T,J} - z_l p_{T,i}) + \mathcal{O}(R^2)$$

$$= \sum_i \frac{\mathrm{d}\tilde{\sigma}^{(0)}_{pp\to Hi}}{\mathrm{d}p_{T,i}} \otimes J_{s,i} + \mathcal{O}(R^2)$$

Simply the $pp \to H + i$ cross section at LO.

Phenomenology: setup

Study the $p_{T,J}$ spectrum in exclusive H+1 jet production.
At NLO:

$$\frac{\mathrm{d}\sigma^{\mathrm{sing,NLO}}_{pp\to HJ}}{\mathrm{d}p_{T,J}}(p_T^{\mathrm{veto}}) = \sum_i \frac{\mathrm{d}\tilde{\sigma}^{(0)}_{pp\to Hi}}{\mathrm{d}p_{T,i}} \otimes J^{(0)}_{s,i}$$

$$+\frac{\alpha_s}{\pi}\left[\sum_i \frac{\mathrm{d}\tilde{\sigma}^{(0)}_{pp\to Hi}}{\mathrm{d}p_{T,i}} \otimes J^{(1)}_{s,i} + \sum_{i,j} \frac{\mathrm{d}\tilde{\sigma}^{(1)}_{pp\to Hij}}{\mathrm{d}p_{T,i}\mathrm{d}p_{T,j}} \otimes J^{(0)}_{s,i} \otimes J^{(0)}_{s,j}\right]$$

Phenomenology: setup

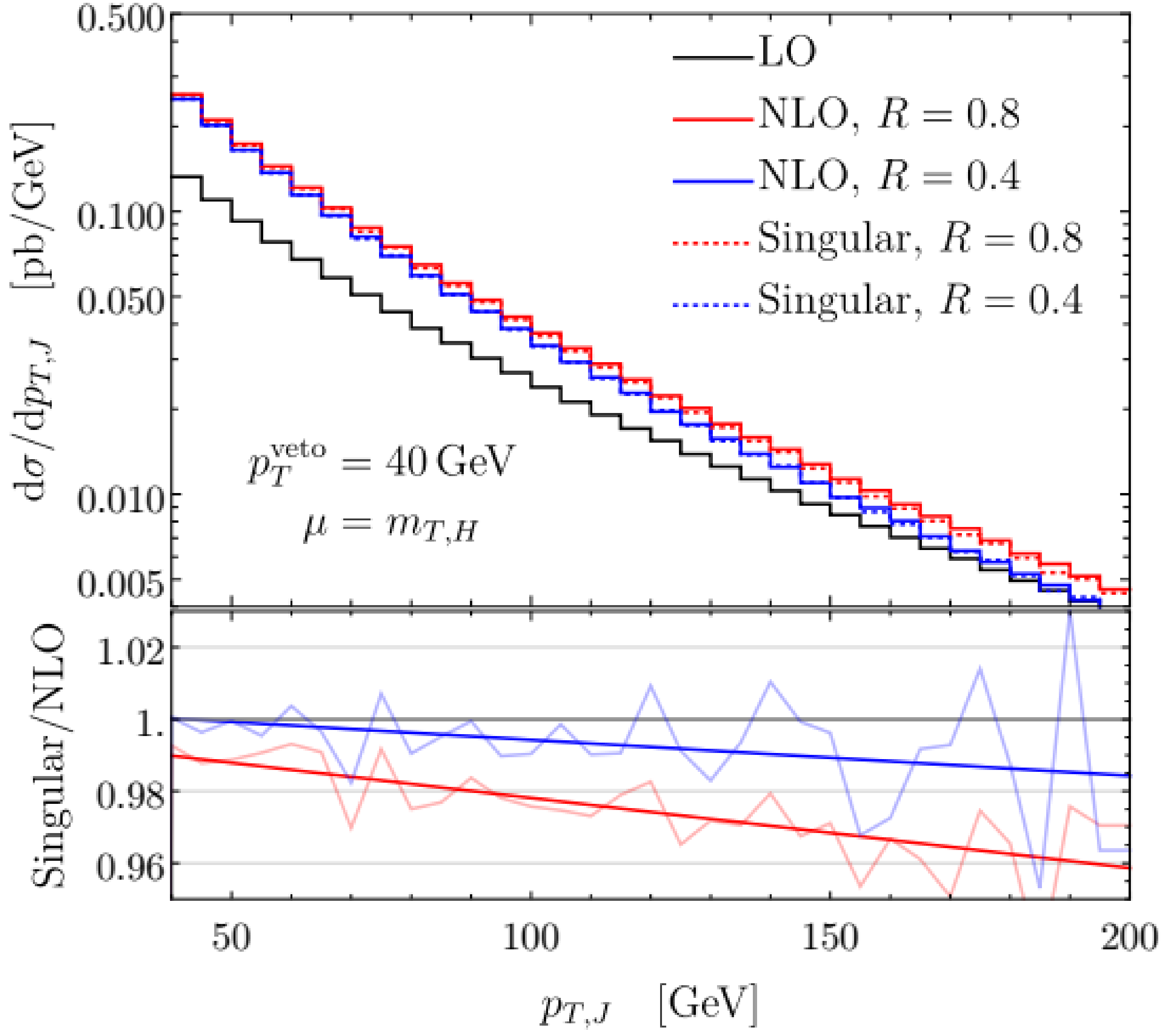
Study the $p_{T,J}$ spectrum in exclusive H+1 jet production.
At NLO:

$$\frac{\mathrm{d}\sigma^{\mathrm{sing,NLO}}_{pp\to HJ}}{\mathrm{d}p_{T,J}}(p_T^{\mathrm{veto}}) = \sum_i \frac{\mathrm{d}\tilde{\sigma}^{(0)}_{pp\to Hi}}{\mathrm{d}p_{T,i}} \otimes J^{(0)}_{s,i} + \frac{\alpha_s}{\pi}\left[\sum_i \frac{\mathrm{d}\tilde{\sigma}^{(0)}_{pp\to Hi}}{\mathrm{d}p_{T,i}} \otimes J^{(1)}_{s,i} + \sum_{i,j} \frac{\mathrm{d}\tilde{\sigma}^{(1)}_{pp\to Hij}}{\mathrm{d}p_{T,i}\mathrm{d}p_{T,j}} \otimes J^{(0)}_{s,i} \otimes J^{(0)}_{s,j}\right]$$

Not just real correction to $pp \to H + i$

- Contains virtual corrections
- Contains two *widely separated* partons
- Independent of R
- Can be extracted from known full NLO predictions

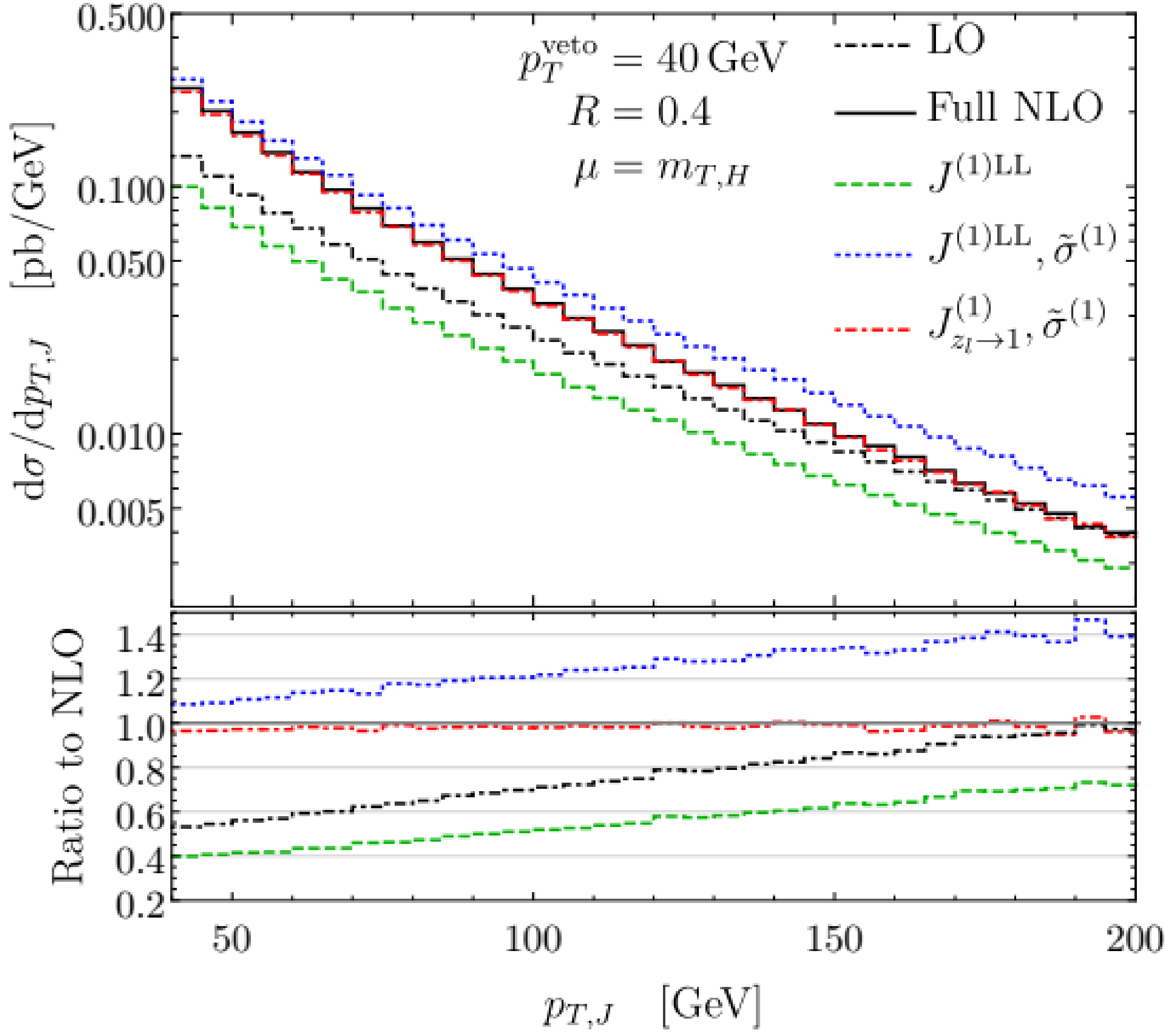
Phenomenology: singular cross section



Singular in good agreement with full NLO.

Is this captured by the $\ln R$ terms?

Phenomenology: singular cross section



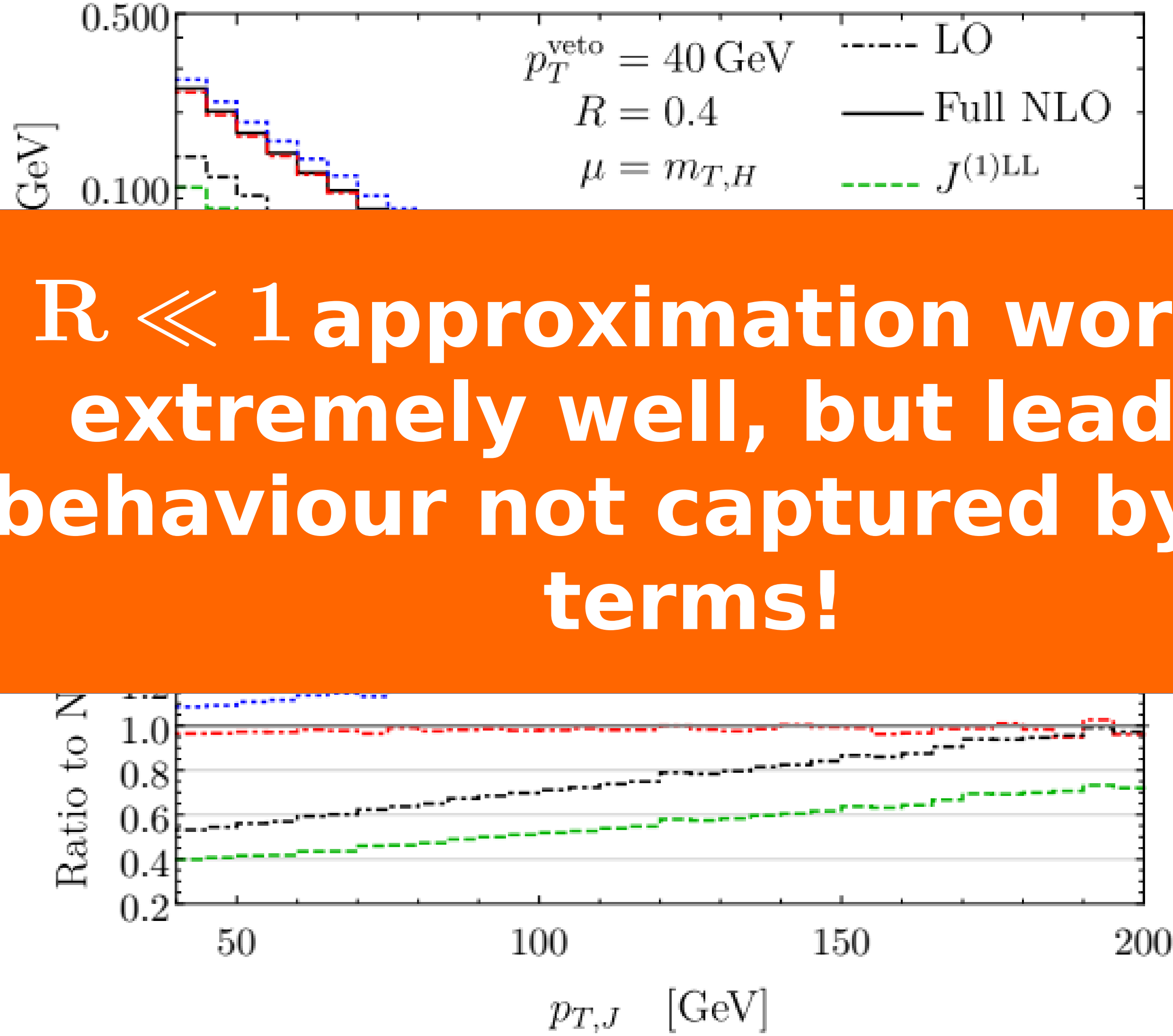
Look at $R = 0.4$

Only LL terms:
Poor approximation

LL + $\tilde{\sigma}^{(1)}$:
Better. Large contribution from "hard" function

Soft J + $\tilde{\sigma}^{(1)}$:
Excellent agreement

Phenomenology: singular cross section



Look at $R = 0.4$

Only LL terms:

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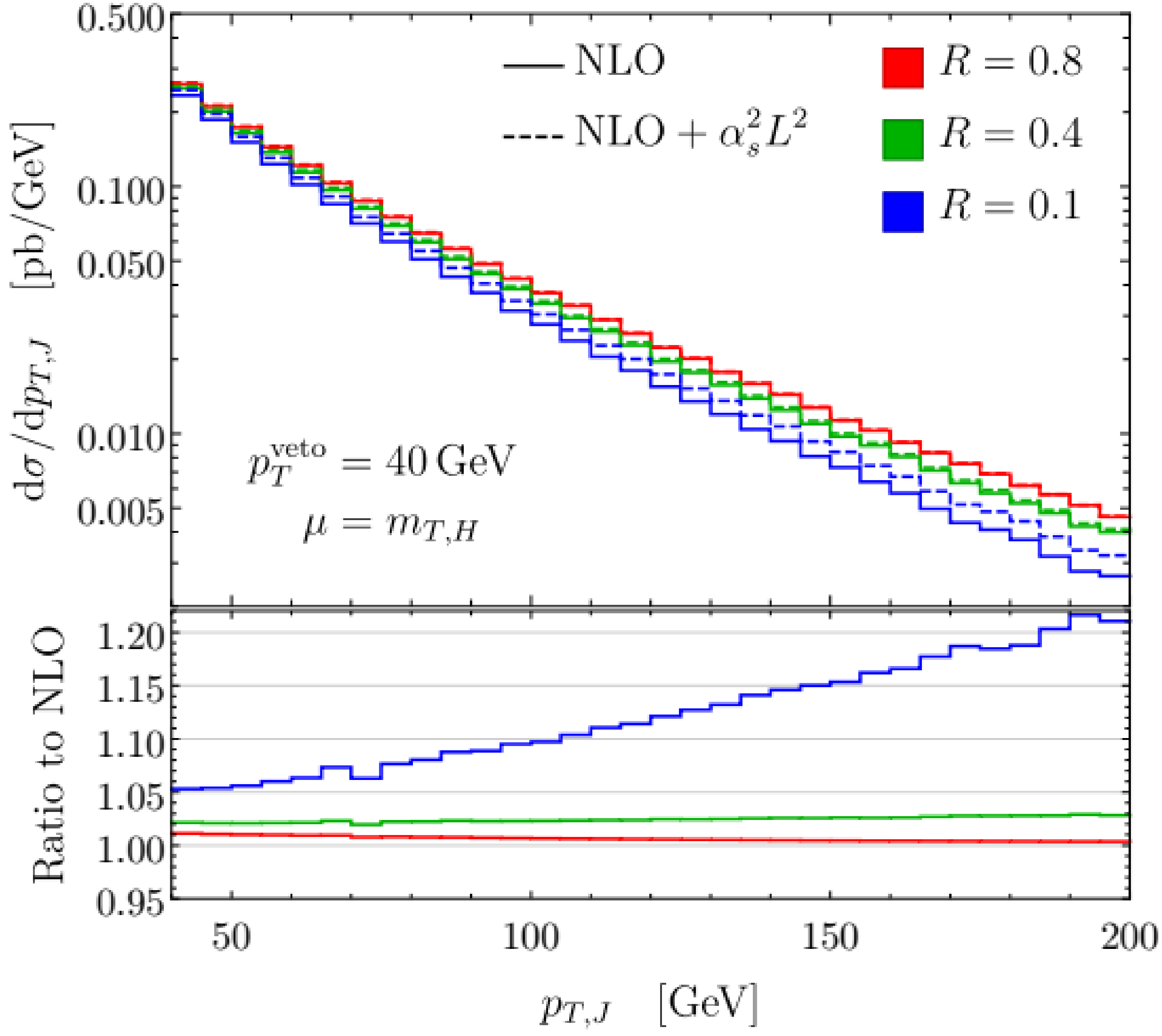
Soft J + $\tilde{\sigma}^{(1)}$:
Excellent agreement

$\mathrm{R} \ll 1$ approximation works extremely well, but leading behaviour not captured by $\ln \mathrm{R}$ terms!

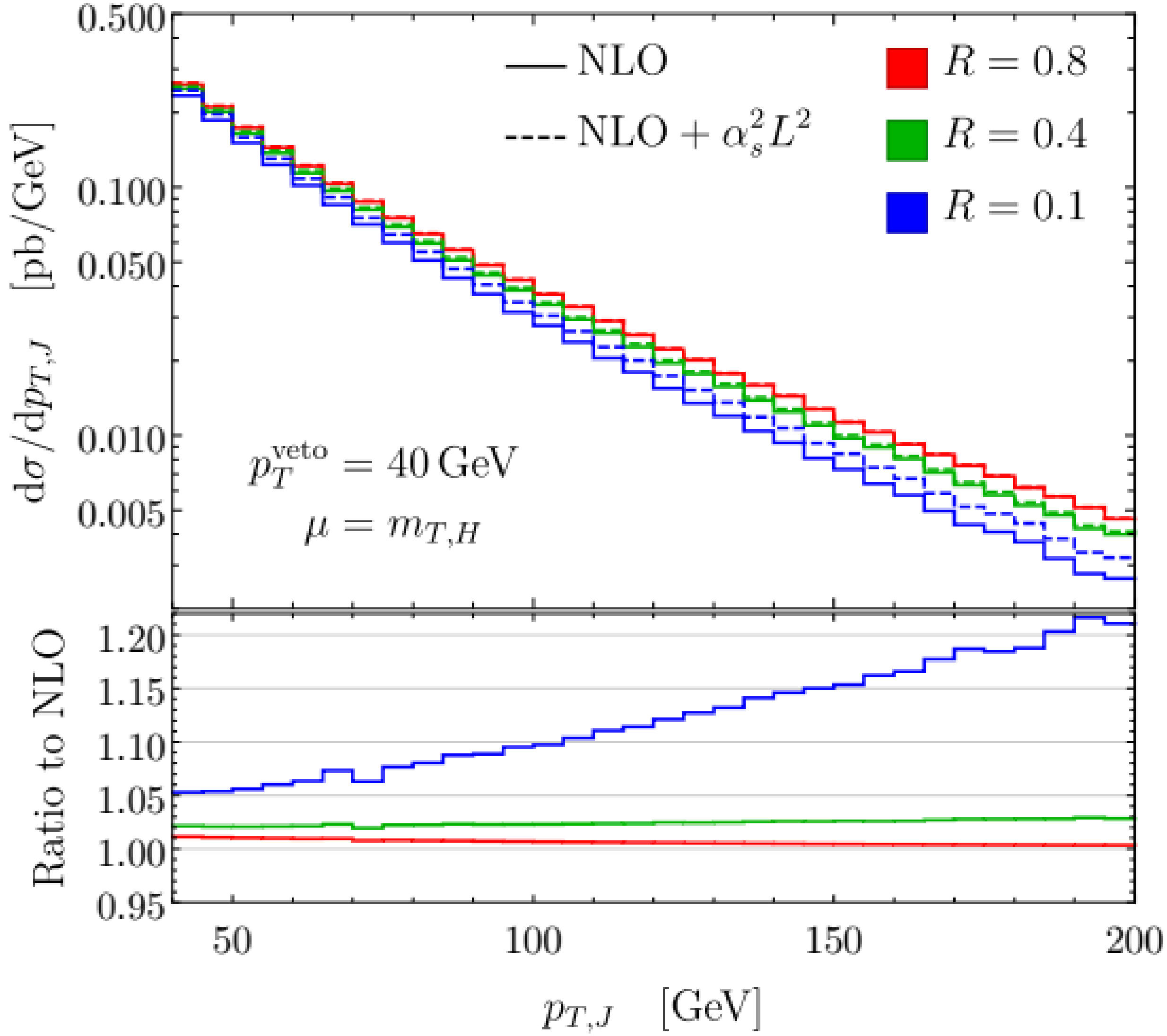
Effect of NNLO LL terms?



Effect of $\ln^2 R$ term is minimal for $R = \{0.4, 0.8\}$

Only for more extreme $R = 0.1$ do we see large effects

Effect of NNLO LL terms?



Effect of $\ln^2 R$ term is minimal for $R = \{0.4, 0.8\}$

Only for more extreme $R = 0.1$ do we see large effects

$\alpha_s^2 \ln^2 R$ terms are a larger fraction of the NLO predictions at high $p_{T,J}$. Effect greater at smaller R

Conclusions

- Explored the $R \ll 1$ limit and impact of $\alpha_s \ln R$ terms
- Introduced jet functions describing leading & subleading jet
- Parton shower approach to construct non-linear RGE structure
- Obtained analytic perturbative solutions for (sub)leading jet functions at LL accuracy
  - Well approximated in soft limit, but not by leading $\ln R$
- Examined the impact of these jet functions on exclusive $pp \to H + J$ production
  - Collinear approximation works well (even for `large R')
  - Corrections from hard scattering essential

Backup: Full NNLO LL leading-jet function

$$J_{l,q}^{(2)\mathrm{LL}}(z_l,\mu) = \ln^2\left(\frac{\mu}{p_T R}\right)\left\{\Theta\left(z_l - \frac{1}{2}\right)\frac{\beta_0}{4}\left[P_{qq}(z_l) + P_{gq}(z_l)\right]\right.$$
$$+\,\Theta\left(z_l - \frac{1}{2}\right)\left(C_F^2 A_{q,1} + C_F C_A A_{q,2} + C_F n_f T_F A_{q,3}\right)$$
$$\left.+\,\Theta\left(\frac{1}{2} - z_l\right)\Theta\left(z_l - \frac{1}{3}\right)\left(C_F^2 B_{q,1} + C_F C_A B_{q,2} + C_F n_f T_F B_{q,3}\right)\right\}$$

$$A_{q,1} = 4\left[\frac{\ln(1-z_l)}{1-z_l}\right]_+ + \frac{3}{[1-z_l]_+} + \left(\frac{9}{8} - \frac{\pi^2}{3}\right)\delta(1-z_l) + \frac{\left(1 - 2z_l^2 - 3z_l\right)}{2(1-z_l)}\ln z_l$$
$$- \left(z_l - \frac{2}{z_l} + 4\right)\ln(1-z_l) - \frac{3}{2}\left(1 + \frac{z_l}{2}\right)$$

$$A_{q,2} = -\frac{2\left(z_l^2 + z_l + 1\right)}{z_l}\ln z_l + \left(z_l + \frac{2}{z_l} - 2\right)\ln(1-z_l) + \frac{8z_l^3 + 17z_l^2 + 26z_l - 40}{12 z_l}$$

$$A_{q,3} = \frac{2 + 5z_l - 4z_l^2 - 4z_l^3}{3z_l} + 2(1+z_l)\ln(z_l)$$

Backup: Full NNLO LL leading-jet function

$$
\begin{aligned}
J_{l,q}^{(2)\mathrm{LL}}(z_l,\mu) = \ln^2\left(\frac{\mu}{p_T R}\right)\Bigg\{ &\Theta\left(z_l - \frac{1}{2}\right)\frac{\beta_0}{4}\left[P_{qq}(z_l) + P_{gq}(z_l)\right] \\
&+ \Theta\left(z_l - \frac{1}{2}\right)\left(C_F^2 A_{q,1} + C_F C_A A_{q,2} + C_F n_f T_F A_{q,3}\right) \\
&+ \Theta\left(\frac{1}{2} - z_l\right)\Theta\left(z_l - \frac{1}{3}\right)\left(C_F^2 B_{q,1} + C_F C_A B_{q,2} + C_F n_f T_F B_{q,3}\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
B_{q,1} =& -\frac{2\left(3z_l^2 - 3z_l + 2\right)}{z_l(1-z_l)}\ln(1-2z_l) + \frac{\left(15z_l^2 - 15z_l + 8\right)}{2z_l(1-z_l)}\ln z_l + \frac{3}{2}\ln\left(\frac{1-z_l}{2}\right) \\
&+ \frac{3(1-3z_l)(2-z_l)^2}{4z_l(1-z_l)} \\
B_{q,2} =& \frac{3z_l - 3z_l^2 - 2}{z_l(1-z_l)}\ln(1-2z_l) + \frac{-z_l^3 + z_l + 2}{z_l(1-z_l)}\ln z_l - (z_l+4)\ln\left(\frac{1-z_l}{2}\right) \\
&+ \frac{-45z_l^6 + 96z_l^5 + 30z_l^4 - 352z_l^3 + 459z_l^2 - 236z_l + 40}{12(1-z_l)^4 z_l} \\
B_{q,3} =& 2(1+z_l)\ln\left(\frac{1-z_l}{2z_l}\right) + \frac{18z_l^6 - 87z_l^5 + 186z_l^4 - 152z_l^3 + 36z_l^2 + 11z_l - 4}{6z_l(1-z_l)^4}
\end{aligned}
$$