Techniques for solving two-loop massive Feynman integrals

Ekta Chaubey

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IPPP
Durham University
United Kingdom

Outline

Introduction

Precision physics

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2. Precise theoretical computations very crucial.

$\Longleftrightarrow$

From cross sections to Feynman integrals

Matching of observables like cross sections, calculated perturbatively in QFT.

Precision target:

1. Experimental uncertainties are reaching few % level.
2. Matching this level of precision requires computation of NNLO QCD and mixed QCD-EW corrections.

$$\text{Feynman diagrams} \xrightarrow{\text{Feynman rules}} \text{scattering amplitude} \longrightarrow \text{observables.}$$

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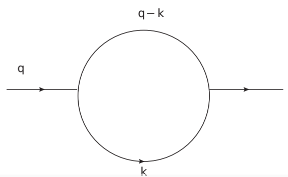
Two-loop (virtual) corrections to processes with massive particles, like top quark and electroweak bosons very important.

Mnemonics



$$\Longrightarrow \qquad I = \int \frac{d^D k}{(2\pi)^D} \, \frac{1}{(-k^2+m_1^2)\,(-(q-k)^2+m_2^2)}$$

One-loop Feynman integral

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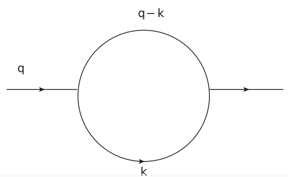

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One-loop Feynman integral

Generalize to multi-loop:

$$I = \int \prod_{r=1}^{l} \frac{d^D k_r}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{\underbrace{(-q_j^2+m_j^2)^{\nu_j}}_{\text{Massive propagators (P)}}}$$

Multi-loop Feynman integral

Big deal?

1. The point: Feynman integrals becomes more and more difficult with growing loops and legs, as well as with the inclusion of masses.
2. Using the available tools to calculation 2-loop 'massive' Feynman integrals is a difficult task.

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Mathematical motivation

1. Helps solving a particular integral efficiently along with aiding multi-scale generalizations.
2. Algebraic structure of Feynman integrals has proven to be a great help, massive Feynman integrals even often contain elliptic curve(s).

Mathematical preliminaries

Multiple Polylogarithms (MPLs)

1. Logarithm:

$$Li_1(x) = -\ln(1-x) = \sum_{i=1}^{\infty} \frac{x^i}{i}.$$

2. Generalizing to classical polylogarithm:

$$Li_n(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^n}.$$

3. Further generalization brings us to the MPLs:

$$Li_{n_1,...,n_k}(x_1,...,x_k) = \sum_{i_1>i_2>...>i_k>0} \frac{x_1^{i_1}}{i_1^{n_1}} \cdots \frac{x_k^{i_k}}{i_k^{n_k}}.$$

Multiple Polylogarithms (MPLs)

1. Integral representation: for $z_k \neq 0$,

$$G(z_1, ..., z_k; y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} ... \int_0^{t_k - 1} \frac{dt_k}{t_k - z_k}.$$

2. Relation between both. Introducing a short hand notation:

$$G_{m_1,...,m_k}(z_1, ..., z_k; y) = G(\underbrace{0, ..., 0}_{m_1 - 1}, z_1, ..., z_{k-1}, \underbrace{0, ..., 0}_{m_k - 1}, z_k; y).$$

$$Li_{m_1,...,m_k}(x_1, ..., x_k) = (-1)^k G_{m_1,...,m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, ..., \frac{1}{x_1 ... x_k}; 1\right).$$

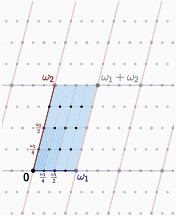
Elliptic curves

With lattice

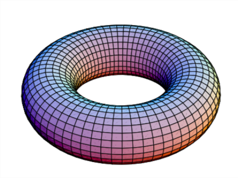
$$L = \{m\omega_1 + n\omega_2 | m, n \in Z\},$$

we can define a meromorphic function f such that

$$f(z + \omega_i) = f(z); \quad z \in \mathbb{C}.$$



$$\overset{\mathbb{C}/L}{\Longrightarrow}$$



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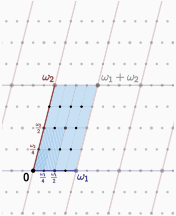

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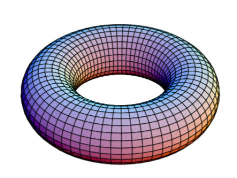

Elliptic curve defined by:

$$E : \omega^2 - (z - z_1)(z - z_2)(z - z_3)(z - z_4) \ = \ 0.$$

Machinery

Differential equations

1. Tool to tackle these Feynman integrals: Differential equations (DE) [Kotikov '90], [Remiddi '97], [Gehrmann and Remiddi '99].

2. Integration by parts identities (IBP) [Tkachov '81], [Chetyrkin '81].

$$\int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_l}{(2\pi)^D} \frac{\partial}{\partial k_i^\mu} v^\mu \prod_{j=1}^{n} \frac{1}{(q_j^2 - m_j^2)^{\nu_j}} = 0.$$

3. Integral with general integer powers related to a finite set of 'Master integrals' (MI).

4. Laporta algorithm and computer implementations [Laporta '01], [REDUZE, FIRE, KIRA]

5. Computing these MI corresponds to computing the family of Feynman integral.

Aim:

Interested in the Laurent expansion of these integrals in ϵ, where $\epsilon = (4 - D)/2$ is the dimensional regularisation parameter.

Setting up the differential equations

1. Consider a Feynman integral with N MI.
2. Let t be an external invariant (or an internal mass) and let $I_i \in \{I_1, ..., I_N\}$ be a MI.

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4. Repeating this for every MI and every kinematic variable we get a system of DE

$$d\vec{I} = A\vec{I},$$

where A is a matrix-valued one-form $A = \sum_{i=1}^{N} A_i dx_i$.

Canonical form for the differential equations

The system of DE is simple if we find canonical or 'epsilon' form [J. Henn, '14],

$$d\vec{J} \; = \; \epsilon A' \vec{J}, \; A' = \sum_{k=1}^{N_L} C_k \omega_k$$

where

1. C_k has only rational or integer entries.
2. ϵ completely factorizes.
3. differential forms ω_k have only simple poles.

When this happens the system of DE is easily solved in terms of MPLs.

Transformations

1. Change the basis of the MIs $\vec{J} = U\vec{I}$, so the DE becomes

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2. Perform a coordinate transformation (particularly useful in case of square roots). As an example, we often encounter
$$\frac{dx}{\sqrt{-x(4-x)}}$$
Here, a change of variable as
$$x = -\frac{(1-x')^2}{x'}$$
will rationalize the square root and transform
$$\frac{dx}{\sqrt{-x(4-x)}} = \frac{dx'}{x'}.$$

Transformations in case of MPLs

1. For cases where rational transformation is sufficient : several algorithms exist [R.N. Lee, '14], [C. Meyer '18] [Lee, Pomeransky '17].

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1. For cases where rational transformation is sufficient : several algorithms exist [R.N. Lee, '14], [C. Meyer '18] [Lee, Pomeransky '17].
2. For algebraic cases (involving roots): not many transformations well known. Algorithms to rationalize square roots. [Becchetti, Bonciani '17] [Besier, van Straten, Weinzierl '18]

When canonical form is not possible!

1. Problem for elliptic cases: DE coupled at order ϵ^0, cannot be removed away.
2. To the rescue: Picard–Fuchs equation. One way of trying to bring down the coupled system of equations into blocks of sizes 2×2 at worst.
3. For elliptic cases, new incarnation of canonical form. One such algorithm

from rational functions in kinematic variables

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rational functions in the kinematic variables,
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4. Some elliptic integrals can be expressed as iterated integrals using modular forms [L. Adams, S. Weinzierl, '17].

5. An important tool: the maximal cut.

Maximal cuts

Simultaneous cutting of all the propagators.

$$I_{\nu_1\nu_2\ldots\nu_n}(D) = \left(\mu^2\right)^{\nu - lD/2} \int \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_l}{(2\pi)^D} \prod_{j=1}^{n} \frac{1}{P_j^{\nu_j}}$$

Maximal cut mathematically means taking the $n-$fold residue at

$$P_1 = \ldots = P_n = 0$$

of the integrand in the complex plane.

Anything special?

- Maximal cuts are solutions of the homogeneous DE.
- For integrals evaluating to MPLs, maximal cut is an algebraic function.
- For elliptic integrals they contain transcendental functions.

An example

1. For elliptic cases $\rightarrow$ search for Feynman integrals whose maximal cuts are periods of an elliptic curve.

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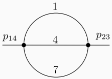
2. For sunrise: 

$$MaxCut_{\mathcal{C}}\ I_{1001001}(2-2\epsilon)=\frac{\mu m^2}{\pi^2}\int_{\mathcal{C}}\frac{dP}{\sqrt{(P-t)}\sqrt{(P-t+4m^2)}\sqrt{(P^2+2m^2P-4m^2t+m^4)}}+O(\epsilon).$$

3. To get the elliptic curve, we aim for an integral representation having a square root of a quartic polynomial in the denominator along with a constant in the numerator.

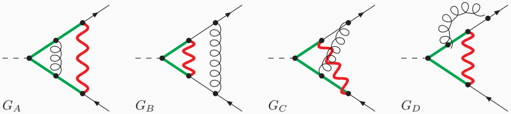
Choice of coordinate system

1. Choosing good coordinates is one of the key tricks to solve the DE.
2. Need of a system of DE in which all occurring square roots are rationalized (not possible for elliptic cases).

$$\frac{d}{ds}\begin{bmatrix} I_1 \\ . \\ . \\ . \\ I_n \end{bmatrix} = \begin{bmatrix} \sqrt{s+t+4m^2}\,ds \ldots . \\ \ldots .. \\ \ldots . \sqrt{s+t-4m^2}\,ds \\ \ldots .. \\ \ldots .. \end{bmatrix} \begin{bmatrix} I_1 \\ . \\ . \\ . \\ I_n \end{bmatrix}$$

Higgs decay

H $\rightarrow b\bar{b}$



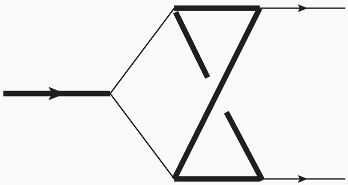
Physical relevance:

1. H $\rightarrow b\bar{b}$ has highest branching ratio.
2. Experimentally observed at ATLAS and CMS recently.

Topology	Number of master integrals
A	18
B	15
C	31
D	14

In total we get 39 MIs.

Elliptic hint



- [Manteuffel, Tancredi, '17] an example of a non planar 2-loop three point function.
- Contributes to 2-loop amplitudes for $t\bar{t}$ production and $\gamma\gamma$ production in gluon fusion.
- Differs from topology C only in external momenta putting $m_W = m_t$.
- Contains an elliptic curve in the top topology.

Setting up the DEs

1. Setting $\mu^2 = m_t^2$, MI depend kinematically on 2 dimensionless quantities.
2. Naive (bad) choice:

$$v = \frac{p^2}{m_t^2}, \qquad w = \frac{m_W^2}{m_t^2},$$

3. We encounter the square roots

$$\sqrt{-v\,(4-v)} \qquad \text{and} \qquad \sqrt{\lambda\,(v,w,1)}$$

   where Källen function is defined by

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx.$$

4. Problem: Rationalizing both square roots simultaneously.

Rationalizing square roots

- We found the parametization [Besier, Van Straten, Weinzierl, '18] :

$$\frac{p^2}{m_t^2} = v = -\frac{(1-x)^2}{x}, \quad \frac{m_W^2}{m_t^2} = w = \frac{(1-y+2xy)(x-2y+xy)}{x(1-y^2)}.$$

that simultaneously rationalizes the two square roots.

So our system is free of any elliptic curves! All the master integrals are expressible in terms of MPLs [**E. Chaubey**, S. Weinzierl, '19].

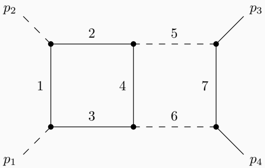
Topbox

Physical relevance

- Top-quark pair production one of the most important topics at LHC.
- Important source of irreducible background to many SM measurements and BSM searches.
- Top-pair production can be used to determine top-quark mass and α_s $\Longrightarrow$ highly important to have a precise understanding of this process.

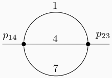
The Topbox

- Solid lines $\rightarrow$ massive propagators,
  all external momenta $\rightarrow$ outgoing and on-shell.
  $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$.
- 44 MI.

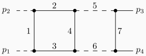


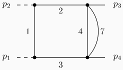
Elliptic Curve a

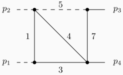
$$E^{(a)} : w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4\right),$$



Elliptic Curve b

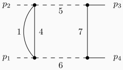






$$E^{(b)} : w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+2m^2z-4m^2t+m^4-\frac{4m^2\left(m^2-t\right)^2}{s}\right),$$

Elliptic Curve c



$$E^{(c)} : w^2 = (z-t)\left(z-t+4m^2\right)\left(z^2+\frac{2m^2(s+4t)}{(s-4m^2)}z+\frac{sm^2\left(m^2-4t\right)-4m^2t^2}{s-4m^2}\right).$$

The Picard–Fuchs operator

- Not expressible in terms of MPLs (elliptic generalisations required).
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- Several MI coupled together at order ϵ^0 in one topology by DE.
- Idea $\rightarrow$ a system of first-order DE easily converted to a higher order DE for a single MI in this sector.
- **Aim** $\rightarrow$ try to transform to a suitable basis of MI which decouples the original system of DE at order ϵ^0 to a system of maximal block size of 2.
- Can be done by exploiting the factorisation properties of the Picard-Fuchs operator [L. Adams, **E. Chaubey**, S. Weinzierl, '17].

Main results from Topbox:

1. Analytic results for the planar double box calculated [L. Adams, **E. Chaubey**, S. Weinzierl, '18].

2. Presence of 3 distinct elliptic curves (for the first time!).

3. Two special points for this system of DE. It simplifies
   - for $t = m^2 \rightarrow$, MI expressible in terms of MPLs
   - as well as for $s = \infty$, MI expressible in terms of iterated integrals of modular forms.

Conclusions

Conclusions and Outlook

1. Higher order loop calculations crucial to make precise theoretical predictions.

2. For mostly massless processes, virtual corrections expressible in terms of MPLs. Starting from 2-loops, MPLs not sufficient.

3. Outlined the computation of MI using DE for :
   - mixed QCD-EW corrections for Higgs decay with a $Ht\bar{t}$ coupling,
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