Multiparton webs at 4-loops

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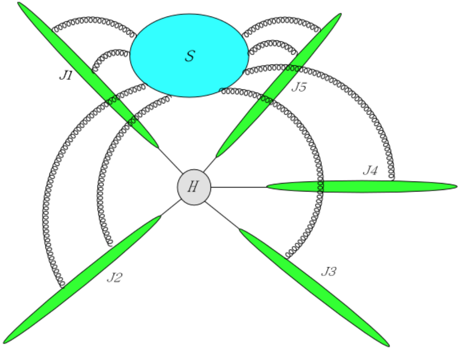
Outline of the talk

- Introduction
- Soft function and Webs
- Multiparton webs at 4-loops
- Summary

Introduction

Factorization of Multileg amplitudes



ref: Mueller (81), Sen (83), Botts Sterman (89), Kidonakis Oderda Sterman (98), Catani (98), Tejeda-Yeomans Sterman (02), Kosower (03), Aybat Dixon Sterman (06), Becher Neubert (09), Gardi Magnea (09)

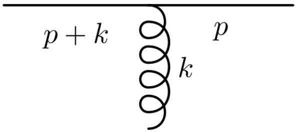
Continued..

The soft singularities can be factorized from the hard part of the scattering amplitudes as-

$$\mathcal{M}_L\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \sum_K S_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon)$$
$$\times H_K\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_i \frac{J_i\left(\frac{(2p_i \cdot n_i)^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}.$$

Hard function H is finite after UV renormalization.

Soft approximation and Wilson lines



$$M^{\mu} = ig_s T^a \frac{p^{\mu}}{2(p \cdot k)} = ig_s T^a \frac{\beta^{\mu}}{2(\beta \cdot k)}$$

A Wilson line is defined as

$$\phi_n(\lambda_2, \lambda_1) = Pexp[\int_{\lambda_1}^{\lambda_2} d\lambda n \cdot A(\lambda n)]$$

$$S =< 0 \mid \phi_{\beta_2}(\infty, 0)\phi_{\beta_1}(0, -\infty)|0 >= g_s^2 \frac{\beta_1 \cdot \beta_2}{(\beta_1 \cdot k)(\beta_2 \cdot k)} + \dots$$

Why study infrared singularities

Dimensional regularization = finite! with large logs

- These logs have large values and can disturb the convergence of expansion in α. We need to do resummation.

- Knowing the IR singularities at all orders, resummation is easy.

Soft function evolution

$$S_{LK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) \to \frac{S_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon)}{\prod_i \mathcal{J}_i(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon)}.$$

$$\mu \frac{d}{d\mu} S_{IK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) = -\Gamma^S_{IJ}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) \times S_{JK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon)$$

Solution:

$$S_{IJ}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) = Pexp\left\{ -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \Gamma^S_{IJ}(\rho_{ij}, \alpha(\lambda^2, \epsilon)) \right\}$$

- Γ^S is finite soft anomalous dimension.

$$\Gamma^s = \sum_{n=1}^{\infty} \Gamma^{(n)} \alpha_s^n$$

Dipole formula

$$\Gamma^S(\rho_{ij}, \alpha_s) \;=\; -\frac{1}{8}\hat{\gamma}_K(\alpha_s)\sum_{i=1}^{n}\sum_{j,j\neq i} ln(\rho_{ij})\sum_a T_i^a T_j^a + \frac{1}{2}\hat{\delta}_{\overline{S}}\sum_{i=1}^{n}\sum_a T_i^a T_i^a$$

Quadratic Casimir:

$$C_i = \sum_a T_i^{(a)} T_i^{(a)}$$

$$\gamma_k^{(i)} = C_i \widehat{\gamma}_k(\alpha_s)$$

This is a result true upto two loops.

ref: Gardi Magnea (09), Becher Neubert (09)

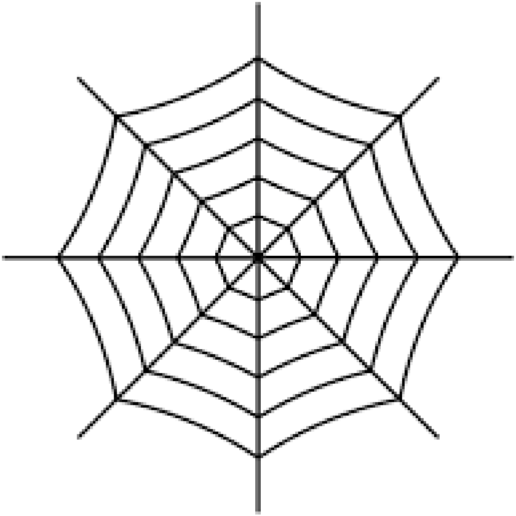
Beyond Dipole formula

Correction to the dipole formula can arise from 2 sources:

- Conformally invariant cross-ratios, which can appear starting at 3 loops.[ref: Gardi et. al. 2015]
- Violations of 'Casimir Scaling' in the cusp anomalous dimension which starts appearing at 4-loops. [ref: Henn, Grozin (18)]

Soft function and Webs

Spider Webs



Diagrammatic representation of Soft function

$$S = exp(W)$$

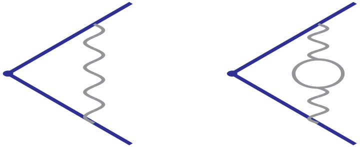
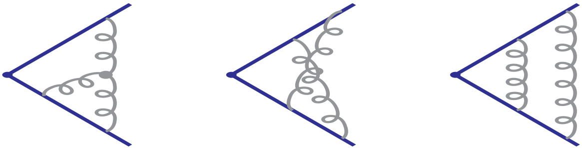


Figure: Webs in abelian theory

In abelian theory only connected diagrams appear in webs.

[ref: Gatheral, Frenkel Taylor (60)]

Webs in non-abelian theory



- It was observed that only the first two diagrams contribute to the exponent, first with same color factor and the second one with modified color factors.
- The color of the 3rd diagram is absent in the exponent as it can be reduced by cutting the Wilson line.

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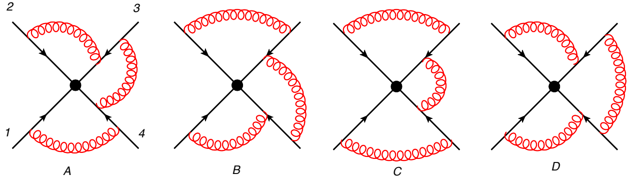
- In multiparton case, the concept of webs generalizes non-trivially.
- A web in the multiparton case is a set of diagrams which differ only by the order of the gluon attachment on each Wilson line.



- If a diagram is $D = F(D)C(D)$ a Web W is expressed as a sum of diagrams in terms of mixing matrix R.

$$W = \sum_{D} F(D)\tilde{C}(D) = \sum_{D,D'} F(D)R_{DD'}C(D)$$

[ref: Gardi et. al. (10)]

Recent progress

- All exponentiated color factors and kinematics at 3-loops order are calculated.
- Color conservation technique $\sum_i T_i^a = 0$ is used to the exponentiated color factors at 3-loops to determine the 3-loop soft anomalous dimension. [ref: Almelid et. al. (15)]

Correction for Γ^S at 3 loops and 4 legs is calculated recently by E.Gardi and O.Almelid

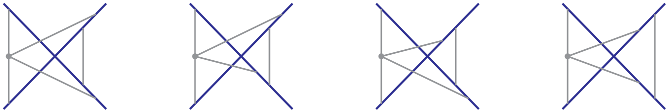
$$\Delta^{(3)} = 16 f_{abe} f_{cde} \Big\{ \sum_{1<i<j<k<l<n} \Big[\mathbf{T}_i^a \mathbf{T}_k^b \mathbf{T}_k^c \mathbf{T}_l^d F(\rho_{ijkl}, \rho_{iljk}) + \mathbf{T}_i^a \mathbf{T}_k^b \mathbf{T}_j^c \mathbf{T}_l^d F(\rho_{ijkl}, \rho_{ilkj}) + \mathbf{T}_i^a \mathbf{T}_l^b \mathbf{T}_j^c \mathbf{T}_k^d F(\rho_{ijlk}, \rho_{iklj}) - C \sum_{i=1}^{n} \sum_{1\leq j<k\leq n; j,k\neq i} \{\mathbf{T}_i^a, \mathbf{T}_i^d\} \mathbf{T}_j^b \mathbf{T}_k^c$$

Multiparton webs at 4-loops

Challenges at 4-loop

- At 4-loop we have 20 webs at 4-legs.
- The largest dimension of the mixing matrix for the web is 24×24
- Results available for 3-loops has largest dimension of mixing matrix as 16×16

4-loop webs



C_1 C_2 C_3 C_4

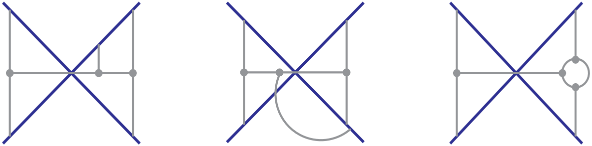
$S = (1, 0, 0, 1)$

$$R = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad Y = \begin{pmatrix} -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad r = 3$$

$$\begin{aligned} W &= F^T R C \\ &= \sum_{H=1}^{r} (F^T Y^{-1})_H (YC)_H \end{aligned}$$

Continued..

$$
\begin{aligned}
\tilde{C}_1 &= if^{abg} f^{cdg} f^{edh} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^e \mathbf{T}_3^c \mathbf{T}_4^h - if^{abg} f^{cdg} f^{cej} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^j \mathbf{T}_4^d \mathbf{T}_4^e \\
\tilde{C}_2 &= -if^{abg} f^{cdg} f^{cej} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^j \mathbf{T}_4^d \mathbf{T}_4^e \\
\tilde{C}_3 &= if^{abg} f^{cdg} f^{edh} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^e \mathbf{T}_3^c \mathbf{T}_4^h - f^{abg} f^{cdg} f^{cej} f^{edh} \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^j \mathbf{T}_4^h
\end{aligned}
$$



(h) (i) (j)

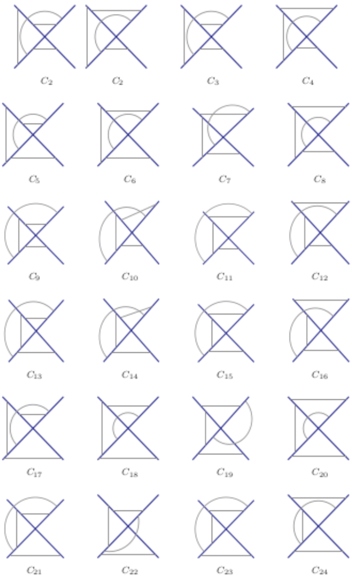
Connected color factors appearing in the exponent of soft function.

[ref: Agarwal, Danish, Magnea , SP, Tripathi (Under preparation)]

Properties of the web mixing matrix R

- Idempotence: $R^2 = R$, eigenvalues 1 or 0.
- Zero-sum rows[ref: Gardi et. al. (10)]
- Conjecture:$\sum_D c(D)s(D) = 0$ [ref: Gardi et. al. (11)]

Appearance of Quartic Casimir



$$R = \begin{pmatrix}
\frac{1}{6} & -\frac{1}{12} & 0 & -\frac{1}{12} & -\frac{1}{12} & 0 & 0 & \frac{1}{12} & 0 & 0 & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & 0 & 0 & \frac{1}{12} & -\frac{1}{12} & \frac{1}{12} & 0 & 0 & \frac{1}{12} & 0 & \frac{1}{12} & -\frac{1}{6} \\
-\frac{1}{6} & \frac{1}{4} & 0 & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{6} & 0 & \frac{1}{12} & 0 & 0 & \frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & 0 & \frac{1}{6} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{12} & 0 & 0 & \frac{1}{12} & 0 & -\frac{1}{4} & \frac{1}{6} \\
-\frac{1}{3} & \frac{1}{12} & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{12} & \frac{1}{6} & 0 & -\frac{1}{12} & \frac{1}{6} & 0 & -\frac{1}{4} & \frac{1}{12} & \frac{1}{12} & 0 & -\frac{1}{6} & \frac{1}{12} & \frac{5}{12} & -\frac{1}{4} & -\frac{1}{2} & \frac{1}{3} & -\frac{1}{4} & 0 & \frac{5}{12} & -\frac{1}{6} \\
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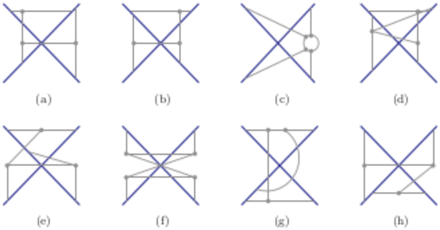
\end{pmatrix}$$

(a)

Figure: Mixing matrix of one of the largest web at 4-loops



(a)

Figure: All the compleletly connected factor arise from the web

ref: Agarwal, Danish, Magnea , SP, Tripathi (Under preparation)

General color structures

Figure: Connected pieces at 4-loops

Summary of results

Summary

- We have developed an in-house mathematica code which can do the following
  - It can generate all the webs at 4-loops connecting 4 and 5 lines.
  - It can generate all the diagrams of the webs.
  - It can calculate the mixing matrix of the webs for 4 and 5 lines.
- We checked correctness of the code by checking the following properties:
  - All the mixing matrices we generated have zero-sum rows
  - All the mixing matrices we generated are idempotent

Continued..

- Claim about column sum conjuctre: We claim that the conjecture for column sum rule is true at 4-loops web mixing matrices.
- The Quartic Casimir which was expected to be found at 4-loops appears in our result, which agrees with Becher and Neubert 2019.

Outlook

- The next step in order to calculate the 4-loop soft anomalous dimension is to apply color conservation to all the exponentiated color factors.

- We need to determine the kinematic contribution of only 2 diagrams that are present in the exponent.

- We can calculate the exponentiated color factors present at higher orders using our in house code.

Thank You

Backup slides

Jet functions

- Partonic jet : $J_i(\frac{(2p_i \cdot n_i)^2}{\mu^2}, \alpha_s(\mu^2), \epsilon) =< p|\bar{\psi}(0)\phi_{\beta_1}(0,-\infty)|0 >$
- Eikonal jet : $\mathcal{J}_i(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon) =< 0 \mid \phi_\beta(\infty,0)\phi_n(0,-\infty)|0 >$

Relation between Γ^{S} and $\Gamma^{\overline{S}}$

$$\Gamma^{\overline{S}}_{IJ} = \Gamma^{S}_{IJ} - \delta_{IJ} \sum_{k=1}^{n} \gamma_{\mathcal{J}_k}$$

RG evolution equation

In d dimensions

$$\mu\frac{d\alpha}{d\mu} = \beta$$
$$\frac{\mu}{m_R}\frac{dm_R}{d\mu} = \gamma$$

Reduced soft function

$$\overline{S}_{LK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) = \frac{S_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon)}{\prod_i \mathcal{J}_i(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon)}.$$

$$\rho_{ij} = \frac{n_i^2 n_j^2 (\beta_i \cdot \beta_j)^2}{4(\beta_i \cdot n_i)^2(\beta_j \cdot n_j)^2}$$

RG equation of different quantities

$$
\begin{aligned}
\mu\frac{d}{d\mu} lnJ_i\left(\frac{(2p_i\cdot n_i)^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) &= -\gamma_{J_i}\\
\mu\frac{d}{d\mu} ln\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\alpha_s(\mu^2),\epsilon\right) &= -\gamma_{\mathcal{J}_i}\\
\mu\frac{d}{d\mu} S_{IK}(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon) &= -\Gamma^{S}_{IJ}(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon)\\
&\quad\times S_{JK}(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon)\\
\mu\frac{d}{d\mu} \overline{S}_{IK}(\rho_{ij},\alpha_s(\mu^2),\epsilon) &= -\Gamma^{\overline{S}}_{IJ}(\rho_{ij},\alpha_s(\mu^2),\epsilon)\\
&\quad\times \overline{S}_{JK}(\rho_{ij},\alpha_s(\mu^2),\epsilon)
\end{aligned}
$$

Factorization in terms of reduced soft function

$$\mathcal{M}_L\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \sum_K S_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon)$$
$$\times H_K\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_i \frac{J_i(\frac{(2p_i \cdot n_i)^2}{\mu^2}, \alpha_s(\mu^2), \epsilon)}{\mathcal{J}_i(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon)}.$$
$$= \overline{S}_{LK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon)$$
$$\times H_K\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right)$$
$$\times J_i(\frac{(2p_i \cdot n_i)^2}{\mu^2}, \alpha_s(\mu^2), \epsilon)$$

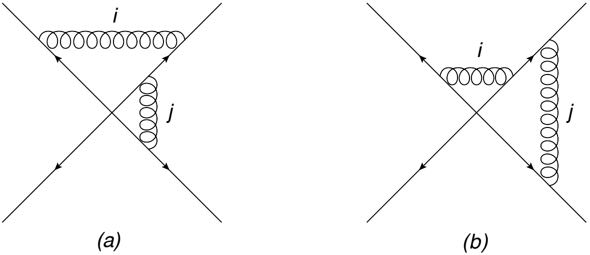
Replica trick Algorithm

- Identify distinct connected pieces in the diagrams. Associate a replica variable to each connected piece which can take value from 1 to N.
- Determine the hierarchy between the replica variables, for example if there are two connected pieces then a hierearchy determines weather i¿j, i=j, i¡j.
- Define an operator R which will arraneg the gluons on a prticular leg according to the hierarchy. Given two attachments i and j to a leg l, the $\mathcal{R}$ operation
  - does nothing if i=j
  - will order them as $T_i T_j$ if i<j and as $T_j T_i$ if i >j.

Continued..

- For each hieracrchy compute its multiplicity, namely a combinational factor counting how many times this hierarchy arise when all i 's go over the range 1 to N. It is equal to the number of ways of choosing n(h) different replica numbers from N. $M_N(h) = NCn(h)$.
- $\tilde{C}(D) = \sum_h M_n(h) R[C(D)|h]$ at order N^1

Calculation of R using replica trick



h	$\mathcal{R}[(a)\|h]$	$M_N(h)$	$\mathcal{O}(N^1)$ part of $M_N(h)$
$i=j$	$C(a)$	N	1
$i>j$	$C(a)$	$N(N-1)/2$	$-\frac{1}{2}$
$i<j$	$C(b)$	$N(N-1)/2$	$-\frac{1}{2}$

$$\widetilde{C}(a) = \frac{1}{2}\left[C(a) - C(b)\right]$$

$$\widetilde{C}(b) = \frac{1}{2}\left[C(b) - C(a)\right]$$

$$\rho_{ijkl} = \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)}$$