Quark masses from lattice QCD

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Ultimate Precision at Hadron Colliders
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• Quark masses – fundamental parameters of the Standard Model.

• Many applications to phenomenology and BSM physics. Example: Higgs partial widths.
  - Couplings proportional to quark masses.
  - Main source of uncertainty in partial widths from \( m_b, m_c, \alpha_s \) \[1404.0319\].

• Focus on precision results using three independent methods.
Higgs couplings

Estimated final ILC precision in $h c \bar{c}$ coupling $\sim 0.7\%$. 
Outline

• Background
  ▶ Lattice simulations
  ▶ Mass determinations

• Quark mass methods
  ▶ Current-current correlator moments
  ▶ Regularisation Invariant (RI) methods
  ▶ Minimal renormalon subtraction (MRS) masses

• Summary & Outlook
Bare quark masses are input parameters to lattice simulations. These parameters are tuned to reproduce physical quantities, e.g.

- $m_{ud0} \to m_{\pi}^2$
- $m_{s0} \to m_{K}^2$
- $m_{c0} \to m_{\eta_c}$

Tuning performed at multiple lattice spacings, defining a continuum trajectory for which $a^2 \to 0$ limit can be taken.

- Rest of physics is then prediction of QCD.
- Parameters can be varied away from physical values to understand effect of quark mass, quantify systematics, etc.
Meson masses – summary plot
MILC ensembles

- HISQ fermion action.
  - Discretization errors begin at $O(\alpha_s a^2)$.
  - Designed for simulating heavy quarks ($m_c$ and higher at current lattice spacings).

- Symanzik-improved gauge action, takes into account $O(N_f \alpha_s a^2)$ effects of HISQ quarks in sea. [0812.0503]

- Multiple lattice spacings down to $\sim 0.045$ (now 0.03) fm.

- Effects of $u/d$, $s$, and $c$ quarks in the sea.

- Multiple light-quark input parameters down to physical pion mass.
  - Chiral fits.
  - Reduce statistical errors.
MILC ensemble parameters

\[ a^2 \approx (\text{fm}^2) \]

\[ M_\pi \approx 135 \text{ MeV} \]

\[ 1712.09262 \]

\[ (0.03)^2 \]

\[ (0.042)^2 \]

\[ (0.06)^2 \]

\[ (0.09)^2 \]

\[ (0.12)^2 \]

\[ (0.15)^2 \]
Quark mass definitions

- Quarks are not asymptotic (physical) states due to confinement – mass cannot be measured directly.
- Quark masses are scheme and scale dependent, $m_q^{\text{scheme}}(\mu)$.
- Generally will quote results $m_q^{\overline{\text{MS}}}(\mu_{\text{ref}})$.
- Lattice input quark masses are non-universal (depend on discretisation), but can be connected to quark masses defined in a continuum scheme.
$\langle JJ \rangle$-correlator moments
Current-current correlators

Calculate time-moments of \( J_5 \equiv \bar{\psi}_h \gamma_5 \psi_h \) correlators:

\[
G(t) = a^6 \sum_x (am_{0h})^2 \langle J_5(t, x)J_5(0, 0) \rangle
\]

- Currents are absolutely normalized (no Zs required).
- \( G(t) \) is UV finite \( \rightarrow G(t)_{\text{cont}} = G(t)_{\text{latt}} + \mathcal{O}(a^2) \).
The time-moments $G_n = \sum_t (t/a)^n G(t)$ can be computed in perturbation theory. For $n \geq 4$,

$$G_n = \frac{g_n(\alpha_{\overline{MS}}, \mu)}{a m_h(\mu)^{n-4}}.$$ 

Basic strategy:

1. Calculate $G_{n,\text{latt}}$ for a variety of lattice spacings and $m_{h0}$.

2. Compare continuum limit $G_{n,\text{cont}}$ with $G_{n,\text{pert}}$ (at reference scale $\mu = m_h$, say).

3. Determine best-fit values for $\alpha_{\overline{MS}}(m_h), m_h(m_h)$. 

Results for $n_f = 4$  

\[ m_c(3m_h) = \frac{r_n(\alpha_{\overline{MS}}, \mu = 3m_h)}{R_n} \]

- Discretization effects grow with $am_h$ and decrease with $n$.
- Grey band shows best-fit $m_c(3m_c)$ evolved perturbatively.

\[ m_c^{\overline{MS}}(3 \text{ GeV}) = 0.9851(63) \text{ GeV} \]
RI intermediate schemes
NPR method

Trying to determine $Z_{m}^{\text{MS}}(\mu, 1/a)$ st

$$m^{\text{MS}}(\mu) = Z_{m}^{\text{MS}}(\mu, 1/a) m_0$$

Options:

- Lattice perturbation theory. – difficult!
- Alternatively, use two steps:
  - latt $\leftrightarrow$ intermediate(continuum-like) $\leftrightarrow$ $\overline{\text{MS}}$
NPR method

General idea is to renormalize operators using a scheme that is well-defined both in the continuum and on the lattice, e.g. the RI schemes:

Calculate off-shell Green’s functions of operator-of-interest with external quark states.

\[ G_{ij}^\Gamma(p) = \langle q^i(p) \left( \sum_x \bar{q}(x) \Gamma q(x) \right) \bar{q}^j(-p) \rangle_{\text{amp}} \]

Require that the trace of the renormalized operator takes its tree-level value:

\[ \Lambda_\Gamma(p) \equiv \frac{1}{12} \text{Tr} \left[ \Gamma G_\Gamma(p) \right] \simeq \frac{Z_q(p)}{Z_\Gamma(p)} \]
The RI (and $\overline{\text{MS}}$) schemes satisfy $Z_m = Z_S^{-1} = Z_P^{-1}$. $Z_m$ can be extracted from the scalar correlator provided

$$\Lambda_{\text{QCD}} \ll |p| \ll \pi/a$$

After determining $Z_m^{RI}(p)$, a perturbative calculation can be used to convert $Z^{\overline{\text{MS}}}(p) = C^{\overline{\text{MS}}\leftarrow RI}(p) Z_m^{RI}(p)$. 
RI/SMOM scheme

- Momentum flow suppresses infrared effects.
  \[ p_1^2 = p_2^2 = (p_1 - p_2)^2 \]

- \( p_1 \sim (x, x, 0, 0), \)
  \( p_2 \sim (0, x, x, 0) \) for \( x = 2, 3, 4 \)

- Other advantages:
  - Reduced mass dependence.
  - SMOM → \( \overline{\text{MS}} \) matching factors closer to 1.
Continuum extrapolations

\[ a^2 \quad \text{[fm}^2\text{]} \]

\[ \langle JJ \rangle \quad [1408.4169] \]
$m_c$ comparison plot

$\bar{m}_c(m_c, n_f = 4)$ (GeV)

HPQCD HISQ RI-SMOM
FNAL/MILC/TUM HISQ MRS
HPQCD HISQ JJc
ETMC RI-MOM

JLQCD DW JJa
MP(hotQCD) HISQ JJa
$\chi$QCD overlap RI-MOM
HPQCD HISQ JJb
HPQCD+ HISQ JJa
Renormalon subtracted masses
HQET masses

Mass of a heavy meson $H$ in heavy quark effective theory (HQET)

\[ M_H = m_Q + \bar{\Lambda} + \frac{\mu_\pi^2}{2m_Q} - \frac{\mu_G^2(m_Q)}{2m_Q} + \ldots, \]

where

- $m_Q$: Pole mass of the heavy quark $Q$
- $\Lambda$: Energy of light quarks and gluons
- $\frac{\mu_\pi^2}{2m_Q}$: Kinetic energy of heavy quark
- $\frac{\mu_G^2(m_Q)}{2m_Q}$: Hyperfine energy due to heavy quark spin

Want to relate pole mass to \MSbar mass,

Meson mass $\leftrightarrow$ quark pole mass $\leftrightarrow$ quark \MSbar mass
Perturbative series connecting the pole mass to the $\overline{\text{MS}}$ mass (known to four loops) diverges due to renormalons,

$$m_{\text{pole}} = \bar{m} \left( 1 + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}(\bar{m}) \right),$$

with

$$r_n \propto (2\beta_0)^n \Gamma(n + b + 1) \text{ as } n \to \infty$$

but can be interpreted using Borel summation. After subtracting the (leading) renormalon from the pole mass, there is a well-behaved connection between the subtracted mass and the $\overline{\text{MS}}$ mass.

$$m_{\text{pole}} \to m_{\text{MRS}} + \mathcal{O}(\Lambda_{\text{QCD}})$$
\[ m_Q \leftrightarrow m^{\overline{\text{MS}}} \]

\[
m_{\text{pole}} + \overline{\Lambda} = \overline{m} \left( 1 + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}(\overline{m}) \right) + \overline{\Lambda} \rightarrow \\
\overline{m} \left( 1 + \sum_{n=0}^{\infty} [r_n - R_n] \alpha_s^{n+1}(\overline{m}) \right) + J_{\text{MRS}}(\overline{m}) + [\delta_m + \overline{\Lambda}] \\
= m_{\text{MRS}} + \overline{\Lambda}_{\text{MRS}}
\]

\[
r_n = (0.4244, 1.0351, 3.6932, 17.4358, \ldots) \\
R_n = (0.5350, 1.0691, 3.5966, 17.4195, \ldots) \\
r_n - R_n = (-0.1106, -0.0340, 0.0966, 0.0162, \ldots)
\]
Measure meson mass $M_{Hs}$ varying heavy input mass $am_{h,0}$.

$$m_{h}^{\overline{\text{MS}}} (\mu) = m_{r}^{\overline{\text{MS}}} (\mu) \frac{am_{h,0}}{am_{r,0}} + \mathcal{O}(a^2),$$

with $m_{r}^{\overline{\text{MS}}} (\mu)$ treated as a fit parameter.

- Fit data including discretization artifacts as well as HQET parameters $\overline{\Lambda}_{\text{MRS}}$, $\mu^2_{\pi}$, $\mu^2_G(\mu)$.
- Evaluate fit at $M_{D_s}, M_{B_s}$ to obtain $\overline{m}_c, \overline{m}_b$. 

![Graph showing fit data and parameters](image1.png)

![Graph showing differences between $M_H$ and $m_{h,MRS}$](image2.png)
\[
m_s^{\overline{\text{MS}}} (2 \text{ GeV}) = 92.47(39)_{\text{stat}}(18)_{\text{sys}}(52) \alpha_s (11) f_{\pi, \text{PDG}} \text{ MeV}
\]
\[
\overline{m_c} = 1273(4)_{\text{stat}}(1)_{\text{sys}}(10) \alpha_s (0) f_{\pi, \text{PDG}} \text{ MeV}
\]
\[
\overline{m_b} = 4201(12)_{\text{stat}}(1)_{\text{sys}}(8) \alpha_s (1) f_{\pi, \text{PDG}} \text{ MeV}
\]

These results can be compared e.g. with current-correlator results:

\[
m_s^{\overline{\text{MS}}} (2 \text{ GeV}) = 93.6 (8) \text{ MeV} \quad \text{[1408.4169]}
\]
\[
\overline{m_c} = 1271 (10) \text{ MeV}
\]
\[
\overline{m_b} = 4196 (23) \text{ MeV} \quad \text{[1408.5768]}
\]
• Bare input mass parameters can be tuned to reproduce hadron masses measured in experiment, and can also be varied away from physical values.

• Now several independent and complementary techniques which establish strange, charm, and bottom quark masses at the (sub-)percent level.

• It is increasingly feasible to perform relativistic simulations with $b$ quarks – currently some form of effective theory is used or an extrapolation to $m_b$ is required – these techniques can then be applied in the same way as for charm [already the case for MRS].
• RI/SMOM intermediate scheme
  ▶ Perturbative and IR (condensate) uncertainties decrease with lattice spacing.
  ▶ Main uncertainty comes from tuning uncertainties - need improved determinations of lattice spacings and input masses.

• Current-current correlators
  ▶ Main uncertainty from perturbation theory.
  ▶ Finer lattice means reference scale $a m_h$ can be increased.
  ▶ See talk by A. Veernala (FNAL/MILC) Lattice 2017.

• MRS subtracted masses
  ▶ Calculation already includes $a \sim 0.045, 0.03$ fm lattices.
  ▶ Uncertainty in $\alpha_s$ is a major source of error.

The main results presented here use MILC lattice ensembles – important to calculate with additional fermion formulations!
Thank you!
Regulate QCD using a (Euclidean) spacetime lattice.

Integrate out fermionic degrees of freedom.

\[ Z = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-\int (\mathcal{L}_{YM} + \bar{\psi}D\psi)} \]

\[ = \int \mathcal{D}U (\det D) e^{-\int \mathcal{L}_{YM}} \]

Generate gluon configurations using Monte Carlo techniques.

Effects of sea quarks are included in the determinant of the Dirac matrix.
Calculate valence quark propagators on gluon field configurations.

\[ D^{-1} = \] 

Tie together the quark propagators to create correlation functions.

\[ \langle \pi \pi^\dagger \rangle = \pi \]

\[ \pi \]
Energies and matrix elements are determined by fitting (sums of) exponentials.

\[
\langle \pi(t) \pi^\dagger(0) \rangle \xrightarrow{\text{large } t} \frac{|\langle 0 | \pi | \pi \rangle|^2}{2m_\pi} e^{-m_\pi t} \propto f_\pi^2 e^{-m_\pi t}
\]
New lattice result from ALPHA collaboration using Schrödinger Functional and step-scaling:

\[ \alpha_s^{\overline{\text{MS}}} (m_Z) = 0.1185(8) \]
HISQ $\langle JJ \rangle$ error budget

<table>
<thead>
<tr>
<th>Source of Uncertainty</th>
<th>$m_c(3)$</th>
<th>$\alpha_{\overline{MS}} (M_Z)$</th>
<th>$m_c/m_s$</th>
<th>$m_b/m_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perturbation theory</td>
<td>0.3</td>
<td>0.5</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Statistical errors</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>$a^2 \to 0$</td>
<td>0.3</td>
<td>0.3</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$\delta m_{\text{sea}}^{\text{uds}} \to 0$</td>
<td>0.2</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$\delta m_{\text{sea}}^{\text{c}} \to 0$</td>
<td>0.3</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$m_h \neq m_c$ (Eq. (15))</td>
<td>0.1</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Uncertainty in $w_0$, $w_0/a$</td>
<td>0.2</td>
<td>0.0</td>
<td>0.1</td>
<td>0.4</td>
</tr>
<tr>
<td>$\alpha_0$ prior</td>
<td>0.0</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Uncertainty in $m_{\eta_s}$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.4</td>
<td>0.0</td>
</tr>
<tr>
<td>$m_h/m_c \to m_b/m_c$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.4</td>
</tr>
<tr>
<td>$\delta m_{\eta_c}$: electromag., annih.</td>
<td>0.1</td>
<td>0.0</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$\delta m_{\eta_b}$: electromag., annih.</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>0.64%</strong></td>
<td><strong>0.63%</strong></td>
<td><strong>0.55%</strong></td>
<td><strong>1.20%</strong></td>
</tr>
</tbody>
</table>

[1408.4169]
### Projected $h \rightarrow AA$ uncertainties

<table>
<thead>
<tr>
<th></th>
<th>$\delta m_b(10)$</th>
<th>$\delta \alpha_s(m_Z)$</th>
<th>$\delta m_c(3)$</th>
<th>$\delta_b$</th>
<th>$\delta_c$</th>
<th>$\delta_g$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>current errors</strong></td>
<td>0.70</td>
<td>0.63</td>
<td>0.61</td>
<td>0.77</td>
<td>0.89</td>
<td>0.78</td>
</tr>
<tr>
<td>+ PT</td>
<td>0.69</td>
<td>0.40</td>
<td>0.34</td>
<td>0.74</td>
<td>0.57</td>
<td>0.49</td>
</tr>
<tr>
<td>+ LS</td>
<td>0.30</td>
<td>0.53</td>
<td>0.53</td>
<td>0.38</td>
<td>0.74</td>
<td>0.65</td>
</tr>
<tr>
<td>+ LS$^2$</td>
<td>0.14</td>
<td>0.35</td>
<td>0.53</td>
<td>0.20</td>
<td>0.65</td>
<td>0.43</td>
</tr>
<tr>
<td>+ PT + LS</td>
<td>0.28</td>
<td>0.17</td>
<td>0.21</td>
<td>0.30</td>
<td>0.27</td>
<td>0.21</td>
</tr>
<tr>
<td>+ PT + LS$^2$</td>
<td>0.12</td>
<td>0.14</td>
<td>0.20</td>
<td>0.13</td>
<td>0.24</td>
<td>0.17</td>
</tr>
<tr>
<td>PT + LS$^2$ + ST</td>
<td>0.09</td>
<td>0.08</td>
<td>0.20</td>
<td>0.10</td>
<td>0.22</td>
<td>0.09</td>
</tr>
<tr>
<td><strong>ILC goal</strong></td>
<td></td>
<td></td>
<td></td>
<td>0.30</td>
<td>0.70</td>
<td>0.60</td>
</tr>
</tbody>
</table>
Topological “freezing” observed in MC time series of ultrafine ($a \sim 0.045$ fm) ensembles.

The effect of fixed topology on masses and decay constants was analysed using $\chi$PT in [1707.05430].
Percentage error in a heavy-light ($Hq$) decay constant:

\[
\frac{\delta f}{f} \approx \frac{1}{2 \chi TV} \cdot \frac{1}{16} \frac{m_{l,\text{sea}}}{m_q^2} \cdot \left[ 1 - \frac{\langle Q^2 \rangle_{\text{sample}}}{\chi TV} \right]
\]

Effect enhanced on ‘uf-5’ ensemble, where $V$ is small and $m_{l,\text{sea}} = m_s/5$. (here $\frac{\langle Q^2 \rangle_{\text{sample}}}{\chi TV} \approx 1.3$). Numerically,

\[
\frac{\delta f_D}{f_D} \sim \frac{\delta f_B}{f_B} \approx 1%
\]

\[
\frac{\delta f_{D_s}}{f_{D_s}} \sim \frac{\delta f_{B_s}}{f_{B_s}} \approx 0.002%
\]
Systematics for $Z_m$

Effect of varying charm, strange, and light sea masses:

Finite volume effects:
MILC heavy quark masses

\begin{align*}
  m_{hl} &= m_{l}^s \\
  m_{ll} &= m_{l}^s / 5 \\
  m_{ll} &= m_{l}^s / 10 \\
  m_{l} &= \text{physical}
\end{align*}