Hadronic light-by-light contribution to a_μ: dispersive approach

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Outline

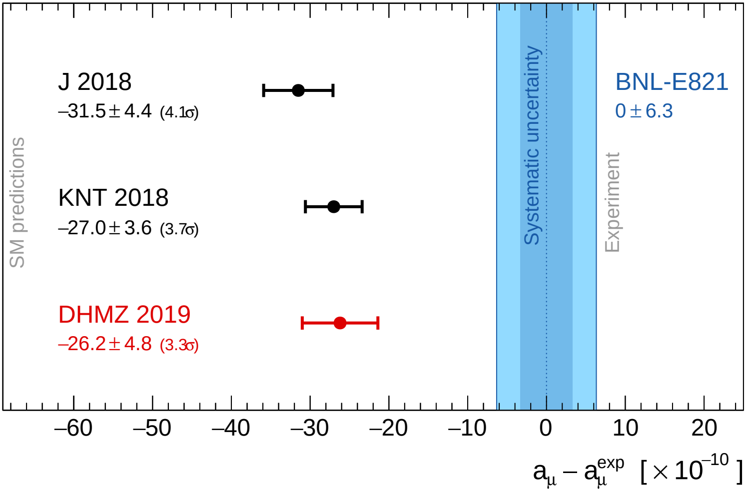
Introduction: $(g-2)_\mu$ and hadronic contributions

Hadronic light-by-light contribution to $(g-2)_\mu$
- Master Formula
- A dispersion relation for HLbL
- Short-distance constraints

Summary and outlook

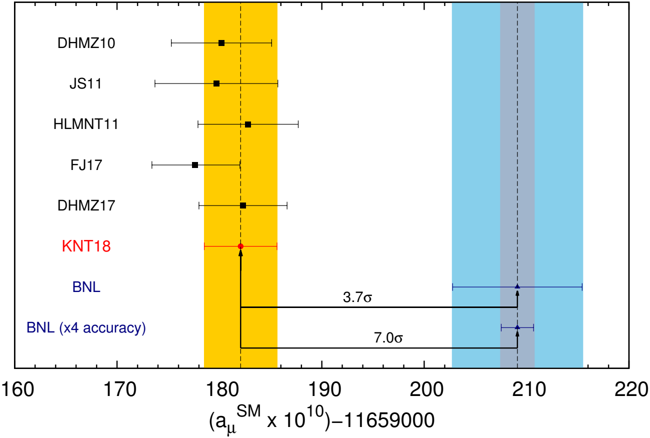
Status of $(g-2)_\mu$, experiment vs SM

Davier, Hoecker, Malaescu, Zhang 2019



Status of $(g-2)_\mu$, experiment vs SM

Keshavarzi, Nomura, Teubner, 2018 (KNT18)



Fermilab experiment's goal: error $\times 1/4$, should be matched by theory:

$\Rightarrow$ Muon "$(g-2)$ Theory Initiative" lead by A. El-Khadra and C. Lehner

Status of $(g-2)_\mu$, experiment vs SM

KNT 18

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.97	0.07
electroweak, total	153.6	1.0
HVP (LO) [KNT 18]	6 932.7	24.6
HVP (NLO) [KNT 18]	−98.2	0.4
HLbL [update of Glasgow consensus–KNT 18]	98.0	26.0
HVP (NNLO) [Kurz, Liu, Marquard, Steinhauser 14]	12.4	0.1
HLbL (NLO) [GC, Hoferichter, Nyffeler, Passera, Stoffer 14]	3.0	2.0
theory	116 591 820.5	35.6

Status of $(g-2)_\mu$, experiment vs SM

KNT 18

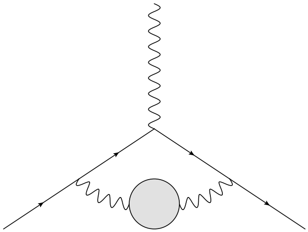
$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 268.5 \pm 72.4 \qquad [3.7\sigma]$$

Keshavarzi, Nomura, Teubner, 2018

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) can be systematically improved

$\rightarrow$ talk by Hoecker



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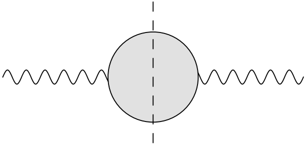

- basic principles: unitarity and analyticity
- direct relation to experiment: $\sigma_{\text{tot}}(e^+e^- \to \gamma^* \to \text{hadrons})$
- dedicated e^+e^- program: BaBar, Belle, BESIII, CMD3, KLOE2, SND
- alternative approach: lattice (ETMC, Mainz, HPQCD, BMW, RBC/UKQCD)

→ talk by Gérardin

Theory uncertainty comes from hadronic physics

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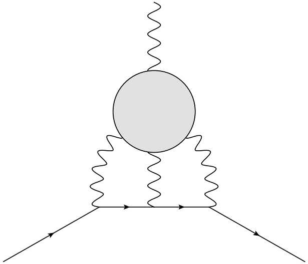
→ talk by Hoecker

- Hadronic light-by-light (HLbL) is more problematic:



  - 4-point fct. of em currents in QCD
  - "*it cannot be expressed in terms of measurable quantities*"
  - until recently, only model calculations
  - lattice QCD is making fast progress

→ talk by Gérardin

Muon $g-2$ Theory Initiative

Steering Committee:
GC
Michel Davier
Simon Eidelman
Aida El-Khadra (co-chair)
Christoph Lehner (co-chair)
Tsutomu Mibe (J-PARC E34 experiment)
Andreas Nyffeler
Lee Roberts (Fermilab E989 experiment)
Thomas Teubner

Workshops:

- First plenary meeting, Q-Center (Fermilab), 3-6 June 2017
- HVP WG workshop, KEK (Japan), 12-14 February 2018
- HLbL WG workshop, U. of Connecticut, 12-14 March 2018
- Second plenary meeting, Mainz, 18-22 June 2018
- Third plenary meeting, Seattle, 9-13 September 2019

Different model-based evaluations of HLbL

Jegerlehner-Nyffeler 2009

Contribution	BPaP(96)	HKS(96)	KnN(02)	MV(04)	BP(07)	PdRV(09)	N/JN(09)
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
" " + subl. in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

Legenda: B=Bijnens Pa=Pallante P=Prades H=Hayakawa K=Kinoshita S=Sanda Kn=Knecht N=Nyffeler M=Melnikhov V=Vainshtein dR=de Rafael J=Jegerlehner

- large uncertainties (and differences among calculations) in individual contributions
- pseudoscalar pole contributions most important
- second most important: pion loop, *i.e.* two-pion cuts (*Ks are subdominant*)
- heavier single-particle poles decreasingly important

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Advantages of the dispersive approach

- model independent
- unambiguous definition of the various contributions
- makes a data-driven evaluation possible (in principle)
- First attempts: GC, Hoferichter, Procura, Stoffer (14) Pauk, Vanderhaeghen (14)
- similar philosophy, with a different implementation: Schwinger sum rule Hagelstein, Pascalutsa (17)
- why hasn't this been adopted before?

The HLbL tensor

HLbL tensor:

$$\Pi^{\mu\nu\lambda\sigma} = i^3 \int dx \int dy \int dz\, e^{-i(x\cdot q_1 + y\cdot q_2 + z\cdot q_3)} \langle 0 | T\{j^\mu(x) j^\nu(y) j^\lambda(z) j^\sigma(0)\} | 0 \rangle$$

$$q_4 = k = q_1 + q_2 + q_3 \qquad k^2 = 0$$

General Lorentz-invariant decomposition:

$$\Pi^{\mu\nu\lambda\sigma} = g^{\mu\nu} g^{\lambda\sigma} \Pi^1 + g^{\mu\lambda} g^{\nu\sigma} \Pi^2 + g^{\mu\sigma} g^{\nu\lambda} \Pi^3 + \sum_{i,j,k,l} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma \Pi^4_{ijkl} + \ldots$$

consists of 138 scalar functions $\{\Pi^1, \Pi^2, \ldots\}$, but in $d = 4$ only 136 are linearly independent

Eichmann *et al.* (14)

Constraints due to gauge invariance? (see also Eichmann, Fischer, Heupel (2015))

$\Rightarrow$ Apply the Bardeen-Tung (68) method+Tarrach (75) addition

Gauge-invariant hadronic light-by-light tensor

Applying the Bardeen-Tung-Tarrach method to $\Pi^{\mu\nu\lambda\sigma}$ one ends up with:

GC, Hoferichter, Procura, Stoffer (2015)

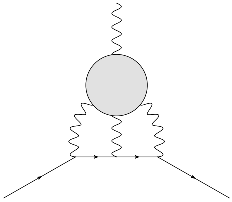
- 43 basis tensors (BT) — in $d=4$: 41=no. of helicity amplitudes
- 11 additional ones (T) — to guarantee basis completeness everywhere
- of these 54 only 7 are distinct structures
- all remaining 47 can be obtained by crossing transformations of these 7: manifest crossing symmetry
- the dynamical calculation needed to fully determine the LbL tensor concerns these 7 scalar amplitudes

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

Master Formula

$$a_\mu^{\text{HLbL}} = -e^6 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p) \hat{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1+q_2)^2 [(p+q_1)^2 - m_\mu^2][(p-q_2)^2 - m_\mu^2]}$$

- $\hat{T}_i$: known kernel functions
- $\hat{\Pi}_i$: linear combinations of the Π_i
- the Π_i are amenable to a dispersive treatment: their imaginary parts are related to measurable subprocesses
- 5 integrals can be performed with Gegenbauer polynomial techniques



GC, Hoferichter, Procura, Stoffer (2015)

Master Formula

After performing the 5 integrations:

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1^4 \int_0^\infty dQ_2^4 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

where Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta:

$$Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$$

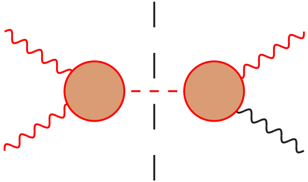
The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

GC, Hoferichter, Procura, Stoffer (2015)

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion pole: imaginary parts = δ-functions
Projection on the BTT basis: easy ✓
Our master formula=explicit expressions in the literature ✓
Input: pion transition form factor Hoferichter et al. (18)
First results of direct lattice calculations Gerardin, Meyer, Nyffeler (16,19)

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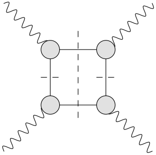
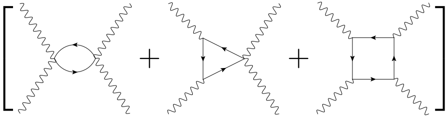


$$\equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2)$$

$\times$



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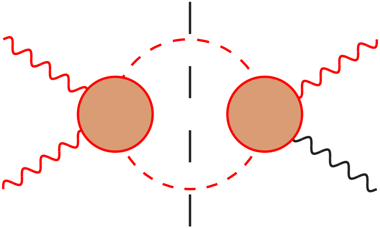


The "rest" with 2π intermediate states has cuts only in one channel and will be calculated dispersively after partial-wave expansion

Setting up the dispersive calculation

We split the HLbL tensor as follows:

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Cuts with states $\neq \pi$ or 2π are neglected

η, η' or the K-loop contribution can be calculated analogously

Pion-pole contribution

Latest complete analyses:

- Dispersive calculation of the pion TFF — Hoferichter et al. (18)

$$a_\mu^{\pi^0} = 63.0^{+2.7}_{-2.1} \times 10^{-11}$$

- Padé-Canterbury approximants — Masjuan & Sanchez-Puertas (17)

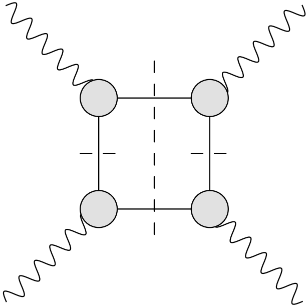
$$a_\mu^{\pi^0} = 63.6(2.7) \times 10^{-11}$$

- Lattice — Gérardin, Meyer, Nyffeler (19)

$$a_\mu^{\pi^0} = 62.3(2.3) \times 10^{-11}$$

Pion-box contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



Pion-box contribution

The only ingredient needed for the pion-box contribution is the vector form factor

$$\hat{\Pi}_i^{\pi\text{-box}} = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, I_i(x,y),$$

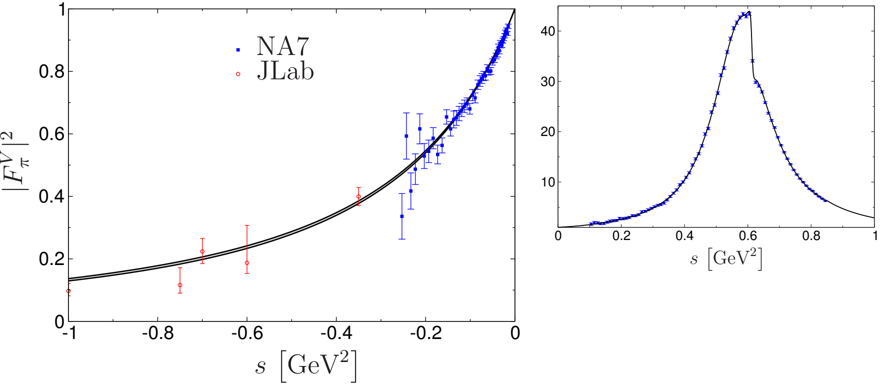
where

$$I_1(x,y) = \frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}},$$

and analogous expressions for $I_{4,7,17,39,54}$ and

$$\Delta_{123} = M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2,$$
$$\Delta_{23} = M_\pi^2 - x(1-x)q_2^2 - y(1-y)q_3^2$$

Pion-box contribution

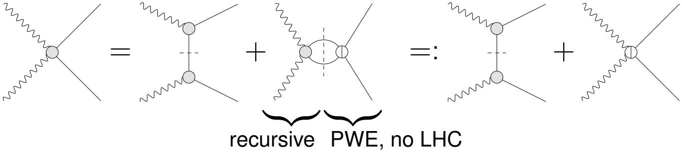


Uncertainties are negligibly small:

$$a_\mu^{\text{FsQED}} = -15.9(2) \cdot 10^{-11}$$

First evaluation of S- wave 2π-rescattering

Omnès solution for $\gamma^*\gamma^* \to \pi\pi$ provides the following:

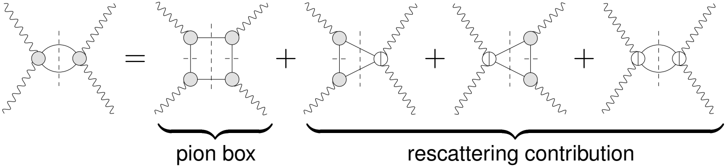


S-wave contribution below ~ 1 GeV:

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$

Two-pion contribution to $(g-2)_\mu$ from HLbL

Two-pion contributions to HLbL:



$$a_\mu^{\pi-\text{box}} + a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -24(1)\cdot 10^{-11}$$

Contributions from above 1 GeV and higher partial waves are work in progress

Short-distance contraints

Two different kinematic configurations for large Q_i^2:

1. All momenta large — Melnikov-Vainshtein (04), Bijnens et al (19)

$$\lim_{Q\to\infty} Q^4\, \bar{\Pi}_1(Q,Q,Q) = -\frac{4}{9\pi^2}.$$

2. $Q^2 \equiv Q_1^2 \sim Q_2^2 \gg Q_3^2$: — Melnikov-Vainshtein (04)

$$\lim_{Q_3\to\infty}\lim_{Q\to\infty} Q^2 Q_3^2\, \bar{\Pi}_1(Q,Q,Q_3) = -\frac{2}{3\pi^2}$$

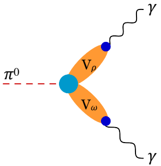
In fact, in the chiral (and large-N_c) limit

$$\lim_{Q\to\infty} Q^2\, \bar{\Pi}_1(Q,Q,Q_3) = -\frac{2}{3\pi^2 Q_3^2}$$

the Q_3^2 dependence is known exactly

Contributions discussed so far do not satisfy these constraints

A Regge-like large-N_C inspired model



$$F_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2) = \sum_{V_\rho,V_\omega} \frac{F_{V_\rho}(q_1^2)\,F_{V_\omega}(q_2^2)\,G_{\pi V_\rho V_\omega}(q_1^2,q_2^2)}{(q_1^2+M_{V_\rho}^2)(q_2^2+M_{V_\omega}^2)} + \left\{q_1 \leftrightarrow q_2\right\}$$

where

$$M_{V_{\rho,\omega}}^2 = M_{\rho,\omega}^2(i_{\rho,\omega}) = M_{\rho,\omega}^2(0) + i_{\rho,\omega}\,\sigma_{\rho,\omega}^2$$

Masjuan, Broniowski, Ruiz Arriola (12)

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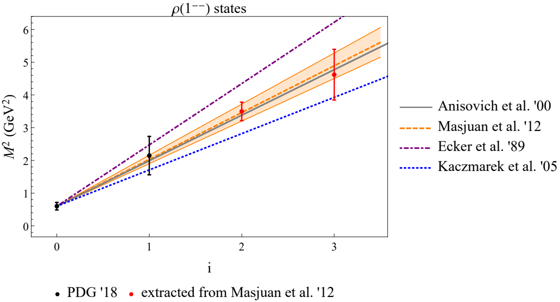

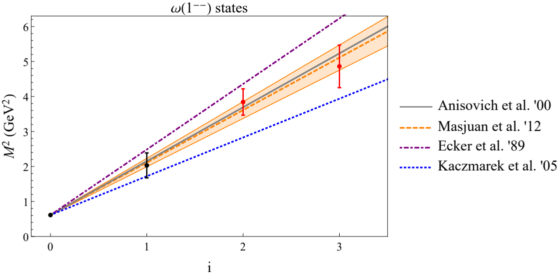

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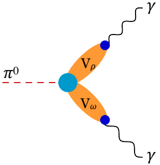

$$F_{\pi^{(n)}\gamma^*\gamma^*}(q_1^2,q_2^2) = \sum_{V_\rho,V_\omega} \frac{F_{V_\rho}(q_1^2)\, F_{V_\omega}(q_2^2)\, G_{\pi^{(n)}V_\rho V_\omega}(q_1^2,q_2^2)}{(q_1^2+M_{V_\rho}^2)(q_2^2+M_{V_\omega}^2)} + \left\{q_1 \leftrightarrow q_2\right\}$$

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Masjuan, Broniowski, Ruiz Arriola (12)

similarly for “excited pions”, described by a Regge-like model:

$$m_\pi^2(n) = \begin{cases} m_{\pi^0}^2 & n=0, \\ m_0^2 + n\,\sigma_\pi^2 & n \geq 1, \end{cases}$$

Satisfying short-distance constraints

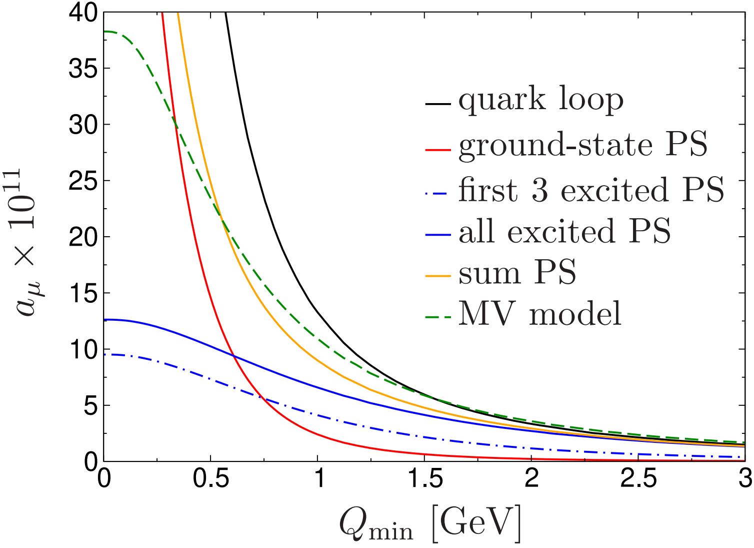
$$\lim_{Q_3\to\infty}\lim_{\tilde{Q}\to\infty}\sum_{n=0}^{\infty}\frac{F_{\pi^{(n)}\gamma^*\gamma^*}(\tilde{Q}^2,\tilde{Q}^2)\,F_{\pi^{(n)}\gamma\gamma^*}(Q_3^2)}{Q_3^2+m^2_{\pi^{(n)}}}=$$
$$=\frac{1}{6\pi^2}\frac{1}{\tilde{Q}^2}\frac{1}{Q_3^2}+\mathcal{O}\left(\tilde{Q}^{-2}Q_3^{-4}\right),$$

where $F_{\pi^{(n)}\gamma^*\gamma^*}$ is the TFF of the n-th radially-excited pion

The infinite sum over excited pions changes the large-Q_3^2 behaviour from Q_3^{-4} (single pion pole) to Q_3^{-2}

The model only works away from the chiral limit

Matching to the quark loop



Matching to the quark loop

Final result:

$$\Delta a_\mu^{\text{LSDC}} = \left[8.7(5.5)_{\text{PS-poles}} + 4.6(9)_{\text{pQCD}}\right] \times 10^{-11}$$
$$= 13(6) \times 10^{-11}$$

Significantly smaller than estimated by Melnikov-Vainshtein :

$$\bar{\Pi}_1(Q_1, Q_2, Q_3) = \frac{F_{\pi\gamma^*\gamma^*}(Q_1^2, Q_2^2) F_{\pi\gamma\gamma}}{Q_3^2 + M_\pi^2}$$

$$\Delta a_\mu^{\text{MV}} = 24 \times 10^{-11} \qquad \text{(using 2004 TFF)}$$
$$\to 38 \times 10^{-11} \qquad \text{(using today's TFF)}$$

White Paper Summary of HLbL (preliminary!)

Contributions to $10^{11} \cdot a_\mu^{\mathrm{HLbL}}$

- Pseudoscalar poles $= 93.8^{+4.0}_{-3.6}$
- pion box (kaon box ~ -0.5) $= -15.9(2)$
- S-wave $\pi\pi$ rescattering $= -8(1)$

Partial total: 69.9 ± 4.1

- scalars and tensors with $M_R > 1$ GeV $\sim -2(3)$
- axial vectors $\sim 6(5)$
- short-distance contribution $\sim 15(10)$
- c-quark loop $\sim 3(1)$

Final result: 91 ± 18

Uncertainties added in quadrature: $\longrightarrow 12$

Improvements obtained with the dispersive approach

Contribution	PdRV(09)	N/JN(09)	J(17)	Our estimate
π^0, η, η'-poles	114 ± 13	99 ± 16	95.45 ± 12.40	93.8 ± 4.0
π, K-loop/box	-19 ± 19	-19 ± 13	-20 ± 5	-16.4 ± 0.2
S-wave $\pi\pi$	-7 ± 7	-7 ± 2	-5.98 ± 1.20	-8 ± 1
subtotal	88 ± 24	73 ± 21	69.5 ± 13.4	69.4 ± 4.1
scalars	–	–	–	$\}\ -2 \pm 3$ (scalars + tensors)
tensors	–	–	1.1 ± 0.1	
axial vectors	15 ± 10	22 ± 5	7.55 ± 2.71	6 ± 5
q-loops / SD	–	21 ± 3	20 ± 4	15 ± 10
c-loop	2.3	–	2.3 ± 0.2	3 ± 1
total	105 ± 26	116 ± 39	100.4 ± 28.2	91 ± 18

HLbL in units of 10^{-11}.
PdRV = Prades, de Rafael, Vainshtein ("Glasgow consensus");
N = Nyffeler; J = Jegerlehner

Conclusions

- The HLbL contribution to $(g-2)_\mu$ can be expressed in terms of measurable quantities in a dispersive approach
- master formula: HLbL contribution to a_μ as triple-integral over scalar functions which satisfy dispersion relations
- relevant observable depends on the intermediate state:
  - single-pion contribution: pion transition form factor
  - pion-box contribution: pion vector form factor
  - 2-pion rescattering: $\gamma^*\gamma^{(*)} \to \pi\pi$ helicity amplitudes

  these have been calculated with very small uncertainties
- short-distance constraints can be satisfied (impact: $\sim$ 15%)
- The goal of matching the experimental reduction of the uncertainty with a similar reduction on the theory side is being achieved (work in progress...)

Backup Slides

The Melnikov-Vainshtein model at low q^2

Pion pole in HLbL: $$\hat{\Pi}_1 = \frac{F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi\gamma\gamma^*}(q_3^2)}{s - M_\pi^2} + \dots .$$

gives for $q_4 \to 0$, which implies $s = q_3^2$:

$$\begin{aligned}\hat{\Pi}_1 &= F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) \frac{F_{\pi\gamma\gamma^*}(q_3^2)}{q_3^2 - M_\pi^2} + G(q_1^2, q_2^2, q_3^2) \\ &= F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) \left[\frac{F_{\pi\gamma\gamma}}{q_3^2 - M_\pi^2} + \frac{F_{\pi\gamma\gamma^*}(q_3^2) - F_{\pi\gamma\gamma}}{q_3^2 - M_\pi^2} \right] + G(q_1^2, q_2^2, q_3^2)\end{aligned}$$

Asymptotic limit, $q_1^2 = q_2^2 = \hat{q}^2 \gg \Lambda_{\rm QCD}^2$ exact in q_3^2:

$$\hat{\Pi}_1 \Big|_{m_q=0} = -\frac{1}{6\pi^2} \frac{1}{\hat{q}^2 q_3^2} + \mathcal{O}(\hat{q}^{-3})$$

implies

$$G(\hat{q}^2, \hat{q}^2, q_3^2)\Big|_{m_q=0} = \frac{2F_\pi}{3\hat{q}^2} \frac{F_{\pi\gamma\gamma^*}(q_3^2) - F_{\pi\gamma\gamma}}{q_3^2}\Big|_{m_q=0} + \mathcal{O}(\hat{q}^{-3}) .$$

The Melnikov-Vainshtein model at low q^2

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The model of Melnikov-Vainshtein is:

$$\hat{\Pi}_1 = \frac{F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi\gamma\gamma}}{q_3^2 - M_\pi^2}$$

which implies the assumption

$$G(q_1^2, q_2^2, q_3^2)\Big|_{m_q=0} = -F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) \frac{F_{\pi\gamma\gamma^*}(q_3^2) - F_{\pi\gamma\gamma}}{q_3^2}\Big|_{m_q=0} .$$

for which there is no justification whatsoever