Determining the strong coupling from Lattice QCD

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together with: T. Onogi, R. Sommer (FLAG4 WG)

[Ultimate precision at Hadron Colliders, Nov. 2019, Saclay, France]

Flavour Lattice Averaging Group (FLAG)

- Appears every $\sim 2 - 3$ years, since 2011, present incarnation FLAG4, (FLAG19) in press
- Review of lattice results, present review ~ 500 pages
- Develops (mostly lattice-specific, but also some non-lattice) criteria:

$$\bigstar, \ \circ, \ \blacksquare$$

- Gives averages based on these criteria

$$\blacksquare, \ \square, \ \square$$

- α_s – since 2013 (FLAG2)

Flavour Lattice Averaging Group (FLAG)

The list of FLAG members (for FLAG19) and their Working Group assignments is:

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- Editorial Board (EB): G. Colangelo, A. Jüttner, S. Hashimoto, S.R. Sharpeand U. Wenger
- Working Groups (coordinator listed first):
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  - $f_{B_{(s)}}$, $f_{D_{(s)}}$, B_B D. Lin, Y. Aoki, and M. Della Morte
  - $B_{(s)}$, D semileptonic and radiative decays E. Lunghi, D. Becirevic, S. Gottlieb and C. Pena
  - α_s R. Sommer, R. Horsley, and T. Onogi
  - NME R. Gupta, S. Collins, A. Nicholson and H. Wittig

$\alpha_s(\mu) \equiv g_s(\mu)^2/4\pi$

The 'running' of the QCD coupling constant as the scale changes is controlled by the β function,

$$\frac{\partial g_s(\mu)}{\partial \log \mu} = \beta^s(g_s(\mu))$$

with $[n_l + 1$ loops in β-function$]$

$$\beta^s(g_s) = -b_0 g_s^3 - b_1 g_s^5 - b_2^s g_s^7 - b_3^s g_s^9 \ldots - b_{n_l}^s g_s^{3+2n_l} \ldots$$

Integrating

$$\frac{\Lambda_s}{\mu} = \exp\left(-\frac{1}{2b_0 g_s^2}\right)\left(b_0 g_s^2\right)^{-\frac{b_1}{2b_0^2}} \exp\left\{-\int_0^{g_s} d\xi \left[\frac{1}{\beta^s(\xi)} + \frac{1}{b_0 \xi^3} - \frac{b_1}{b_0^2 \xi}\right]\right\}$$

with (scheme dependent) integration constant Λ_s

To leading order

$$\alpha_s(\mu) \sim \frac{4\pi}{b_0 \ln(\mu/\Lambda_s)^2}$$

I) Lattice determination of $\alpha_{\overline{\rm MS}}(M_Z)$

Basic method: 'measure' some short distance quantity
$\mathcal{O}(\mu) = \lim_{a\to 0} \mathcal{O}_{\rm lat}(a, \mu)$ and match to a perturbative expansion

$$\mathcal{O}(\mu) = c_1 \alpha_s(\mu) + c_2 \alpha_s^(\mu) + \cdots$$

and need conversion from scheme S (relevant to process) to (pert.) $\overline{\rm MS}$

$$g^2_{\overline{\rm MS}}(\mu) = g_s^2(\mu)(1 + c_g^{(1)} g_s^2(\mu) + \ldots + c_g^{(n_l)} g_s^{2n_l}(\mu) + \ldots) \quad [n_l \text{ loops known}]$$

- Lattice
  - \+ Can 'design' (Euclidean) $\mathcal{O}$
  - \+ No hadronisation issues
  - \+ Can simulate at at range of parameters that do not exist in nature
  - \- Lattice specific problems, eg continuum limit
  - \- $N_f = 3$ perhaps 4 to (pert) cross quark theshold(s) $\to \alpha_{\overline{\rm MS}}(M_Z)$
  - \- Need 2 disparate scales - hadron mass (M_N, r_0, ... for setting overall scale) and also high energies (for perturbative matchings)
- Main question:
  Can we get to a perturbative regime?

Roughly need scale 'μ' large (so α_s small); lattice spacing 'a' small

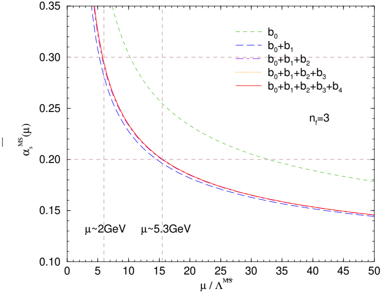
Lattice requirements: need to compromise!

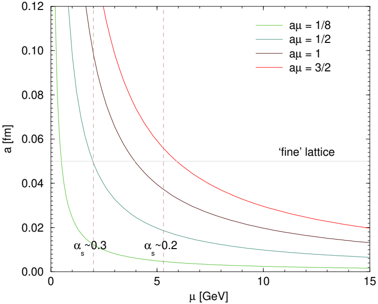
- $1/a \gg \mu$ ie $a\mu \ll 1$
- But also $L \gg$ hadron size $\sim \Lambda_{\rm QCD}^{-1}$ giving $L/a \ggg \mu\Lambda_{\rm QCD}^{-1}$ or

$$\mu \lll \frac{L}{a} \times \Lambda_{\rm QCD} \sim 10 - 30\,\text{GeV} \qquad L/a \sim 32 - 96$$

so $\mu \sim$ few GeV at most

Is $a\mu \ll 1$? Consider $\alpha_{\overline{\rm MS}} \sim 0.2 - 0.3$





Perturbative considerations

$$\Lambda_{\overline{\rm MS}} = \Lambda_s \exp[c_g^{(1)}/(2b_0)] \quad \Rightarrow \quad \text{regard errors in } \alpha_{\overline{\rm MS}} \text{ from } \Lambda_s$$

$$\left(\frac{\Delta\Lambda_s}{\Lambda_s}\right)_{\Delta\alpha_S} = \frac{\Delta\alpha_s(\mu)}{8\pi b_0 \alpha_s^2(\mu)} \times [1 + O(\alpha_s(\mu))] \qquad \text{error in } \alpha_s \quad (1)$$

$$\left(\frac{\Delta\Lambda_s}{\Lambda_s}\right)_{\rm trunc} = k\alpha_s^{n_l}(\mu) + O(\alpha_s^{n_l+1}(\mu)) \quad \text{trunc error in } \beta_s(g_s) \quad (2)$$

[n_l loops in g_s conversion; $n_l + 1$ loops in $\beta_s(g_s)$]

Want perturbative error $\ll$ (determined) error in α_s, (1) + (2) $\Rightarrow$

$$\alpha_s^{n_l}(\mu) \ll \frac{\Delta\alpha_s(\mu)}{8\pi b_0 \alpha_s^2(\mu)}$$

Also simple consequence is error at two different scales, (1) $\Rightarrow$

$$\frac{\Delta\alpha_s(\mu_2)}{\Delta\alpha_s(\mu_1)} \approx \frac{\alpha_s^2(\mu_2)}{\alpha_s^2(\mu_1)}$$

Criteria (1,2):

- $1/\mu$ small
  α_s small

- Renormalization scale
  - ★ all points relevant in the analysis have $\alpha_{\rm eff} < 0.2$
  - ○ all points have $\alpha_{\rm eff} < 0.4$ and at least one $\alpha_{\rm eff} \leq 0.25$
  - ■ otherwise

- $a\mu \ll 1$

- Continuum extrapolation

  At a reference point of $\alpha_{\rm eff} = 0.3$ (or less)
  - ★ three lattice spacings with $\mu a < 1/2$ and full $O(a)$ improvement,
    or three lattice spacings with $\mu a \leq 1/4$ and 2-loop $O(a)$ improvement,
    or $\mu a \leq 1/8$ and 1-loop $O(a)$ improvement
  - ○ three lattice spacings with $\mu a < 3/2$ reaching down to $\mu a = 1$ and full $O(a)$ improvement,
    or three lattice spacings with $\mu a \leq 1/4$ and 1-loop $O(a)$ improvement
  - ■ otherwise

Compromise between renormalisation scale and lattice spacing

Criterion (3):

- n_l large
- Computation checks pert. behaviour
  or
  pert (trunc) error $\ll$ error in α_s
- Non-perturbative effects $\sim \exp(-\gamma/\alpha_s)$ negligible

Need to compromise

- Perturbative behaviour
  - ★ verified over a range of a factor 4 change in $\alpha_{\rm eff}^{n_l}$ without power corrections or alternatively $\alpha_{\rm eff}^{n_l} \le \frac{1}{2}\Delta\alpha_{\rm eff}/(8\pi b_0 \alpha_{\rm eff}^2)$ is reached
  - ○ agreement with perturbation theory over a range of a factor $(3/2)^2$ in $\alpha_{\rm eff}^{n_l}$ possibly fitting with power corrections or alternatively $\alpha_{\rm eff}^{n_l} \le \Delta\alpha_{\rm eff}/(8\pi b_0 \alpha_{\rm eff}^2)$ is reached
  - ■ otherwise

I) Lattice determinations of $\alpha_{\overline{\mathrm{MS}}}$

- Heavy (static) Q-$\bar{Q}$ – potential at short distances $\quad \mu = 2/r$
- Vacuum polarisation, $\langle JJ \rangle$, at large Q^2 $\quad \mu = Q$
  [$\mathcal{O}(\mu) = D(Q^2)$, Adler function, $J \sim V + A$]
- Current $\langle JJ \rangle$ functions, moments of heavy quarks($\sim c$) $\quad \mu = 2\bar{m}_h$
- Ghost–gluon . . . vertices, fixed gauge $\quad \mu = Q$
- Eigenvalue density of Dirac operator $\quad \mu = 1/\lambda$
  [$\mathcal{O}(\mu) = \partial \ln(\rho(\lambda))/\partial \ln \lambda$, where ρ is the spectral density]

1) Determination of α_s from the potential at short distances

Force/potential between (infinitely) massive quark–anti-quark pair

$$F(r) = \frac{dV(r)}{dr} = C_F \frac{\alpha_{qq}(r)}{r^2}$$

- Determine $V(r)$ from Wilson loops

$$\langle W(r,t) \rangle = |c_0|^2 e^{-V(r)t} + \sum_{n \neq 0} |c_n|^2 e^{-V_n(r)t}$$

- Force needs (numerical) gradient
- If use $V(r)$ need to fix at some $r = r_{\rm ref}$ - introduces new renormalisation scale (renormalon $1/Q^2$ term, need to subtract)
- $n_l = 3$ (with $\alpha^4_{\overline{\rm MS}} \ln \alpha_{\overline{\rm MS}}$) higher order partially known

2) Determination of α_s from current two-point functions

[moment method]

$$G(t) = \sum_{\vec{x}} \langle J^\dagger(x) J(0) \rangle \qquad J = m_h \overline{q}_h \gamma_5 q_{h'}$$

q_h, $q_{h'}$ mass degen. m_h

heavy valence quarks

$h \sim$ charm

- Consider (finite) moments ($n \geq 4$)

$$G_n = \sum_{t=-(T/2-a)}^{t=T/2-a} t^n G(t)$$

- Moments dominated by $t \sim 1/m_h$, ie short distances, large μ
  – discretisation errors
- Moments become increasingly perturbative for decreasing n
- Continuum PT

$$R_n \sim \frac{G_n}{G_n^{(0)}} \sim 1 + r_{n1}\alpha_{\overline{\mathrm{MS}}} + r_{n2}\alpha_{\overline{\mathrm{MS}}}^2 + r_{n3}\alpha_{\overline{\mathrm{MS}}}^3 + \ldots$$

- $r_{ni}(\mu/m_h^{\overline{\mathrm{MS}}}(\mu))$ known (continuum PT)
- Leads to a determination of both $\alpha_{\overline{\mathrm{MS}}}$ and $m_h^{\overline{\mathrm{MS}}}$ (heavy quark mass)

II) Lattice determinations of $\alpha_{\overline{\mathrm{MS}}}$ at the lattice spacing scale, a

- Evaluate a short distance quantity $\mathcal{O} \sim W$ Wilson loop (generalise to $m \times n$ where m, n fixed) at the scale of the lattice spacing $\sim 1/a$

$$\mu = \frac{d}{a} \qquad \text{or} \qquad a\mu = d \gtrsim 1$$

- Then determine its relationship to $\alpha_{\overline{MS}}$ via a power series expansion in bare (lattice) coupling
-

$$\begin{aligned} W(a,a) &= c_0 + c_1\alpha_0(1/a) + c_2\alpha_0^2(1/a) + \ldots + O(a^2) \\ &= \tilde{c}_0 + \tilde{c}_1\alpha_{\overline{\mathrm{MS}}}(\mu) + \tilde{c}_2\alpha_{\overline{\mathrm{MS}}}^2(\mu) + \ldots + O(a^2) \end{aligned}$$

- Continuum extrapolation necessary
- Replace Criterion (2) [$a\mu \ll 1$] by several lattice spacings available
- (bare) PT can be complicated (ie only low n_l available)

III) Lattice determinations of $\alpha_{\overline{\rm MS}}$ using step scaling

- Set (Schrödinger Functional, SF)

$$\mu = \frac{1}{L} \qquad \text{as can take } L/a = 8, 16, 32, \ldots \;\Rightarrow\quad a\mu \ll 1$$

- Series of steps (including cont. limit) to reach high scale ($s = 2$)

$$\mu \to s\mu \to s^2\mu \to \ldots \to s^N\mu \qquad \mu \sim 200\,\text{MeV} \to s^N\mu \sim 100\,\text{GeV}$$

- Cleanly separates hadron scale $\sim 200\,$MeV from PT scale $\sim 100\,$GeV
- Can check running of $\alpha_{\rm SF}$ over large range
-

$$\Lambda^{(3)}_{\overline{\rm MS}} = \underbrace{\frac{\Lambda^{(3)}_{\overline{\rm MS}}}{\mu_{\rm PT}}}_{\rm pert.\ theory} \times \underbrace{\frac{\mu_{\rm PT}}{\mu_{\rm had}}}_{\rm step-scaling} \times \underbrace{\frac{\mu_{\rm had}}{f_{\pi K}}}_{\rm large\ vol\ simulation} \times \underbrace{f_{\pi K}}_{\rm expt\ data}$$

Raw data

- Looked at $\sim$ 70 papers
- Can only perturbatively cross quark thresholds (ie charm) to $Q = M_Z$, so drop $N_f = 0$, 2 data, only consider $N_f = 3$, 4
- About $\sim$ 20 publications determine $\alpha_{\overline{\mathrm{MS}}}(M_Z)$
  - 8 enter final analysis

FLAG19: $\alpha_{\overline{\mathrm{MS}}}(M_Z)$ pre-ranges

- (Roughly) similar to PDG procedure
- Applies to
  - Heavy (static) Q-$\bar{Q}$ – potential at short distances
    [Bazavov 14]
  - Current $\langle JJ \rangle$ functions, moments of heavy quarks
    [JLQCD 16, HPQCD 14A, HPQCD 10]
  - Wilson loops
    [HPQCD 10, Maltman 08]
  - Step scaling + SF
    [ALPHA 17, PACS-CS 09A]
- Central value is weighted average of these results

FLAG19: $\alpha_{\overline{\rm MS}}(M_Z)$ pre-range error estimates I

- Heavy (static) Q-$\bar{Q}$ – potential at short distances [Bazavov]

Small lattice spacings $\Delta\Lambda_s/\Lambda_s \sim 9\alpha_s^3$ [quenched, Husang 17]

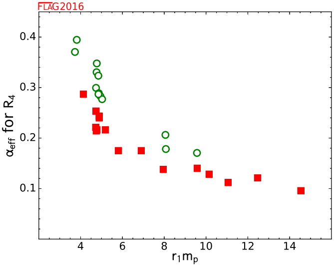
$$\Delta\alpha_s(\mu) \approx 8\pi b_0 \alpha_s^2(\mu) \times 9\alpha_s^3(\mu)$$

Typically $\alpha_s^{\rm min} \sim 0.19$ [Bazavov 14], run to M_Z gives

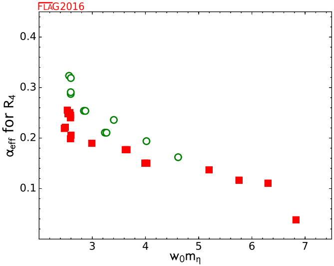
$$\Delta\alpha_{\overline{\rm MS}} \sim 0.0014$$

FLAG19: $\alpha_{\overline{\rm MS}}(M_Z)$ pre-range error estimates II

- Current $\langle JJ \rangle$ functions, moments of heavy quarks
- eg



HPQCD 10

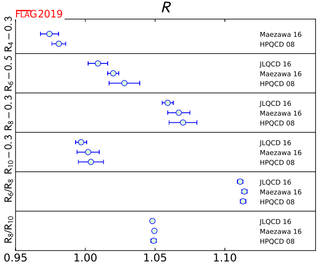


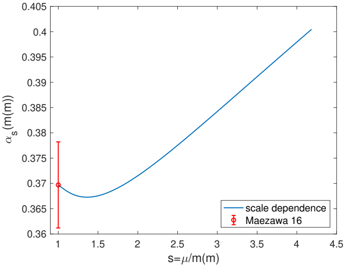
HPQCD 14A

- Possible discretisation errors [★ $a\mu < 1/2$, ○ $a\mu < 3/2$]

FLAG19: $\alpha_{\overline{\rm MS}}(M_Z)$ pre-range error estimates II

- Current $\langle JJ \rangle$ functions, moments of heavy quarks [HPQCD,JLQCD]





Various estimations:

- LH plot: Consider difference in published continuum Rs,

  [eg between Maezawa 16 and JLQCD 16]

  $R_6/R_4 - 1 \sim k\alpha_s \sim 4.5\%$ leading to $\Delta\alpha_{\overline{\rm MS}}(M_Z) \sim 0.0023$

- RH plot: Perturbative uncertainty, vary scale eg $s = 3$ [here $\mu_\alpha = \mu_h$]

  $\Delta\alpha_{\overline{\rm MS}}(M_Z) \sim 0.0017$

- Total difference $|$Maezawa 16 $-$ JLQCD 16$|$, or $\Delta\alpha_{\overline{\rm MS}}(M_Z) \sim 0.0015$
- Settle on $\Delta\alpha_{\overline{\rm MS}}(M_Z) \sim 0.0015$

FLAG19: $\alpha_{\overline{\rm MS}}(M_Z)$ pre-range error estimates III [HPQCD, Maltman et al]

- Wilson loops, perturbative error

$$\Delta\alpha_{\rm s}(\mu) \approx \left|\frac{c_4}{c_1}\right| \alpha_{\rm s}^4(\mu) \qquad \left|\frac{c_4}{c_1}\right| \sim 2 \quad \text{HPQCD 10}$$

Typically $\alpha_s(\mu \sim 5\,\text{GeV}) \sim 0.2$, run to M_Z gives

$$\Delta\alpha_{\overline{\rm MS}} \sim 0.0012$$

- Step scaling + SF [ALPHA]
  Straight weighted average error

$$\Delta\alpha_{\overline{\rm MS}} \sim 0.0008$$

$\alpha_{\overline{\mathrm{MS}}}$

Collaboration	N_f	publication status	renormalization scale	perturbative behaviour	continuum extrapolation	$\alpha_{\overline{\mathrm{MS}}}(M_Z)$	Method	n_l
ALPHA 17	2+1	A	★	★	★	0.11852(84)	step-scaling	2
PACS-CS 09A	2+1	A	★	★	○	0.11800(300)	step-scaling	2
pre-range (average)						0.11848(81)		
Takaura 18	2+1	P	■	○	○	$0.11790(70)(^{+130}_{-120})$	Q-$\bar{Q}$ potential	3
Bazavov 14	2+1	A	○	★	○	0.11660(100)	Q-$\bar{Q}$ potential	3
Bazavov 12	2+1	A	○	○	○	$0.11560(^{+210}_{-220})$	Q-$\bar{Q}$ potential	3
pre-range with estimated pert. error						0.11660(160)		
Hudspith 18	2+1	P	○	★	■	$0.11810(270)(^{+80}_{-220})$	vacuum polarization	3
JLQCD 10	2+1	A	■	○	■	$0.11180(30)(^{+160}_{-170})$	vacuum polarization	2
HPQCD 10	2+1	A	○	★	★	0.11840(60)	Wilson loops	2
Maltman 08	2+1	A	○	○	★	0.11920(110)	Wilson loops	2
pre-range with estimated pert. error						0.11858(120)		
JLQCD 16	2+1	A	○	○	○	0.11770(260)	current two points	2
Maezawa 16	2+1	A	○	■	○	0.11622(84)	current two points	2
HPQCD 14A	2+1+1	A	○	★	○	0.11822(74)	current two points	2
HPQCD 10	2+1	A	○	★	○	0.11830(70)	current two points	2
HPQCD 08B	2+1	A	■	■	■	0.11740(120)	current two points	2
pre-range with estimated pert. error						0.11824(150)		
ETM 13D	2+1+1	A	○	○	■	0.11960(40)(80)(60)	gluon-ghost vertex	3
ETM 12C	2+1+1	A	○	○	■	0.12000(140)	gluon-ghost vertex	3
ETM 11D	2+1+1	A	○	○	■	$0.11980(90)(50)(^{+0}_{-50})$	gluon-ghost vertex	3
Nakayama 18	2+1	A	★	○	■	0.12260(360)	Dirac eigenvalues	2

$\alpha_{\overline{\rm MS}}(M_Z)$ average of pre-ranges

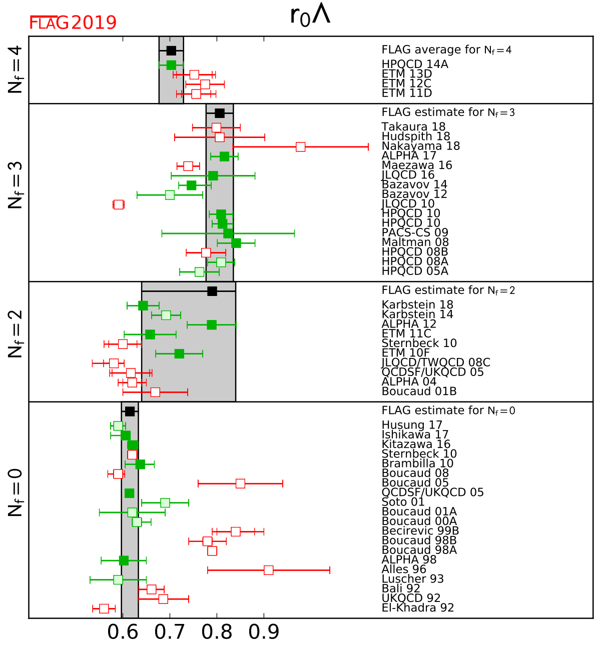
- Mean: weighted average of pre-ranges, 0.1182
- Error:
  1. error of weighted average 0.0006
  2. smallest error of individual pre-ranges 0.0008
  3. mean of errors of individual pre-ranges 0.0013

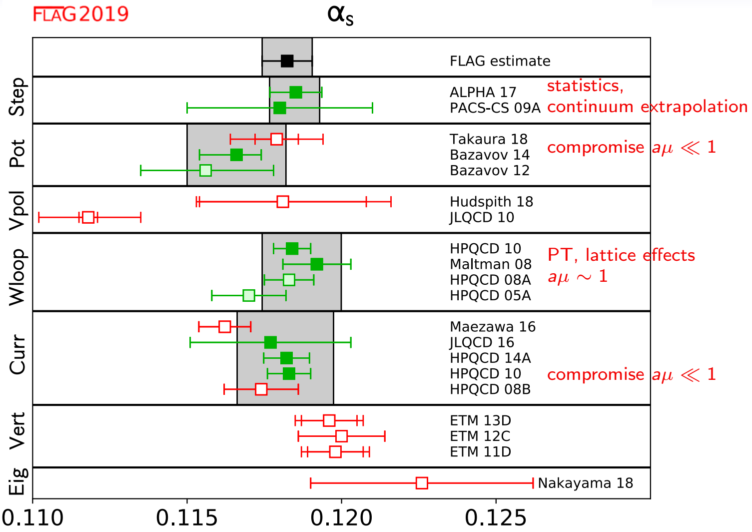
Choose (conservative), middle one, ie No. 2

Lambda-parameter, $\Lambda^{(N_f)}$

- ■ used
- □ superseded
- □ not used

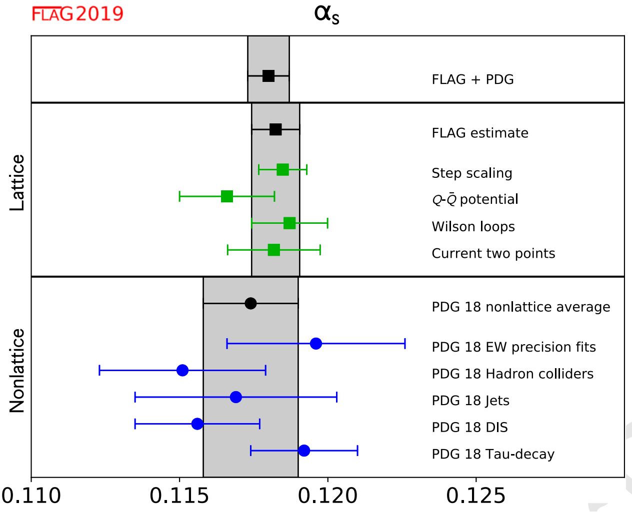
$r_0 \approx 0.5\,\text{fm}$





FLAG2019: $\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z) = 0.1182(8)$

Comparison with PDG:



Lattice results compatible with phenomenological results

FLAG19 + (non-lat)PDG18: $\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1180(7)$

Conclusions

- Possibly $\sim 1\%$ accuracy reached
- Strive for:
  - Always check in PT region
  - Try to avoid mixing computation with NP effects (condensates, renormalons, . . .)
  - More investigation of the continuum limit
- Possibly pheomenologically determinations could also consider minimum value of μ, PT check
  (ie here renormalisation scale / perturbative behaviour criteria)

Additional slides

b coefficients

The first two coefficients are scheme independent:

$$b_0 = \frac{1}{(4\pi)^2}\left(11 - \frac{2}{3}n_f\right), \qquad b_1 = \frac{1}{(4\pi)^4}\left(102 - \frac{38}{3}n_f\right)$$

$\overline{MS}$ scheme: [only defined perturbatively]

$$\begin{aligned}
b_2^{\overline{MS}} &= \frac{1}{(4\pi)^6}\left(\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right) \\
b_3^{\overline{MS}} &= \frac{1}{(4\pi)^8}\left[\frac{149753}{6} + 3564\,\zeta_3 - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n_f \right. \\
&\qquad \left. + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n_f^2 + \frac{1093}{729}n_f^3\right] \\
b_4^{\overline{MS}} &= \text{also known}
\end{aligned}$$

[5-loops: Baikov et al., arXiv:1606.08659; Herzog et al., arXiv:1701.01404]

Mass independent, fixed n_f scheme

$\overline{MS}$ is a mass independent, fixed n_f scheme

- 'Relatively' easy to compute b coefficients
- Cross quark thresholds, need to match $n_f \to n_f + 1$

$$\alpha_{\overline{MS}}^{(n_f)}(\mu) = \alpha_{\overline{MS}}^{(n_f+1)}(\mu) \left\{ 1 + \sum_{k=1}^{\infty} \sum_{n=0}^{k} c_{kn} \left[\frac{\alpha_{\overline{MS}}^{(n_f+1)}(\mu)}{\pi} \right]^k \ln^n \left[\frac{\mu^2}{m_{\overline{MS}}^2(\mu)} \right] \right\}$$

with

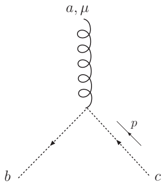
$$c_{10} = 0 , \quad c_{20} = \frac{11}{72} , \quad c_{30} = \frac{564731}{124416} - \frac{82043}{27648}\zeta_3 - \frac{2633}{31104} n_f , \quad \ldots c_{43}$$

- Usually choose $\mu = m_{\overline{MS}}(\mu)$ (ie no logs)
- So 'secret' scale dependence of b coefficients
- Perturbative matching, only trust (?) at charm mass and above, ie $n_f = 3 \to 4$
- In a MOM scheme (more physical), explicit mass dependence – only b_0^{MOM} , b_1^{MOM} known [Jegerlehner et al., hep-ph/9809485]
  But smoother behaviour across quark thresholds

Determination of α_s from QCD vertices

- 'Natural' definition
- Zero incoming ghost momentum in ghost-ghost-gluon vertex
- Simplification: vertex not renormalised (Taylor)
  'T' or 'MM' (minimal mom) scheme

$$\alpha_{\mathrm{T}}(\mu) = D_{\mathrm{lat}}^{\mathrm{ghost}}(\mu, a)^2 D_{\mathrm{lat}}^{\mathrm{gluon}}(\mu, a)\,\frac{g_0^2(a)}{4\pi}$$



- $D_{\mathrm{lat}}^{\mathrm{ghost}}$, $D_{\mathrm{lat}}^{\mathrm{gluon}}$ (bare lattice) dressed ghost/gluon 'form factors' propagator functions in the Landau gauge

$$D^{ab}(p) = -\delta^{ab}\,\frac{D^{\mathrm{ghost}}(p)}{p^2}\,, \quad D_{\mu\nu}^{ab}(p) = \delta^{ab}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{D^{\mathrm{gluon}}(p)}{p^2}$$

[$D^{\mathrm{ghost/gluon}}(p) = D_{\mathrm{lat}}^{\mathrm{ghost/gluon}}(p, 0)$ (continuum)]

- Thus there is now no need to compute the ghost-ghost-gluon vertex, just the ghost and gluon propagators

Determination of α_s from the vacuum polarisation function at short dist.

$$\langle J^a_\mu J^b_\nu \rangle = \delta^{ab}[(\delta_{\mu\nu}Q^2 - Q_\mu Q_\nu)\Pi^{(1)}(Q) - Q_\mu Q_\nu \Pi^{(0)}(Q)]$$

- Q_μ is a space like momentum
- $J_\mu \equiv V_\mu, A_\mu$ for (non-singlet) vector/axial-vector currents

Set $\Pi_J(Q) \equiv \Pi_J^{(0)}(Q) + \Pi_J^{(1)}(Q)$, OPE of the vacuum polarisation function $\Pi_{V+A}(Q) = \Pi_V(Q) + \Pi_A(Q)$:

$$\begin{aligned} &\Pi_{V+A}|_{\mathrm{OPE}}(Q^2, \alpha_s) \\ &= c + C_0(Q^2) + C_m^{V+A}(Q^2)\frac{\bar{m}^2(Q)}{Q^2} + \sum_{q=u,d,s} C_{\bar{q}q}^{V+A}(Q^2)\frac{\langle m_Q \bar{q}q\rangle}{Q^4} \\ &\quad + C_{GG}(Q^2)\frac{\langle \alpha_s GG\rangle}{Q^4} + \mathrm{O}(Q^{-6}) \end{aligned}$$

C_X^{V+A} known up to 4-loops (in $\overline{MS}$ scheme)

- c is Q–independent and divergent ultraviolet cutoff $\to \infty$
- NP condensates eg $\langle \alpha_s GG\rangle$
- terms in C_X which do not have a series expansion in α_s

[Use of Adler function, $D(Q^2) \equiv -Q^2 d\Pi(Q^2)/dQ^2$ is a scheme independent finite quantity, and so avoids some of these problems]