α_s from soft parton-to-hadron fragmentation in jets

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Ultimate precision at hadron colliders
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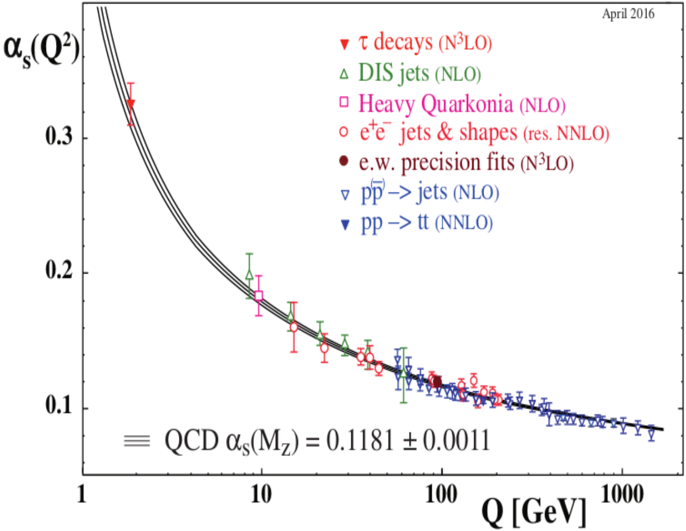
- **Determination of $\alpha_s(M^2_{Z^0})$ from the energy evolution of FF moments**

α_s extractions (PDG 2018)



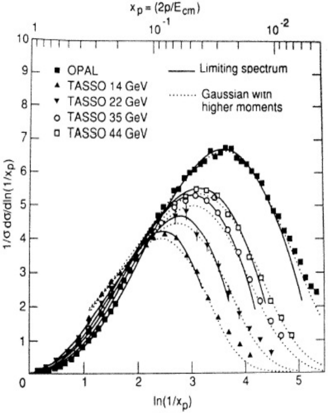
Motivation

The QCD coupling constant α_s

1. Less precisely determined among the coupling constants of the SM of particle physics.
2. Importance: many fundamental SM observables at the LHC and future FCC-ee depend on this key parameter described by QCD.
3. Current uncertainty of the strong coupling world-average value: $\alpha_s(m_Z) = 0.1181 \pm 0.0011$ is about 1%
4. Motivation: reduce the uncertainty by combining current α_s extractions with novel high-precision observables
5. Novel NNLO* $\alpha_s(m_Z)$ determination from the energy-evolution of the FF moments

Motivation

Can we do better than this LO+LL (MLLA) results in the PDG?



19. Fragmentation functions in e^+e^-, ep and pp collisions

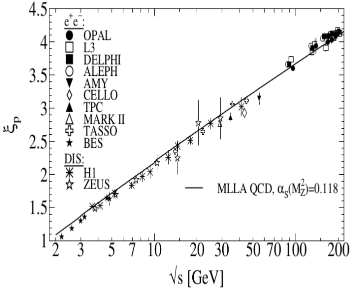
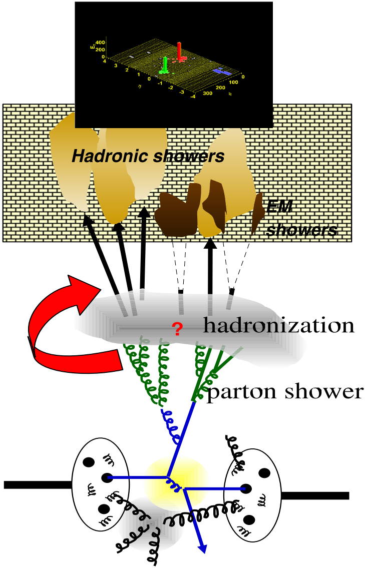


Figure 19.5: Evolution of the peak position, ξ_p, of the ξ distribution with the CM energy $\sqrt{s}$. The MLLA QCD prediction using $\alpha_S(s = M_Z^2) = 0.118$ is superimposed to the data of Refs. [26,28,29,32–34,36,41,55,56,73,74,77–85].

Part I

Jet fragmentation in pQCD

Jets in $e^+e^- \to q\bar{q}$, DIS, p-p, p-pbar



- Partonic cascade: treated in pQCD.
  - Planar gauge: tree amplitudes $\Rightarrow$ parton shower picture (probabilistic interpretation of branching processes).
- Parton shower evolution in the Leading-Log Approx. (and extensions: MLLA, NMLLA,...$D \sim \delta\left(1 - \frac{x}{z}\right)$ for $Q_0 \to \Lambda_{QCD}$)
- Hadronization in the Local Parton Hadron Duality Hypothesis (LPHD):
  - Parton FFs $\simeq$ hadron distributions modulo overall constant factor $\mathcal{K}^{ch}$. [Dokshitzer, Khoze, Mueller]

Soft/collinear res.: DLA, MLLA, next-to-MLLA,...

DLA: $\alpha_s \log(1/x) \log\Theta \sim 1$

1. resummation of soft and collinear gluons
2. main ingredient to the estimation of inclusive observables in jets
3. neglects energy conservation
4. Anomalous dimension: $\gamma \sim \sqrt{\alpha_s}$

$$d\sigma_q^g = \frac{C_F \alpha_s}{\pi} \frac{dz}{z} \frac{dk_\perp^2}{k_\perp^2}$$

Soft/collinear res.: DLA, MLLA, next-to-MLLA,...

MLLA: $\alpha_s \log(1/x) \log\Theta + \alpha_s \log\Theta \sim 1 + \sqrt{\alpha_s}$

1. $\mathcal{O}(\alpha_s)$ collinear splittings (i.e. LLA FFs, PDFs at large $x \sim 1$)
2. partially "restores" energy conservation
3. includes ($\overline{\mathrm{MS}}$) α_s running coupling effects ($\propto \beta_0, \beta_1$)
4. Anomalous dimension: $\gamma \sim \sqrt{\alpha_s} + \alpha_s$

$$d\sigma_q^g = \frac{C_F \alpha_s(k_\perp^2)}{\pi} P_{qg}(z) dz \frac{dk_\perp^2}{k_\perp^2}$$

Soft/collinear res.: DLA, MLLA, next-to-MLLA,...

Next-to-...-MLLA anomalous dimension

After diagonalisation, the $D^h_{q,\bar{q},g}$ FFs can be determined through :

$$\gamma_{++} \sim \sqrt{\alpha_s} + \alpha_s + \alpha_s^{3/2} + \ldots$$

1. Includes higher order running coupling effects $\propto (\beta_0, \beta_1)$
2. Further improve the account of energy conservation
3. Anomalous dimension: Note expansion in half-powers of α_s

Anomalous dimension

NMLLA anomalous dimension

Rate of average multiplicity growth in QCD jets (for $\omega = 0$) :

$$\gamma_\omega = P_{++}^{(0)}(\omega) + P_{++}^{(1)}(\omega) + P_{++}^{*(2)}(\omega)$$

DLA:

$$P_{++}^{(0)} = \frac{1}{2}\omega(s-1) = \mathcal{O}(\sqrt{\alpha_s})$$

with

$$s = \sqrt{1 + \frac{4\gamma_0^2}{\omega^2}}, \text{ where } \gamma_0^2 = \frac{4N_c\alpha_s}{2\pi}$$

Anomalous dimension

NMLLA (NNLL+NLO*) anomalous dimension

Rate of average multiplicity in QCD jets (for $\omega = 0$) :

$$\gamma_\omega = P_{++}^{(0)}(\omega) + P_{++}^{(1)}(\omega) + P_{++}^{*(2)}(\omega)$$

MLLA:

$$P_{++}^{(1)} = \frac{\alpha_s}{2\pi}\left[-\frac{1}{2}a_1(1+s^{-1}) + \frac{\beta_0}{4}(1-s^{-2})\right] = \mathcal{O}(\alpha_s)$$

with

$$s = \sqrt{1+\frac{4\gamma_0^2}{\omega^2}}, \text{ where } \gamma_0^2 = \frac{4N_c\alpha_s}{2\pi}$$

Anomalous dimension

NMLLA (NNLL+NLO*) anomalous dimension

Rate of average multiplicity in QCD jets (for $\omega = 0$) :

$$\gamma_\omega = P_{++}^{(0)}(\omega) + P_{++}^{(1)}(\omega) + P_{++}^{*(2)}(\omega)$$

NMLLA (NNLL+NLO*) :

$$P_{++}^{*(2)} = \frac{\alpha_s^2(\omega s)^{-1}}{104\pi^2}\Big[4a_1^2(1-s^{-2}) + 8a_1\beta_0(1-s^{-3}) + \beta_0^2(1-s^{-2})(3+5s^{-2}) - 64N_c\frac{\beta_1}{\beta_0}\ln(Y+\lambda)\Big] + \frac{N_c\alpha_s}{2\pi}a_2(\omega s)\left[\left(1+s^{-1}\right)^2\right] = \mathcal{O}\left(\alpha_s^{3/2}\right)$$

with

$$s = \sqrt{1+\frac{4\gamma_0^2}{\omega^2}}, \text{ where } \gamma_0^2 = \frac{4N_c\alpha_s}{2\pi}$$

Anomalous dimension

Extension with splittings functions at NLO+resummed (small-x) (based on C.-H. Kom, A. Vogt, K. Yeats JHEP 1210 (2012) 033)

$$
\begin{aligned}
P^T_{qq}(N) &= \frac{4}{3}\frac{C_F n_f}{C_A} a_s \left\{ \frac{1}{2\xi}(S-1)(\mathcal{L}+1)+1 \right\} \\
&+ \frac{1}{18}\frac{C_F n_f}{C_A^3} a_s \bar{N} \Big\{ (-11 C_A^2 + 6 C_A n_f - 20 C_F n_f) \frac{1}{2\xi}(S-1+2\xi) + 10 C_A^2 \frac{1}{\xi}(S-1)\mathcal{L} \\
&- (51 C_A^2 - 6 C_A n_f + 12 C_F n_f) \frac{1}{2}(S-1) + (11 C_A^2 + 2 C_A n_f - 4 C_F n_f) S^{-1}\mathcal{L} \\
&+ (5 C_A^2 - 2 C_A n_f + 6 C_F n_f) \frac{1}{\xi}(S-1)\mathcal{L}^2 + (51 C_A^2 - 14 C_A n_f + 36 C_F n_f)\mathcal{L} \Big\}, \qquad (3.2)
\end{aligned}
$$

$$
P^T_{qg}(N) = \frac{C_A}{C_F} P^T_{qq}(N) - \frac{2}{9}\frac{n_f}{C_A^2} a_s \bar{N} (C_A^2 + C_A n_f - 2 C_F n_f) \left\{ \frac{1}{2\xi}(S-1)(\mathcal{L}+1)+1 \right\}, \qquad (3.3)
$$

$$
\begin{aligned}
P^T_{gg}(N) &= \frac{1}{4}\bar{N}(S-1) - \frac{1}{6C_A} a_s (11 C_A^2 + 2 C_A n_f - 4 C_F n_f)(S^{-1}-1) - P^T_{qq}(N) \\
&+ \frac{1}{576 C_A^3} a_s \bar{N} \Big\{ \big([1193 - 576\zeta_2] C_A^4 - 140 C_A^3 n_f + 4 C_A^2 n_f^2 - 56 C_A^2 C_F n_f + 16 C_A C_F n_f^2 \\
&- 48 C_F^2 n_f^2 \big)(S-1) + \big([830 - 576\zeta_2] C_A^4 + 96 C_A^3 n_f - 8 C_A^2 n_f^2 - 208 C_A^2 C_F n_f \\
&+ 64 C_A C_F n_f^2 - 96 C_F^2 n_f^2 \big)(S^{-1}-1) + (11 C_A^2 + 2 C_A n_f - 4 C_F n_f)^2 (S^{-3}-1) \Big\}, \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
P^T_{gq}(N) &= \frac{C_F}{C_A} P^T_{gg}(N) - \frac{1}{3}\frac{C_F}{C_A^2} a_s (C_A^2 + C_A n_f - 2 C_F n_f) \frac{1}{\xi}(S-1+2\xi) \\
&+ \frac{1}{36}\frac{C_F}{C_A^4} a_s \bar{N} \Big\{ \big(11 C_A^4 + 13 C_A^2 n_f (C_A - 2C_F) + 2 C_A^2 n_f^2 - 8 (C_A - C_F) C_F n_f^2 \big)(1 - S^{-1}) \\
&- \big(48 C_A^4 - 45 C_A^3 C_F - 72 \zeta_2 C_A^3 (C_A - C_F) - 33 C_A^3 n_f + 2 C_A^2 n_f^2 + 48 C_A^2 C_F n_f \\
&- 8 C_F^2 n_f^2 \big) \frac{1}{\xi}(S-1+2\xi) + \big(-54 C_A^4 + 45 C_A^3 C_F + 72 \zeta_2 C_A^3 (C_A - C_F) + 23 C_A^3 n_f \\
&- 28 C_A^2 n_f C_F - 8 (C_A - 2C_F) C_F n_f^2 \big) \frac{1}{\xi}(S-1)\mathcal{L} \Big\} \qquad (3.5)
\end{aligned}
$$

Anomalous dimension

$\mathrm{NNLL} + \mathrm{NLO}$ (NMLLA) anomalous dimension

Rate of average multiplicity in QCD jets (for $\omega = 0$) :

$$\gamma_\omega = P_{++}^{(0)}(\omega) + P_{++}^{(1)}(\omega) + P_{++}^{(2)}(\omega)$$

$\mathrm{NNLL} + \mathrm{NLO}$:

$$P_{++}^{(2)} = \frac{\alpha_s^2}{104\pi^2}(\omega s)^{-1}\left[4a_1^2(1-s^{-2}) + 8a_1\beta_0(1-s^{-3}) + \beta_0^2(1-s^{-2})\right.$$
$$\left.\times\,(3+5s^{-2}) - 64N_c\frac{\beta_1}{\beta_0}\ln(Y+\lambda)\right] + \frac{N_c\alpha_s}{2\pi}a_2(\omega s)\left[\left(1+s^{-1}\right)^2\right.$$
$$\left. + a_3\,(s-1) - a_4\left(1-s^{-1}\right) - a_6\right] = \mathcal{O}\left(\alpha_s^{3/2}\right)$$

with

$$s = \sqrt{1+\frac{4\gamma_0^2}{\omega^2}},\ \text{where } \gamma_0^2 = \frac{4N_c\alpha_s}{2\pi}$$

Anomalous dimension fixed-order+log power counting

Order	LL (DLA)	NLL	NNLL	N^{3}LL	N^{4}LL
LO $P_{ac}^{(0)}$	$\mathcal{O}(\sqrt{\alpha_s})$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO $P_{ac}^{(1)}$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO $P_{ac}^{(2)}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$
LO α_s	$\cdots$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO α_s	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO α_s	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$

MLLA: NLL+LO : $\gamma = \sqrt{\alpha_s} + \alpha_s$

Anomalous dimension fixed-order+log power counting

Order	LL (DLA)	NLL	NNLL	$\mathrm{N^3LL}$	$\mathrm{N^4LL}$
LO $P_{ac}^{(0)}$	$\mathcal{O}(\sqrt{\alpha_s})$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO $P_{ac}^{(1)}$	...	...	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO $P_{ac}^{(2)}$	...	...	...	...	$\mathcal{O}(\alpha_s^{5/2})$
LO α_s	...	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO α_s	...	...	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO α_s	...	...	...	...	$\mathcal{O}(\alpha_s^{5/2})$

NMLA: NNLL+NLO* : $\gamma = \sqrt{\alpha_s} + \alpha_s + \alpha_s^{3/2}$

Anomalous dimension fixed-order+log power counting

Order	LL (DLA)	NLL	NNLL	$\mathrm{N^3LL}$	$\mathrm{N^4LL}$
LO $P_{ac}^{(0)}$	$\mathcal{O}(\sqrt{\alpha_s})$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO $P_{ac}^{(1)}$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO $P_{ac}^{(2)}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$
LO α_s	$\cdots$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO α_s	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO α_s	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$

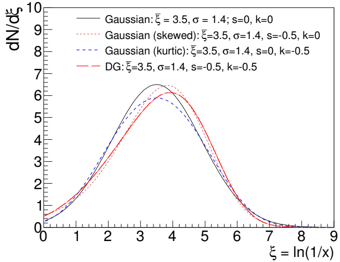
NMLLA: NNLL+NLO : $\gamma = \sqrt{\alpha_s} + \alpha_s + \alpha_s^{3/2}$

Part II

Phenomenology: Generic distorted Gaussian Ansatz for FF

Single inclusive distribution: Distorted Gaussian

- $D^+(\xi, Y, \lambda) = \frac{\mathcal{N}}{\sigma\sqrt{2\pi}} \exp\left[\frac{1}{8}k - \frac{1}{2}s\delta - \frac{1}{4}(2+k)\delta^2 + \frac{1}{6}s\delta^3 + \frac{1}{24}k\delta^4\right]$



- $\delta = \frac{(\xi - \bar{\xi})}{\sigma}$
- Mean multiplicity: $\mathcal{N} = \mathcal{D}^+(\omega = 0, Y, \lambda)$
- Mean peak position: $\bar{\xi}$
- Dispersion (width): σ
- Skewness: s, kurtosis: k

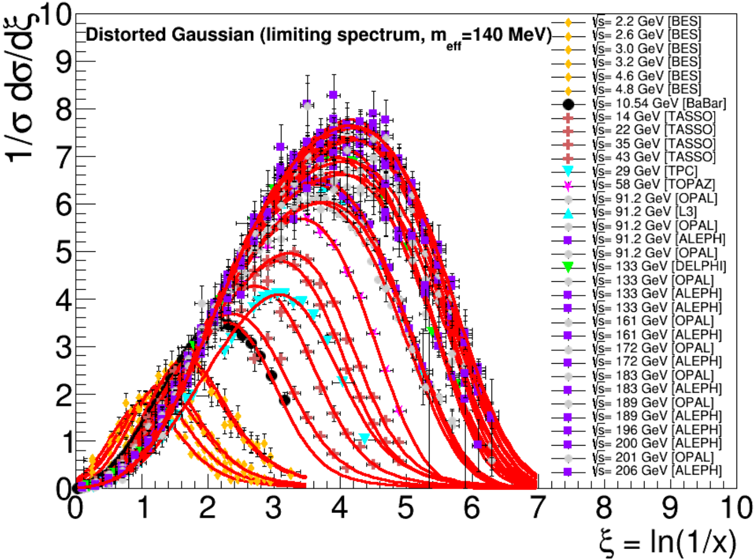
- Moments of the Distorted Gaussian (from anomalous dimension):

$$\mathcal{N} = K_0, \quad \bar{\xi} = K_1, \quad \sigma = \sqrt{K_2}, \quad s = \frac{K_3}{\sigma^3}, \quad k = \frac{K_4}{\sigma^4}$$

$$K_{n\geq 0} = \int_0^Y dy \left(-\frac{\partial}{\partial\omega}\right)^n \gamma_\omega(\alpha_s(y+\lambda))\Big|_{\omega=0}, \; Y = \ln\frac{E\theta}{Q_0}$$

- Skewness and kurtosis (new ingredient) affect tails $\neq$ Gaussian shape!

Example: Distorted Gaussian fits to e^+e^- FFs



Evolution of the NNLL+NNLO* moments of the (DG) FFs

Final expressions as a function of $Y = \ln(E\theta/Q_0)$ and $\lambda = \ln(Q_0/\Lambda_{QCD})$:
(N_f=5) initial jet energy, shower energy cutoff

- **Multiplicity:**

$$\mathcal{N}(Y) = \mathcal{K}^{\rm ch} \exp\left[2.50217\left(\sqrt{Y+\lambda}-\sqrt{\lambda}\right) - 0.491546\ln\frac{Y+\lambda}{\lambda} + (0.0153206 + 0.41151\ln(Y+\lambda))\frac{1}{\sqrt{Y+\lambda}} - (0.0153206+0.41151\ln\lambda)\frac{1}{\sqrt{\lambda}}\right]. \quad (71)$$

- **Average:**

$$\bar{\xi}(Y) = 0.5Y + 0.592722\left(\sqrt{Y+\lambda}-\sqrt{\lambda}\right) + 0.0763404\ln\frac{Y+\lambda}{\lambda}. \quad (73)$$

- **Peak position:**

$$\xi_{\max}(Y) = 0.5Y + 0.592722\left(\sqrt{Y+\lambda}-\sqrt{\lambda}\right) + 0.0763404\ln\frac{Y+\lambda}{\lambda} - 0.355325. \quad (74)$$

- **Width:**

$$\sigma(Y,\lambda) = \left(\frac{\beta_0}{144N_c}\right)^{1/4}\sqrt{(Y+\lambda)^{3/2}-\lambda^{3/2}}\left\{1 - \frac{\beta_0}{64}f_1(Y,\lambda)\sqrt{\frac{16N_c}{\beta_0(Y+\lambda)}} + \left[\frac{3}{16}(3a_2+a_3+2a_4)f_2(Y,\lambda) - \frac{3}{64}\left(\frac{3a_1^2}{16N_c^2}f_2(Y,\lambda) + \frac{a_1\beta_0}{8N_c^2}f_2(Y,\lambda) - \frac{\beta_0^2}{64N_c^2}f_2(Y,\lambda) + \frac{3\beta_0^2}{128N_c^2}f_1^2(Y,\lambda)\right) + \frac{\beta_1}{64\beta_0}(\ln 2(Y+\lambda)-2)f_3(Y,\lambda)\right]\frac{16N_c}{\beta_0(Y+\lambda)}\right\}, \quad (75)$$

$$\sigma(Y) = 0.36499\sqrt{(Y+\lambda)^{3/2}-\lambda^{3/2}}\left\{1 - 0.299739 f_1(Y,\lambda)\frac{1}{\sqrt{Y+\lambda}} - [1.61321 f_2(Y,\lambda) + 0.0449219 f_1^2(Y,\lambda) + (0.32239 - 0.246692\ln(Y+\lambda))\, f_3(Y,\lambda)]\frac{1}{Y+\lambda}\right\}. \quad (76)$$

- **Skewness:**

$$s(Y) = -\frac{1.94704}{\sqrt{(Y+\lambda)^{3/2}-\lambda^{3/2}}}\left[1 - 0.299739 f_1(Y,\lambda)\frac{1}{\sqrt{Y+\lambda}}\right]. \quad (78)$$

- **Kurtosis:**

$$k(Y) = -\frac{2.15812}{\sqrt{Y+\lambda}}\frac{1-\left(\frac{\lambda}{Y+\lambda}\right)^{5/2}}{\left[1-\left(\frac{\lambda}{Y+\lambda}\right)^{3/2}\right]^2}\left\{1 + [1.19896 f_1(Y,\lambda) - 1.99826 f_4(Y,\lambda)]\frac{1}{\sqrt{Y+\lambda}} + [1.07813 f_1^2(Y,\lambda) + 6.45283 f_2(Y,\lambda) + 1.28956 f_3(Y,\lambda) - 2.39583 f_1(Y,\lambda) f_4(Y,\lambda) - 7.13372 f_5(Y,\lambda) + 0.0217751 f_6(Y,\lambda) - (0.986767 f_3(Y,\lambda) - 0.822306 f_6(Y,\lambda))\ln(Y+\lambda)]\frac{1}{Y+\lambda}\right\}. \quad (80)$$

Evolution of the NNLL+NNLO* moments of the (DG) FFs (limiting spectrum)

Expressions evolved down to $\Lambda_{\rm QCD}$ $Q_0 \sim \Lambda_{\rm QCD}$:

$$*\text{Multiplicity}: \quad \mathcal{N}(Y) = \mathcal{K}^{\rm ch} \exp\left[2.50217\sqrt{Y} - 0.491546 \ln Y - (0.06889 - 0.41151 \ln Y)\frac{1}{\sqrt{Y}} + (0.00068 - 0.161658 \ln Y)\frac{1}{Y}\right]$$

$$*\text{Peak position}: \quad \xi_{\max}(Y) = 0.5Y + 0.592722\sqrt{Y} - 0.351319 + 0.002$$

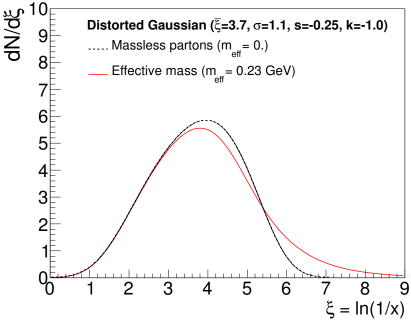
$$*\text{Width}: \quad \sigma(Y) = 0.36499 Y^{3/4}\left[1 - 0.299739\frac{1}{\sqrt{Y}} - (1.4921 - 0.246692 \ln Y)\frac{1}{Y} + \frac{1.98667}{Y^{3/2}}\right]$$

$$*\text{Skewness}: \quad s(Y) = -\frac{1.89445}{Y^{3/4}}\left[1 - 0.312499\frac{1}{\sqrt{Y}} - \frac{1.64009}{Y}\right]$$

$$*\text{Kurtosis}: \quad k(Y) = -\frac{2.15812}{\sqrt{Y}}\left[1 - 0.799305\frac{1}{\sqrt{Y}} + (0.730466 - 0.164461 \ln Y)\frac{1}{Y} - \frac{8.05771}{Y^{3/2}}\right] \qquad (1)$$

$*$ Evolution of all moments depend on 1 single free parameter $\Lambda_{\rm QCD}$, which can be extracted from fits of exp. e^+e^- and $e^-p \to$ jets(hadrons) data.

Hadron mass effects

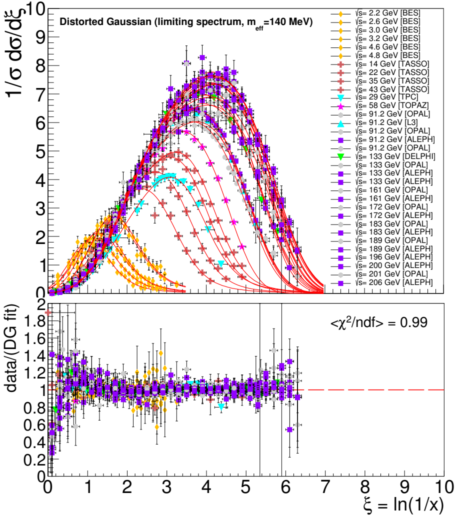


Including hadron mass m_h effects (mixture of pions (65%), kaons (35%) and protons (5%)):

$$\frac{1}{\sigma_{\rm tot}}\frac{d\sigma^{\rm h}}{d\xi_p} \propto \frac{p_h}{E_h} D^+(\xi, Y) \qquad \xi = \ln(1/x) = \ln\left(\frac{\sqrt{s}/2}{\sqrt{(s/4)e^{-2\xi_p} + m_{\rm eff}^2}}\right)$$

$$m_h \sim \mathcal{O}(\Lambda_{\rm QCD}) \;\; E_h = \sqrt{p_h^2 + m_{\rm eff}^2} \;\; p_h = (\sqrt{s}/2)\exp(-\xi_p)$$

Hadron mass effects



- Best agreement reached for $m_h = 0.14$ GeV: consistent with a dominant pion composition of the inclusive charged hadron spectra.

Part III

Extraction of $\alpha_s(M_{Z_0}^2)$ from fits

Fitting procedure

- Experimental distribution will be fitted to the DG parametrization as a function of ξ in the energy range $[0, Y]$.

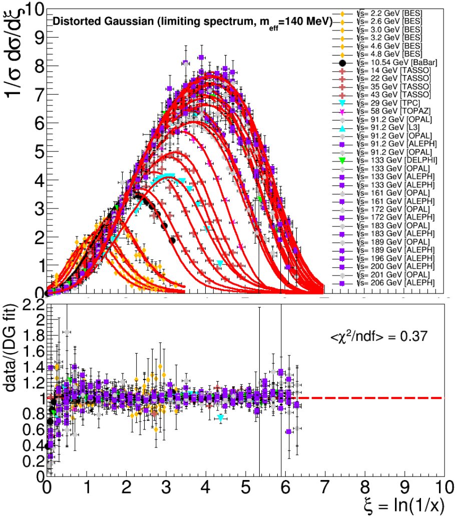
$$\frac{1}{\sigma_{\rm tot}}\frac{d\sigma^{\rm h}}{d\xi} = \mathcal{K}^{\rm ch}\frac{2C_F}{N_c}D^{+}(\xi, Y), \quad Y = \ln\left(\frac{\sqrt{s}}{2\Lambda_{\rm QCD}}\right)$$

- Each fit of the DG has five free parameters: maximum peak position, total multiplicity, width, skewness and kurtosis.

- Each parameter or component of the DG is derived from the fit for each data set.

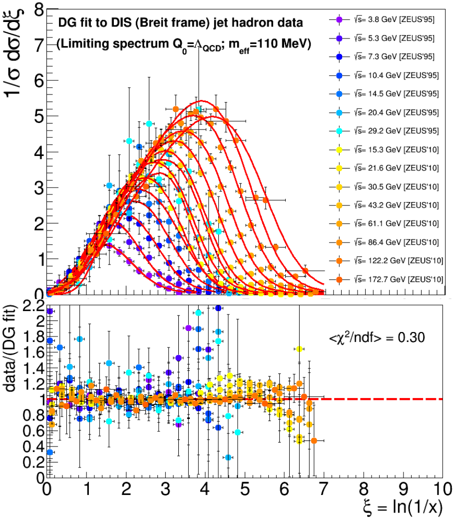
Distorted Gaussian fits to e^+e^- FFs

- 34 e^+e^- data-sets at $\sqrt{s} = 2.2 - 206$ GeV ~ 1200 data points
- For increasing energy : peak shifts to right, width increases, moderate non-Gaussian tails
- Excellent fit at all energies, with 5 free DG parameters : $\mathcal{N}_{ch}$, ξ_{max}, σ, s and k

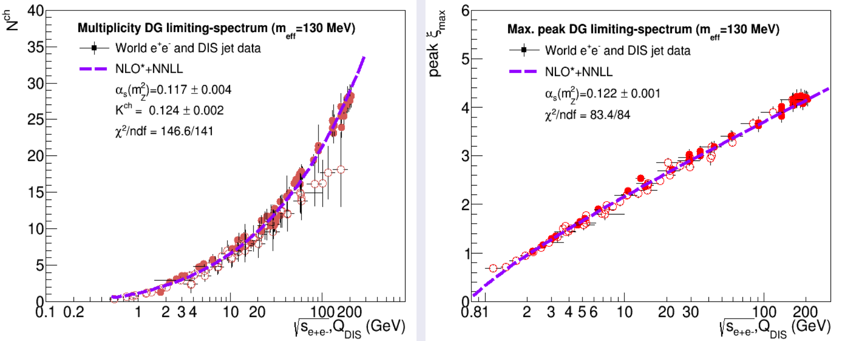


Distorted Gaussian fits to *ep* (DIS) FFs

- Brick wall frame : incoming quarks scatters off photos & returns along the same axis
- 15 ZEUS data-sets at $\sqrt{s} = 3.8 - 173$ GeV ~ 250 data points (other measured H1, ZEUS moments added to global fit)
- Excellent fits to DG but larger uncertainties than e^+e^- measurements

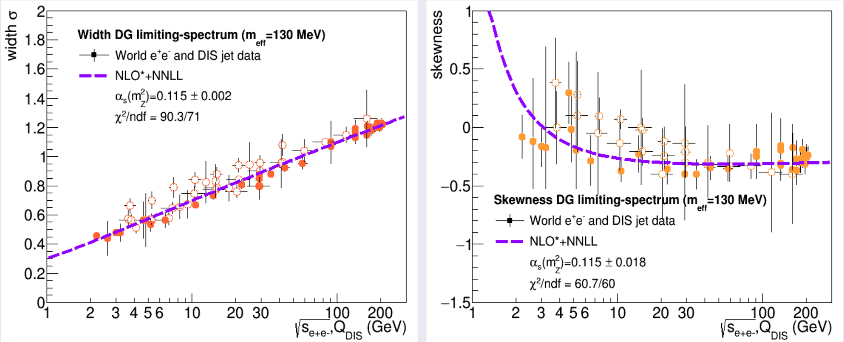


$\mathcal{N}_{ch}$ & FF peak vs. $\sqrt{s}$: Data vs. NNLL+NLO*



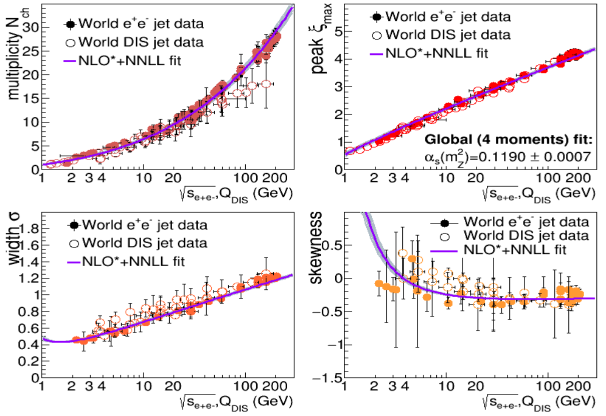
- Theoretical $\mathcal{N}_{ch}$ absolutely normalized to match data (LHPD). QCD coupling drives evolution: care about shape, NOT absolute Nch.
- Very good agreement between e^+e^-, DIS and theory for the FF peak position
- DIS multiplicity lower than e^+e^- but with larger uncertainties

FF width & skewness vs. $\sqrt{s}$: Data vs. NNLL+NLO*



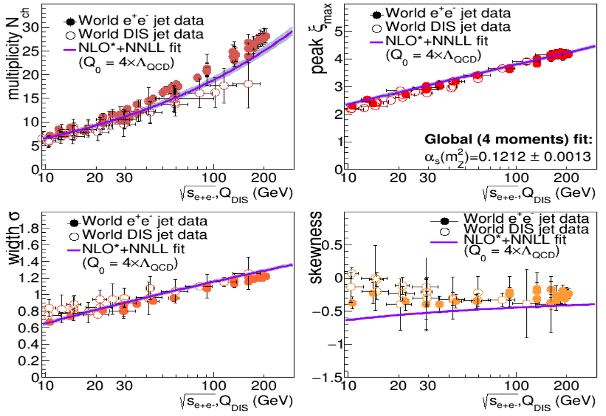
- Good data-theory agreement (skewness has large experimental uncertainties)
- Consistent e^+e^-& e-p moments (but larger DIS uncertainties)

Global fit of FF moments : Data vs. NNLL+NLO*



- χ^2 averaging: increased uncertainty for few point fits to reach $\chi^2/\mathrm{ndf} \sim 1$
- Final α_s uncertainty of $\sim$ 1.2% includes m_{eff}, exp. fits and correlations

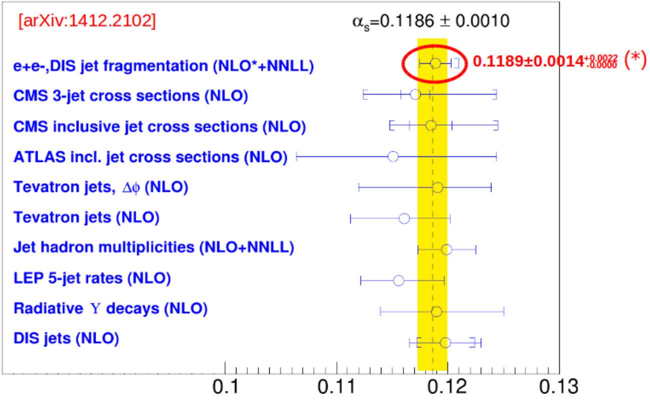
α_s at NNLL+NLO*: scale uncertainty



- Extra uncertainty $\alpha_s(Q_0 = \Lambda_{\rm QCD}) - \alpha_s(Q_0 = 1\text{GeV}) = 2\%$ due to scaling variation

$\alpha_s(M_{Z_0}^2)$ at NNLL+NLO* from low-z FFs evolution

- Most precise measurement of α_s among those at NLO* accuracy (with a different set of systematic uncertainties):



Conclusion

Novel high-precision measurement of α_s at NNLL+NLO* accuracy

Work in progress

Order	LL (DLA)	NLL	NNLL	$\mathrm{N^3LL}$	$\mathrm{N^4LL}$
LO $P_{ac}^{(0)}$	$\mathcal{O}(\sqrt{\alpha_s})$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO $P_{ac}^{(1)}$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO $P_{ac}^{(2)}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$
LO α_s	$\cdots$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO α_s	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO α_s	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$

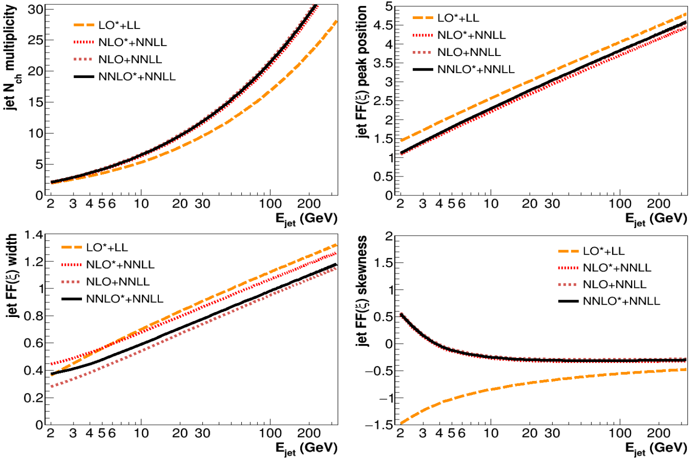
NMLLA: NNLL+NLO : $\gamma = \sqrt{\alpha_s} + \alpha_s + \alpha_s^{3/2}$

Work in progress

Order	LL (DLA)	NLL	NNLL	$\mathrm{N^3LL}$	$\mathrm{N^4LL}$
LO $P_{ac}^{(0)}$	$\mathcal{O}(\sqrt{\alpha_s})$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO $P_{ac}^{(1)}$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO $P_{ac}^{(2)}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$
LO α_s	$\cdots$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NLO α_s	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{3/2})$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^{5/2})$
NNLO α_s	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\mathcal{O}(\alpha_s^{5/2})$

NNLL+NNLO* : $\gamma = \sqrt{\alpha_s} + \alpha_s + \alpha_s^{3/2} + \alpha_s^2 + \alpha_s^{5/2}$

Evolution of FF moments : LO, NLO*, NLO, NNLO*

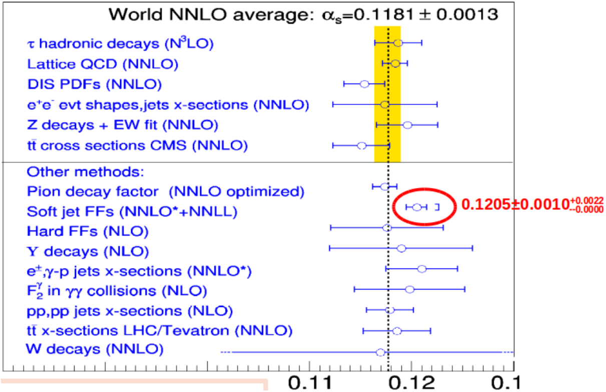


From figures :

Small impact (±10%) corrections beyond NLO* for first 4 FF moments!

$\alpha_s(M^2_{Z_0})$ at NNLL+NLO* from low-z FFs evolution

- Most precise measurement of α_s among those at NNLO* accuracy (with a different set of systematic uncertainties):



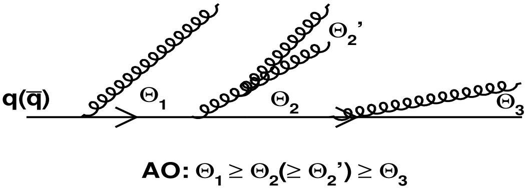
Conclusion

Novel high-precision measurement of α_s at NNLL+NNLO* accuracy

Part IV

Backup slides

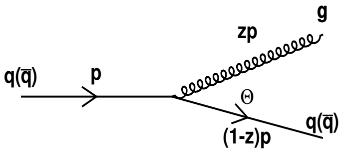
Angular Ordering (MLLA) and k_t-ordering (DGLAP)



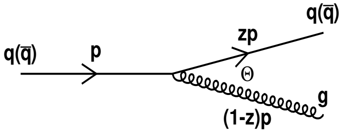
Successive parton decays (soft/collinear and/or hard/collinear) ruled by:

- QCD coherence → Angular Ordering (AO) → MLLA evolution equations for FFs at small $x \ll 1$: ev. time variable $t = \ln \Theta$
  - Soft FF $x < 0.1$: bulk of hadron production in jets
- $k_\perp$-ordering → DGLAP LLA evolution equations at large $x \sim 1$: ev. time variable "$t = \ln k_\perp$"
  - Hard FF $x > 0.1$: hard-hadrons in jets

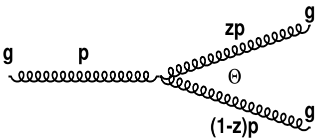
DGLAP LO splitting functions $a[1] \to b[z]c[1-z]$:



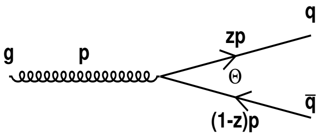
$$P_q^{gq}(z)=C_F\frac{1+(1-z)^2}{z}$$



$$P_q^{qg}(z)=C_F\frac{1+z^2}{1-z}$$



$$P_g^{gg}(z)=2C_A\left(\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right)$$



$$P_g^{q\bar{q}}(z)=T_R\left[z^2+(1-z)^2\right]$$

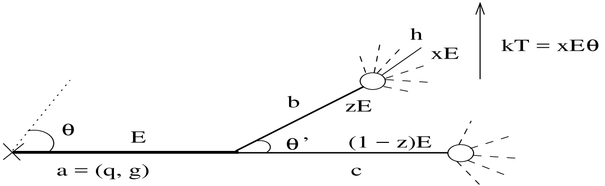
DGLAP versus MLLA evolution equations:

Renormalized QCD evolution equations for $a[1] \to b[z]c[1-z]$:

$$\frac{d}{d\ln\theta}\left[xD_a^b(x,\ln E\theta)\right] = \sum_c \int_0^1 dz\, P_{ac}(z)\frac{\alpha_s(\ln zE\theta)}{\pi}\left[\frac{x}{z}D_c^b\left(\frac{x}{z},\ln zE\theta\right)\right]$$

- z: energy fraction of intermediate parton; x: energy fraction of the hadron.
- Identical but for one detail: for hard partons the shift in $\ln z$ in the argument of D and α_s is negligible:
- for soft/collinear splittings $z \ll 1$: $|\ln z| \gg 1$, $\Theta \ll 1$: DLA.
  - for hard/collinear splittings $z \sim 1$: $\ln z \sim 0$, $\Theta \ll 1$: LLA.
  - for soft/collinear + hard/collinear corrections: Modified-LLA (MLLA).

Solving the evolution equations at small x (or ω)



Mellin transform :

$$\mathcal{D}(\omega, Y) = \int_0^\infty d\xi e^{-\omega\xi} D(\xi, Y), \quad \hat{\xi} = \ln\frac{1}{z}, \quad Y = \ln\frac{E\theta}{Q_0}$$

$$\Rightarrow \frac{\partial}{\partial Y}\mathcal{D}(\omega, Y) = \int_0^\infty d\hat{\xi}\left[e^{-\omega\hat{\xi}}\right] P(\hat{\xi})\frac{\alpha_s(Y-\hat{\xi})}{2\pi}\mathcal{D}(\omega, Y-\hat{\xi}),$$

with $\overline{\mathrm{MS}}$ NLO:

$$\alpha_s(Y) = \frac{2\pi}{\beta_0(Y+\lambda)}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln 2(Y+\lambda)}{Y+\lambda}\right].$$

Solving the evolution equations at small x (or ω)

- Diagonalization of the matrix for $P(\Omega) \to 2$ eigenvalues: $P_{\pm\pm}(\Omega)$.
- Express $\mathcal{D}_q$ and $\mathcal{D}_g$ as the linear combination of the corresponding eigenvectors: $\mathcal{D}^{\pm}$:

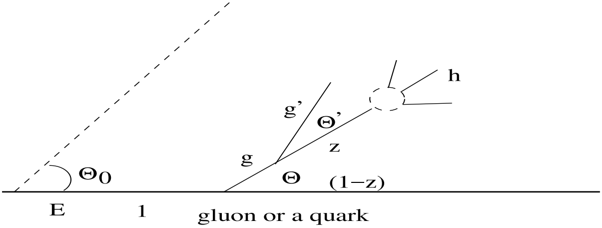
$$\frac{\partial}{\partial Y}\mathcal{D}^{\pm}(\omega, Y, \lambda) = P_{\pm\pm}(\Omega)\frac{\alpha_s(Y)}{2\pi}\mathcal{D}^{\pm}(\omega, Y, \lambda), \quad P_{++}(\Omega) = \frac{4N_c}{\Omega} - a_1 + 4N_c a_2 \Omega$$

- NMLLA evolution equation for $\mathcal{D}^{+}$:

$$\left(\omega+\frac{\partial}{\partial Y}\right)\frac{\partial}{\partial Y}\mathcal{D}^{+} = \left[1-\frac{a_1}{4N_c}\left(\omega+\frac{\partial}{\partial Y}\right)+a_2\left(\omega+\frac{\partial}{\partial Y}\right)^2\right]4N_c\frac{\alpha_s}{2\pi}\mathcal{D}^{+}$$

- $\overline{\mathrm{MS}}$ NLO: $\alpha_s(Y) = \frac{2\pi}{\beta_0(Y+\lambda)}\left[1 - \frac{\beta_1}{\beta_0^2}\frac{\ln 2(Y+\lambda)}{Y+\lambda}\right]$.
- DLA term: $\propto \mathcal{O}(1)$.
- Hard single logs: $\propto a_1 \sim \mathcal{O}(\sqrt{\alpha_s})$ (MLLA or NLL+LO) & $\propto a_2 \sim \mathcal{O}(\alpha_s)$ (NMLLA or NNLL+NLO*).

Exact Angular Ordering



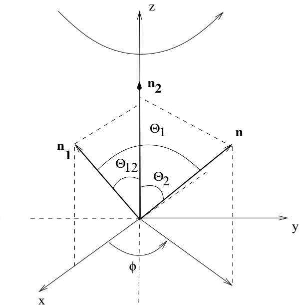
$$d^2\sigma^h_{q\to qg} = \frac{\alpha_s}{\pi} P_q^{qg}(z)dz\, V(\vec{n})\frac{d\Omega}{8\pi}, \quad V^{g'}_{g(q)}(\vec{n}) = \frac{a_{g'q} + a_{gq} - a_{g'g}}{a_{g'g}a_{g'q}}.$$

$$\left\langle V^{g'}_{g(q)} \right\rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} V^{g'}_{g(q)}(\vec{n}) = \frac{2}{a_{g'g}} \vartheta(a_{gq} - a_{g'g})$$

$$a_{gq} = 1 - \cos\Theta, \quad a_{g'g} = 1 - \cos\Theta'$$

- $\Theta' \leq \Theta$ (MLLA exact AO), $\Theta_0 \geq \Theta$ (kinematics)

Computation of the azimuthal average:



- $\vec{n}_1$ and $\vec{n}_2$ are kept fixed, $\vec{n}$ rotates around $\vec{n}_2$.

$$a_{ij} = 1 - \vec{n}_i \cdot \vec{n}_j = 1 - \cos\Theta_{ij}$$

$$< \mathcal{J} >_{azim} = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \frac{\vec{n}_1 \cdot \vec{n}_2 - (\vec{n} \cdot \vec{n}_1)(\vec{n} \cdot \vec{n}_2)}{(1 - \vec{n} \cdot \vec{n}_1)(1 - \vec{n} \cdot \vec{n}_2)}$$

$$\vec{n}_1 \cdot \vec{n}_2 = \cos\Theta_{12}$$

$$\vec{n} \cdot \vec{n}_2 = \cos\Theta_2$$

$$\vec{n} \cdot \vec{n}_1 = \cos\Theta_1 = \vec{n}_z \cdot \vec{n}_{1z} + \vec{n}^{\perp} \cdot \vec{n}_1^{\perp}$$

$$= \cos\Theta_2 \cos\Theta_{12} + \sin\Theta_{12} \sin\Theta_2 \cos\phi$$

$$< \mathcal{J} >_{azim} = \frac{\cos\Theta_2}{1 - \cos\Theta_2} + \frac{\cos\Theta_{12} - \cos\Theta_2}{1 - \cos\Theta_2} \left\langle \frac{1}{1 - \cos\Theta_1} \right\rangle_{\phi}$$

Useful integral: $\displaystyle\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \arctan\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right)$

$$< V_2 >_{azim} = < \mathcal{R}_2 - \mathcal{J} >_{azim} = \frac{2}{a_2} \vartheta(a_{12} - a_2).$$

Method of Generating Functionals

- Introduced in *Jet Calculus* by Konish, Ukawa and Veneziano (1979)

- Taylor expansion of $Z(u) = \exp(u)$ can be said to generate $a_n = 1$ series according to the expansion:

$$a_n \equiv \left(\frac{d}{du}\right)^n Z(u)\bigg|_{\{u=0\}}$$

- natural numbers $a_0 = 0,\ a_1 = 1,\ a_2 = 2 \ldots a_n = n$:

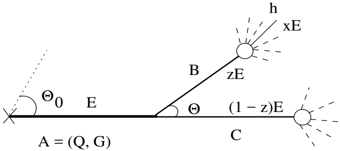
$$Z(u) = u\exp(u)$$

- Hermite polynomials $H_n(x)$:

$$Z(u;x) = \exp(2xu - u^2)$$

- $N-$gluon production cross section as $N-$th term in a series expansion of some "generating" object which contains the overall QCD cascade picture.

- From $Z \to$ MLLA Master Equation (exact Angular Ordering):



$$\frac{d}{d\ln\Theta} Z_A\left(p,\Theta;\{u\}\right) = \frac{1}{2}\sum_{B,C}\int_0^1 dz\, \Phi_A^{B[C]}(z)\, \frac{\alpha_s\left(k_\perp^2\right)}{\pi}$$
$$\Big(Z_B\left(zp,\Theta;\{u\}\right)\, Z_C\left((1-z)p,\Theta;\{u\}\right) - Z_A\left(p,\Theta;\{u\}\right)\Big)$$
$$Z \propto \exp\int^t \gamma(t')dt'; \quad \gamma \simeq 1 + \sqrt{\alpha_s} + \alpha_s + \dots$$

- Exact solution of approached integro-differential equations: MLLA evolution equations at $x \ll 1$:
  - for the one-particle inclusive distributions: $D = \frac{\delta}{\delta u} Z(u)$,
  - for n-particle correlations inside jets: $D^{(n)} = \frac{\delta^n}{\delta u_1 \dots \delta u_n} Z(u)$.