Schwarz maps for the hypergeometric function

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The *hypergeometric differential equation* $E(a,b,c)$:

$$x(1-x)u''+\{c-(a+b+1)x\}u'-abu=0,$$

sometimes written as

$$E_{2,4}(a),\ a=(a_1,\dots,a_4),\ \sum a_i=2.$$

Fundamental solutions, e.g:

$$u_0(x) = \int_0^x \omega,\ u_1(x) = \int_1^x \omega,$$

$$\begin{aligned} \omega &= t^{b-c}(1-t)^{c-a-1}(x-t)^{-b}dt \\ &= t^{a_1-1}(1-t)^{a_2-1}(x-t)^{a_3-1}dt. \end{aligned}$$

Not single valued.

The *Schwarz map* (end of 19C):

$$s : X \ni x \longmapsto u_0(x) : u_1(x) \in \boldsymbol{P}^1,$$

$X = \boldsymbol{C} - \{0, 1\}$,

$\boldsymbol{P}^1$: the complex projective line.

Example: $E(1/2, 1/2, 1) = E_{2,4}(1/2, \ldots, 1/2)$

I will come back to this map later.

By analytic continuation along a loop γ

$$(u_0, u_1) \quad \text{changes into} \quad (u_0, u_1)M_\gamma,$$

where $M_\gamma \in \mathrm{GL}(2)$ is called a circuit matrix along γ.

The circuit matrices form a group called *the monodromy group* Monod.

Easy to get a set of generators,say $M_{\gamma_0}, M_{\gamma_1}$.

The entries of the generators are rational functions in $\{e^{2\pi ia}, e^{2\pi ib}, e^{2\pi ic}\}$.

Easy: if a, b, c are real, then there is a hermitian matrix H, such that

$$MHM^* = H, \qquad M \in \text{Monod}.$$

Though the solutions u_0 and u_1 are not single valued, the quantity

$$(u_0, u_1)H \begin{pmatrix} \overline{u_0} \\ \overline{u_1} \end{pmatrix}$$

is single valued; H is called *the invariant hermitian form.*

This form determins the image of the Schwarz map.

Question: Why such H exists?

For the hypergeometric equation $E(a, b, c)$ there is no need to ask so, because we have explicit expression of M_{γ_0}, M_{γ_1}, and H.

About thirty years ago I encountered an integrable system of linear partial differential equations sometimes called the Aomoto-Gelfand hypergeometric equation

$$E_{3,6}(a),\ a = (a_1, \dots, a_6),\ \sum a_j = 3$$

anyway it is

of rank 6 in 4 variables.

Solutions expressed by integrals:

$$u(x) = \int_D \prod_{j=1}^{5} \ell_j(x)^{a_j - 1} ds dt,$$

$$\ell_j(x): \text{ linear in } (s,t).$$

The Schwarz map is defined by

$$s : \mathbf{C}^4 - D \ni x \mapsto u_0(x) : \cdots : u_5(x) \in \mathbf{P}^5,$$

where D is a divisor, the singular locus of the system.

$E_{3,6}(1/2, \ldots, 1/2)$ is specially interesting algebro-geometrically. In this case, there is an invariant hermitian form H, and the image of s is determined as

$$(z_0, \ldots, z_5) H\, {}^t(z_0, \ldots, z_5) = 0,$$

$$(z_0, \ldots, z_5) H (z_0, \ldots, z_5)^* > 0,$$

a so-called type IV symmetric space.

The inverse map is an automorphic map, which is not the subject today.

Question: For general real parameters a, an invariant hermitian form exists?

We found a set of generators M_j of the Monodromy group of $E_{3,6}(a)$; it is a hard work. And solved the system of linear equations

$$M_j H M_j^* = H, \quad j = 1, 2, ..$$

The result was surprisingly simple.

There must be a reason for the invariant form to exist.

If we can evaluate the *intersection number* $\overline{D}_i \cdot D_j$ for two *loaded* domains D_i and D_j of integration of two bases u_i and u_j, the intersection matrix

$$I := (\overline{D}_i \cdot D_j) = \begin{pmatrix} \overline{D}_1 \\ \overline{D}_2 \\ \vdots \end{pmatrix} (D_1, D_2, \dots)$$

is naturally invariant under small topological changes of the domains, so invariant under the monodromy group:

$$M^* I M = I, \quad M \in \text{Monod}.$$

We found that

$$H = I^{-1}$$

so easy so obvious.

Today's audience know well how to evaluate, I think, because on the poster of this conference, I found a fundamental example:

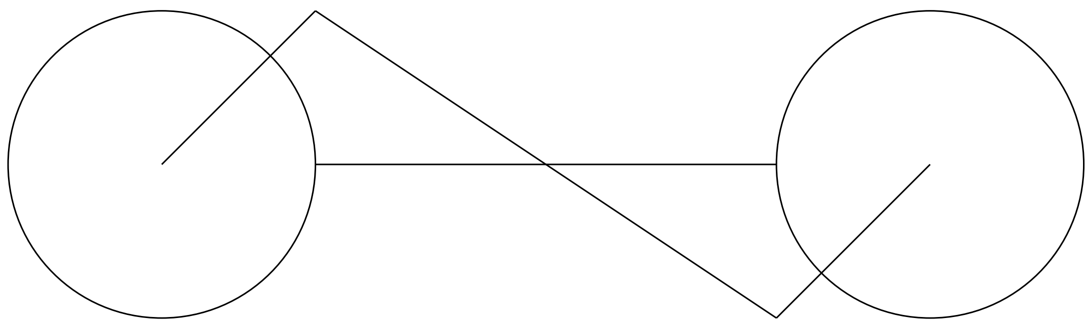

Figure 1: Eevaluating an intersection number

I and my collaborators were happy to find a way to evaluate these intersection numbers.

But among mathematicians this was and has been not popular at all.

So when I got an email from Sebastian Mizera two years ago that this can be used in quantum field theory, I was just surprised,

thank you.

Recall the classical *Schwarz map*:

$$s : X \ni x \longmapsto u_0(x) : u_1(x) \in \boldsymbol{P}^1.$$

For years, I felt: the target is not exactly the correct one.

Even if the monodromy group of s is discrete in $\mathrm{GL}_2(\boldsymbol{C})$, it does not, in general, act properly discontinuously on any non-empty open set of the target $\boldsymbol{P}^1$, and so the image would be chaotic.

The 1-dim complex projective space $\boldsymbol{P}^1$ is also called the Riemann sphere; yes, it is a sphere. The group $\mathrm{PGL}_2(\boldsymbol{C})$ acts naturally on the inside of the sphere not the skin! Inside of the sphere, the ball, is the 3-dim hyperbolic space $\boldsymbol{H}^3$ equipped with the motion group $\mathrm{PGL}_2(\boldsymbol{C})$.

For years, I dreamed a correct Schwarz map with target $\boldsymbol{H}^3$, which should be called the *hyperbolic Schwarz map*.

About twenty years ago, I got it, once it is found, it is simple and natural, of course.

Change the equation $E(a,b,c)$ into the so-called SL-form:

$$u'' - q(x)u = 0,$$

and transform it to the matrix equation

$$\frac{d}{dx}\begin{pmatrix} u \\ u' \end{pmatrix} = \Omega \begin{pmatrix} u \\ u' \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 1 \\ q(x) & 0 \end{pmatrix}.$$

We now define the *hyperbolic Schwarz map*, denoted by S, as the composition of the (multi-valued) map

$$S : X \ni x \longmapsto U(x)^* \, U(x) \in \mathrm{Her}^+(2)$$

and the natural projection

$$\mathrm{Her}^+(2) \to \boldsymbol{H}^3 := \mathrm{Her}^+(2)/\boldsymbol{R}^\times_{>0},$$

where

$$U(x) = \begin{pmatrix} u_0 & u_1 \\ u_0' & u_1' \end{pmatrix} (x)$$

is a fundamental solution of the system, $\mathrm{Her}^+(2)$ the space of positive-definite Hermitian matrices of size 2.

Note that the target of the hyperbolic Schwarz map is $\boldsymbol{H}^3$, whose boundary is $\boldsymbol{P}^1$, which is the target of the Schwarz map.

In this sense, our hyperbolic Schwarz map is a lift-to-the-air of the Schwarz map.

Note also that the monodromy group of the system acts naturally on $\boldsymbol{H}^3$:

$$W = U(x)^* U(x) \to M^* W M, \quad M \in \text{Monod.}$$

The image surface (of X under S) has the following geometrically nice properties:

- It has singularities along the image of the curve

$$C := \{x \in \mathbf{C}; |q| = 1\}.$$

  Generic singularities of flat fronts are cuspidal edges and swallowtail singularities

- From an image point $S(x)$, extend the normal to hit the ideal boundary $\mathbf{P}^1$ at two points: one is the Schwarz image $s(x)$, and the other is the derived Schwarz image $s'(x)$, where

$$s' : X \ni x \longmapsto u_0'(x) : u_1'(x) \in \mathbf{P}^1.$$

- If Monod is discrete in $\mathrm{GL}(2, \mathbf{C})$, then the image is a closed surface in $\boldsymbol{H}^3$.

Examples:

Monod = Dihedral group $D_{2\cdot3}$

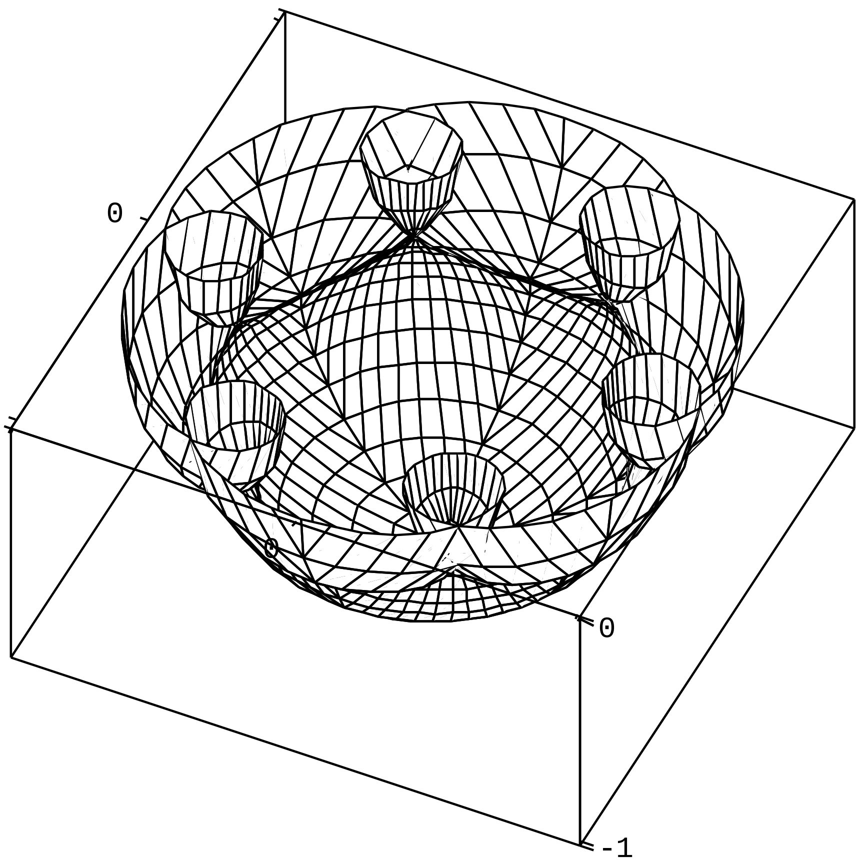

Figure 2: Image under S

Monod = a Fuchsian group $\Gamma(2)$

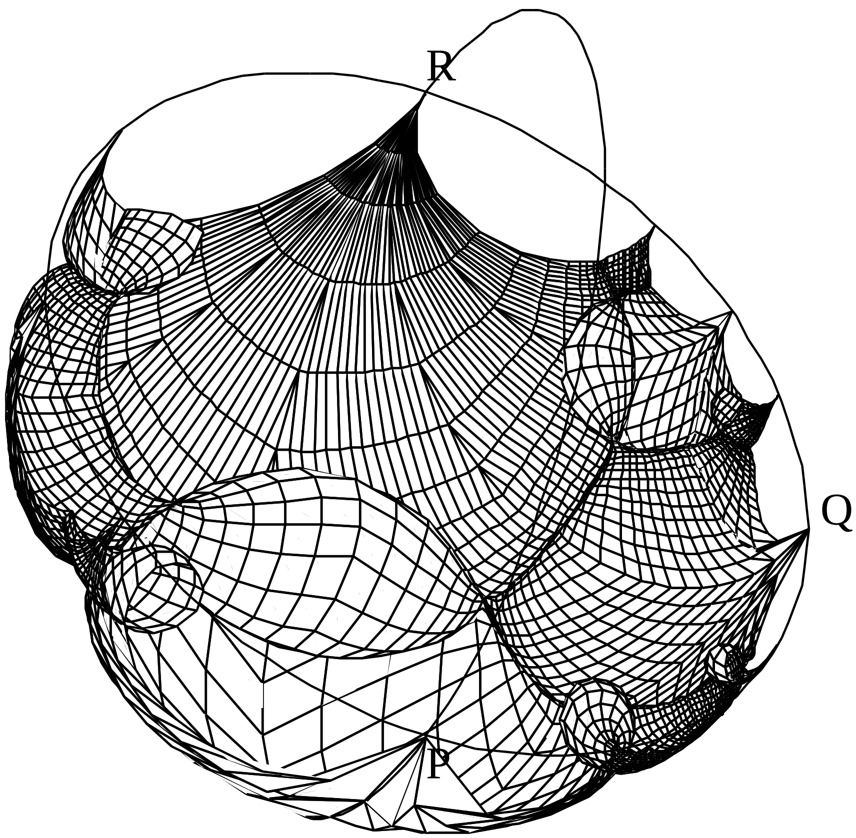

Figure 3: Image under S

Thank you