Massively Parallel Computer Algebra with Applications to Feynman Integrals

Janko Boehm

Technische Universität Kaiserslautern

December 19, 2019

supported by SFB-TRR 195

Outline

- What Computer Algebra can offer

Outline

- What Computer Algebra can offer

- Massively parallel methods
- Application to classical problems

Outline

- What Computer Algebra can offer

- Massively parallel methods
- Application to classical problems

- Integration-by-parts identities for Feynman integrals

Outline

- What Computer Algebra can offer

- Massively parallel methods
- Application to classical problems

- Integration-by-parts identities for Feynman integrals

- Tropical mirror symmetry and Feynman integrals

Singular

- Open Source computer algebra system for polynomial computations, over 30 development teams worldwide, over 140 libraries.

https://www.singular.uni-kl.de/

- Founded by G.-M. Greuel, G. Pfister, H. Schönemann. Current Head: W. Decker

Features of Singular

Commutative Algebra:

- Gröbner bases over fields and integers, free resolutions
- Local computations
- Normalization
- Primary decomposition, factorization
- Invariant theory
- Non-commutative subsystem

Features of Singular

Commutative Algebra:

- Gröbner bases over fields and integers, free resolutions
- Local computations
- Normalization
- Primary decomposition, factorization
- Invariant theory
- Non-commutative subsystem

Algebraic and Tropical Geometry:

- Classification of Singularities
- Resolution of singularities
- Deformation theory
- Sheaf cohomology
- DeRham cohomology
- Rational parametrization
- Tropicalization
- GIT fans

Non-linear algebra: Gröbner Bases

Division with remainder in one variable successively eliminates the highest power.

Non-linear algebra: Gröbner Bases

Division with remainder in one variable successively eliminates the highest power. In more than one variable we have to fix a **monomial ordering** (total ordering compatible with multiplication).

Non-linear algebra: Gröbner Bases

Division with remainder in one variable successively eliminates the highest power. In more than one variable we have to fix a **monomial ordering** (total ordering compatible with multiplication).

Divide $x^2 - y^2$ durch $x^2 + y$ und $xy + x$ with respect to lexicographic ordering.

$$\begin{array}{ll} x^2 - y^2 = & 1 \cdot (x^2 + y) + (-y^2 - y) \\ \underline{x^2 + y} & \\ -y^2 - y & \end{array}$$

Non-linear algebra: Gröbner Bases

Division with remainder in one variable successively eliminates the highest power. In more than one variable we have to fix a **monomial ordering** (total ordering compatible with multiplication).

Divide $x^2 - y^2$ durch $x^2 + y$ und $xy + x$ with respect to lexicographic ordering.

$$\begin{array}{rl} x^2 - y^2 = & 1 \cdot \left(x^2 + y\right) + \left(-y^2 - y\right) \\ x^2 + y & \\ \hline -y^2 - y & \end{array}$$

so remainder $\neq 0$, but

$$x^2 - y^2 = -y\left(x^2 + y\right) + x\left(xy + x\right) \in I := \left\langle x^2 + y, xy + x \right\rangle$$

Non-linear algebra: Gröbner Bases

Division with remainder in one variable successively eliminates the highest power. In more than one variable we have to fix a **monomial ordering** (total ordering compatible with multiplication).
Divide $x^2 - y^2$ durch $x^2 + y$ und $xy + x$ with respect to lexicographic ordering.

$$\begin{array}{rl} x^2 - y^2 = & 1 \cdot (x^2 + y) + (-y^2 - y) \\ x^2 + y & \\ \hline -y^2 - y & \end{array}$$

so remainder $\neq 0$, but

$$x^2 - y^2 = -y\left(x^2 + y\right) + x\left(xy + x\right) \in I := \left\langle x^2 + y, xy + x \right\rangle$$

Problem: Lead terms cancel, division algorithm can't do that.

Non-linear algebra: Gröbner Bases

Division with remainder in one variable successively eliminates the highest power. In more than one variable we have to fix a **monomial ordering** (total ordering compatible with multiplication).
Divide $x^2 - y^2$ durch $x^2 + y$ und $xy + x$ with respect to lexicographic ordering.

$$\begin{array}{rl} x^2 - y^2 = & 1 \cdot (x^2 + y) + (-y^2 - y) \\ x^2 + y & \\ \hline -y^2 - y & \end{array}$$

so remainder $\neq 0$, but

$$x^2 - y^2 = -y\left(x^2 + y\right) + x\left(xy + x\right) \in I := \left\langle x^2 + y, xy + x \right\rangle$$

Problem: Lead terms cancel, division algorithm can't do that.
Solution: Add $y^2 + y$ to the divisor set. The result is a **Gröbner basis** of I. Then

Non-linear algebra: Gröbner Bases

Division with remainder in one variable successively eliminates the highest power. In more than one variable we have to fix a **monomial ordering** (total ordering compatible with multiplication).
Divide $x^2 - y^2$ durch $x^2 + y$ und $xy + x$ with respect to lexicographic ordering.

$$\begin{array}{rl} x^2 - y^2 = & 1 \cdot \left(x^2 + y\right) + \left(-y^2 - y\right) \\ x^2 + y \\ \hline -y^2 - y \end{array}$$

so remainder $\neq 0$, but

$$x^2 - y^2 = -y\left(x^2 + y\right) + x\left(xy + x\right) \in I := \left\langle x^2 + y, xy + x\right\rangle$$

Problem: Lead terms cancel, division algorithm can't do that.
Solution: Add $y^2 + y$ to the divisor set. The result is a **Gröbner basis** of I. Then

$$f \in I \Longleftrightarrow NF(f, G) = 0$$

The Main Computational Tool: Gröbner Bases

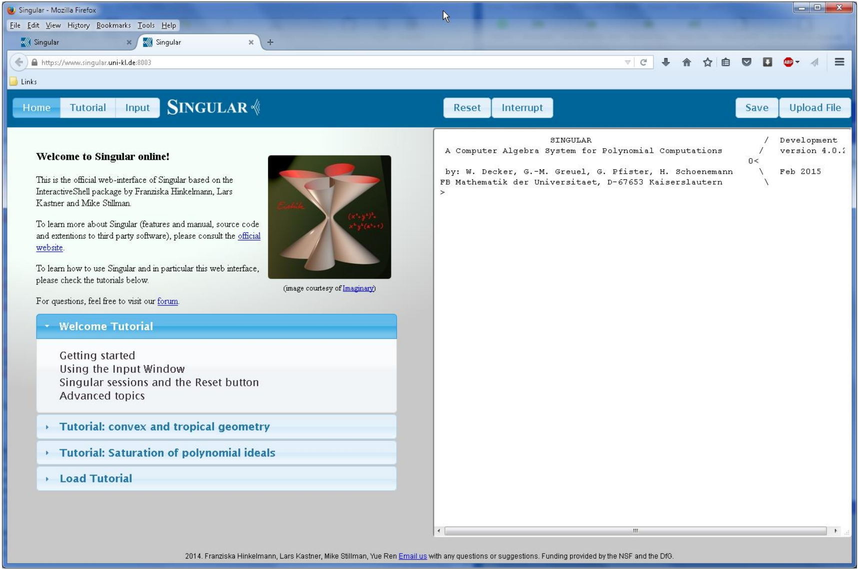
Gröbner Bases can be used for fundamental computations with ideals and modules:

Example

- eliminate variables,
- determine intersections,
- compute syzygies (polynomial relations),
- compute ideal quotients and saturations,
- birational geometry.

Greuel, Pfister: *A Singular Introduction to Commutative Algebra.* Springer.

Singular Online



OSCAR

Cornerstone of next generation Open Source Computeralgebrasystem OSCAR developed in SFB TRR 195 "Symbolic Tools in Mathematics and Their Application":

OSCAR

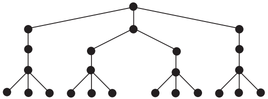
Cornerstone of next generation Open Source Computeralgebrasystem OSCAR developed in SFB TRR 195 "Symbolic Tools in Mathematics and Their Application":

Resolution of Singularities

Hironaka used order sequence of a local Gröbner basis (standard basis) to prove existence of resolution of singularities.

Hironaka (1964)

Can be turned into an effective criterion for smoothness iteratively generating a tree of charts

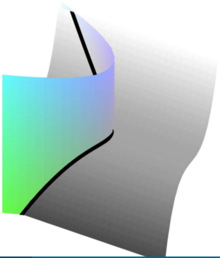


s.t. in every chart the smooth variety is a smooth complete intersection:

JB, Frühbis-Krüger (2017)

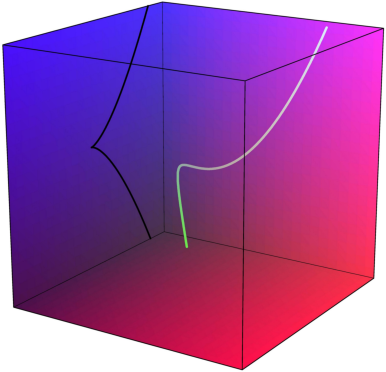
$$\left\{ \begin{array}{r} x_0x_3 - x_1x_2 = 0 \\ x_1^2 - x_0x_2 = 0 \\ x_1x_3 - x_2^2 = 0 \end{array} \right\} \cap \{x_0 \neq 0\}$$

$$= \left\{ \begin{array}{r} x_3 - x_1x_2 = 0 \\ x_1^2 - x_2 = 0 \end{array} \right\}$$



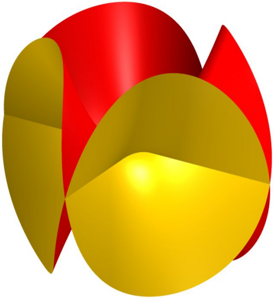
Resolution of Singularities

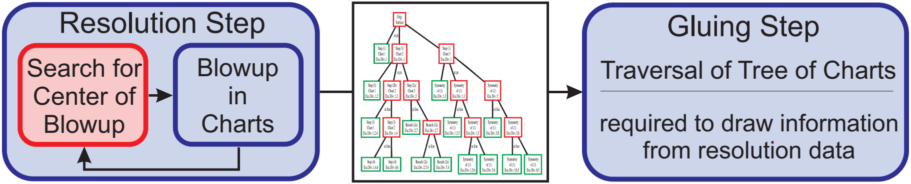
If not smooth, loci for interative blowup can be found:



- Bravo, Encinas, Villamayor (2005).
- Frühbis-Krüger, Pfister (2007).

Resolution of Singularities





Massively parallel methods in computer algebra

Framework for massively parallel computations in computer algebra by joining Singular with the workflow management system GPI-Space, developed at Fraunhofer Institute für Industrial Mathematics ITWM.

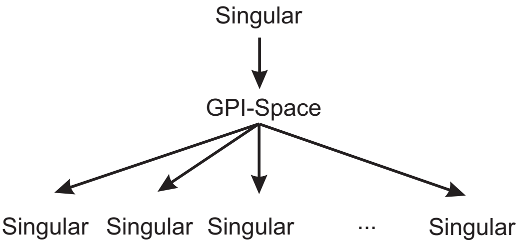
Framework for massively parallel computations in computer algebra by joining SINGULAR with the workflow management system GPI-SPACE, developed at Fraunhofer Institute für Industrial Mathematics ITWM.

JB, Decker, Frühbis-Krüger, Pfreundt, Rahn, Ristau, 2018.

Massively parallel methods in computer algebra

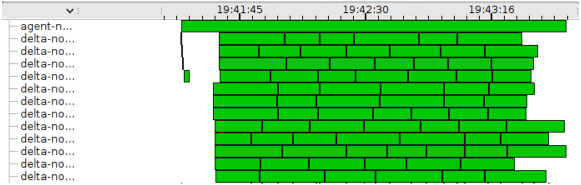
Framework for massively parallel computations in computer algebra by joining SINGULAR with the workflow management system GPI-SPACE, developed at Fraunhofer Institute für Industrial Mathematics ITWM.

JB, Decker, Frühbis-Krüger, Pfreundt, Rahn, Ristau, 2018.



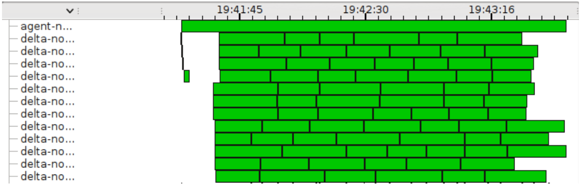
GPI-Space

- Distributed runtime system for parallel computations.



GPI-Space

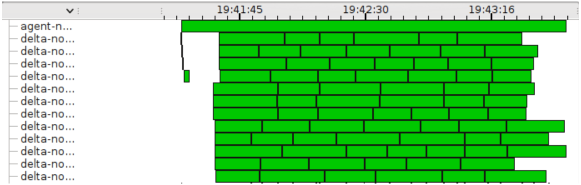
- Distributed runtime system for parallel computations.



- Based on idea of **separation of computation and coordination**.

GPI-Space

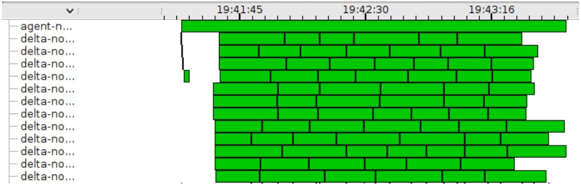
- Distributed runtime system for parallel computations.



- Based on idea of **separation of computation and coordination**.
- Specialized language for coordination layer: **Petri nets**.

GPI-Space

- Distributed runtime system for parallel computations.



- Based on idea of **separation of computation and coordination**.
- Specialized language for coordination layer: **Petri nets**.
- In the computation layer, existing applications (e.g. SINGULAR) can be used and mixed as long as they can be called as a C-library.

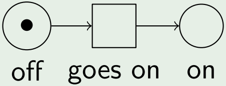
Petri nets

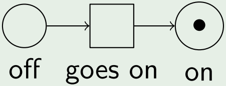
Introduced by Carl Adam Petri (1926–2010) in 1962 as a graphical way to describe concurrent asynchronous systems, Petri nets are bipartite, directed graphs consisting out of **places** and **transitions**. By assigning to places a number of **tokens**, a state is described. Transitions can fire if all input places hold a token, consume one token from each input place, and put one token on each output place.

Petri nets

Introduced by Carl Adam Petri (1926–2010) in 1962 as a graphical way to describe concurrent asynchronous systems, Petri nets are bipartite, directed graphs consisting out of **places** and **transitions**. By assigning to places a number of **tokens**, a state is described. Transitions can fire if all input places hold a token, consume one token from each input place, and put one token on each output place.

Example



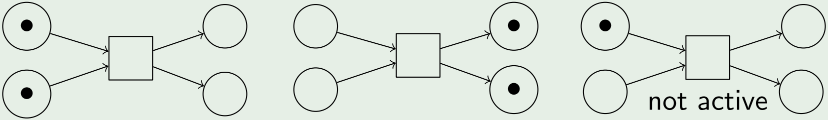


Petri nets

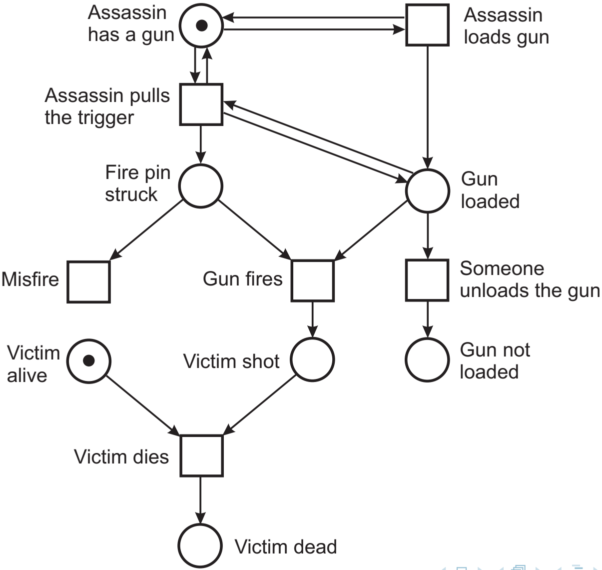
Introduced by Carl Adam Petri (1926–2010) in 1962 as a graphical way to describe concurrent asynchronous systems, Petri nets are bipartite, directed graphs consisting out of **places** and **transitions**. By assigning to places a number of **tokens**, a state is described. Transitions can fire if all input places hold a token, consume one token from each input place, and put one token on each output place.

Example

Example

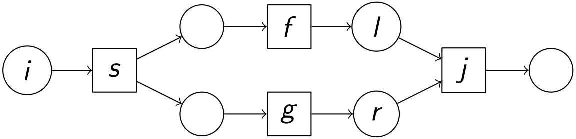


An Example of a Petri Net



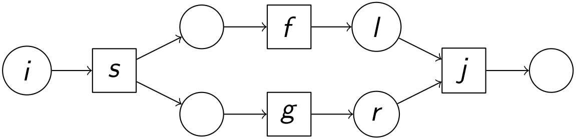
Features of Petri nets

- Task parallelism:
  Transitions f and g can fire in parallel:

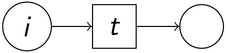


Features of Petri nets

- Task parallelism:
  Transitions f and g can fire in parallel:

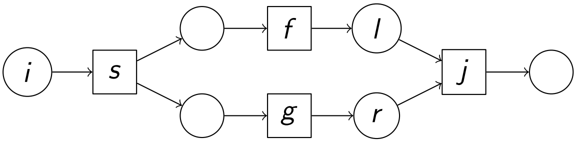


- Data parallelism:
  If i holds multiple tokens, t can fire in parallel:

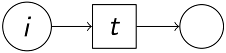


Features of Petri nets

- Task parallelism:
  Transitions f and g can fire in parallel:



- Data parallelism:
  If i holds multiple tokens, t can fire in parallel:



Real world implementation:

- Transitions take time.
- Tokens can be complex data structures.
- Transitions can impose conditions on input tokens.

Applications

- Classical Algebraic Geometry:
  - Determining smoothness of algebraic varieties
    JB, Decker, Frühbis-Krüger, Ristau
  - Resolution of singularities
    Frühbis-Krüger, Ristau, Schober

Applications

- Classical Algebraic Geometry:
  - Determining smoothness of algebraic varieties
    JB, Decker, Frühbis-Krüger, Ristau
  - Resolution of singularities
    Frühbis-Krüger, Ristau, Schober
- Geometric Invariant Theory: Computing GIT-fans with symmetry
  JB, Frühbis-Krüger, Reinbold
- Tropical Geometry: Tropicalization of algebraic varieties with symmetry
  Bendle, JB, Ren
- Physics: Integration-by-parts identities for Feynman integrals
  Bendle, JB, Decker, Georgoudis, Rahn, Pfreundt, Wasser, Zhang
- Algebraic/Tropical Geometry, Physics: Generating functions and Recursions for Gromov-Witten invariants
  JB, Bringmann, Buchholz, Goldner, Markwig, Ristau

Parallelism in Algebraic Geometry

- Key concept in algebraic geometry:
    - Description of schemes and sheaves in terms of coverings by charts.

- Key concept in algebraic geometry:
  - Description of schemes and sheaves in terms of coverings by charts.
  - Global properties are related to local ones in the individual charts.

Parallelism in Algebraic Geometry

- Key concept in algebraic geometry:
  - Description of schemes and sheaves in terms of coverings by charts.
  - Global properties are related to local ones in the individual charts.
- Use this natural parallel structure algorithmically?

Parallelism in Algebraic Geometry

- Key concept in algebraic geometry:
  - Description of schemes and sheaves in terms of coverings by charts.
  - Global properties are related to local ones in the individual charts.
- Use this natural parallel structure algorithmically?
- Computational basis of most algorithms is Buchberger's Algorithm.

Parallelism in Algebraic Geometry

- Key concept in algebraic geometry:
  - Description of schemes and sheaves in terms of coverings by charts.
  - Global properties are related to local ones in the individual charts.
- Use this natural parallel structure algorithmically?
- Computational basis of most algorithms is Buchberger's Algorithm.
- Buchberger's Algorithm has doubly exponential worst case complexity, much faster in many practical examples of interest $\rightarrow$ unpredictable.

Mayr, Meyer (1982)

Parallelism in Algebraic Geometry

- Key concept in algebraic geometry:
  - Description of schemes and sheaves in terms of coverings by charts.
  - Global properties are related to local ones in the individual charts.
- Use this natural parallel structure algorithmically?
- Computational basis of most algorithms is Buchberger's Algorithm.
- Buchberger's Algorithm has doubly exponential worst case complexity, much faster in many practical examples of interest $\rightarrow$ unpredictable.

Mayr, Meyer (1982)

- $\rightarrow$ Single chart may dominate the run-time.

Parallelism in Algebraic Geometry

- Key concept in algebraic geometry:
  - Description of schemes and sheaves in terms of coverings by charts.
  - Global properties are related to local ones in the individual charts.
- Use this natural parallel structure algorithmically?
- Computational basis of most algorithms is Buchberger's Algorithm.
- Buchberger's Algorithm has doubly exponential worst case complexity, much faster in many practical examples of interest $\rightarrow$ unpredictable.

Mayr, Meyer (1982)

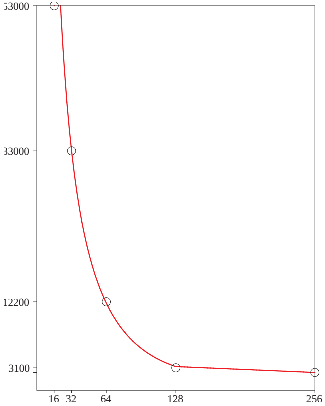
- $\rightarrow$ Single chart may dominate the run-time.
- Solution: Model algorithm in a parallel way s.t. it automatically finds a good cover.

Performance of smoothness certificate

Superlinear speedup for numerical Godeaux surface of codim 11 constructed by

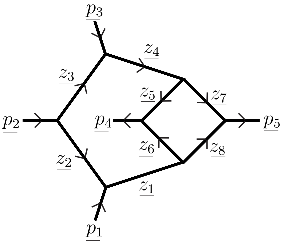
Schreyer, Stenger (2018)

while Jacobian criterion not feasable.



Integration-by-parts identities for Feynman integrals

Feynman integral



$$\longmapsto \quad \int d^D \underline{z_1} \int d^D \underline{z_5} \frac{N(p,z)}{\underline{z_1}^2 \cdot \ldots \cdot \underline{z_8}^2}$$

subject to impuls conservation: $\underline{z_2} + \underline{z_3} - \underline{p_2} = 0, \ldots$

Integration-by-parts identities for Feynman integrals

- In **Baikov representation** in terms of independent variable scalar products

$$x_1 = \underline{z_1}^2, \ldots, x_8 = \underline{z_8}^2, \; x_9, x_{10}, x_{11}$$

between vectors z_i, p_j and independent constant scalar products

$$c_1, \ldots, c_5$$

between p_i,

Integration-by-parts identities for Feynman integrals

- In **Baikov representation** in terms of independent variable scalar products

$$x_1 = \underline{z_1}^2, \ldots, x_8 = \underline{z_8}^2,\ x_9, x_{10}, x_{11}$$

between vectors z_i, p_j and independent constant scalar products

$$c_1, \ldots, c_5$$

between p_i, integral can be expressed as linear combination of integrals of form

$$\int dx_1 \ldots \int dx_{11} \frac{\det(G(c,x))^{\frac{D-L-E-1}{2}}}{x_1^{a_1} \cdot \ldots \cdot x_8^{a_8} \cdot x_9^{a_9} \cdot x_{10}^{a_{10}} \cdot x_{11}^{a_{11}}} \quad \text{with} \quad \begin{array}{l} a_1, \ldots, a_8 \leq -1 \\ a_9, a_{10}, a_{11} \leq 0 \end{array}$$

where G is Gram matrix, $L = 2$ genus of graph, $E = 4$ number of independent external momenta.

Integration-by-parts identities for Feynman integrals

- In **Baikov representation** in terms of independent variable scalar products

$$x_1 = \underline{z_1}^2, \ldots, x_8 = \underline{z_8}^2,\ x_9, x_{10}, x_{11}$$

between vectors z_i, p_j and independent constant scalar products

$$c_1, \ldots, c_5$$

between p_i, integral can be expressed as linear combination of integrals of form

$$\int dx_1 \ldots \int dx_{11} \frac{\det(G(c,x))^{\frac{D-L-E-1}{2}}}{x_1^{a_1} \cdot \ldots \cdot x_8^{a_8} \cdot x_9^{a_9} \cdot x_{10}^{a_{10}} \cdot x_{11}^{a_{11}}} \quad \text{with} \quad \begin{array}{l} a_1, \ldots, a_8 \leq -1 \\ a_9, a_{10}, a_{11} \leq 0 \end{array}$$

where G is Gram matrix, $L = 2$ genus of graph, $E = 4$ number of independent external momenta.

- **Integration-by-parts identities** express huge number of such integrals in a small number master integrals.

Integration-by-parts identities for Feynman integrals

For

$$\int dx_1 \ldots \int dx_k \frac{P^{\frac{D-L-E-1}{2}}}{x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}}$$

with Baikov polynomial $P = \det(G(c,x))$,

Integration-by-parts identities for Feynman integrals

For

$$\int dx_1 \dots \int dx_k \frac{P^{\frac{D-L-E-1}{2}}}{x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}}$$

with Baikov polynomial $P = \det(G(c,x))$, integration-by-parts identities are obtained as

$$0 = \int dx_1 \cdots dx_k \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \left(a_i(x) \frac{P^{\frac{D-L-E-1}{2}}}{x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}} \right)$$

Integration-by-parts identities for Feynman integrals

For

$$\int dx_1 \ldots \int dx_k \frac{P^{\frac{D-L-E-1}{2}}}{x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}}$$

with Baikov polynomial $P = \det(G(c,x))$, integration-by-parts identities are obtained as

$$0 = \int dx_1 \cdots dx_k \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \left(a_i(x) \frac{P^{\frac{D-L-E-1}{2}}}{x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}} \right)$$

To avoid shifts in dimension parameter D and to retain $a_i \leq 0$, require

$$\left(\sum_{i=1}^{k} a_i(x) \frac{\partial P}{\partial x_i} \right) + b(x) P = 0\,,$$

$$a_i(x) = b_i(x) x_i\,, \quad i = 1, \ldots, m$$

Integration-by-parts identities for Feynman integrals

Syzygy equation

$$\left(\sum_{i=1}^{k} a_i(x) \frac{\partial P}{\partial x_i} \right) + b(x) P = 0 \qquad (*)$$

can be solved by Laplace expansion on G.

JB, Georgoudis, Larsen, Schulze, Zhang (2018).

Integration-by-parts identities for Feynman integrals

Syzygy equation

$$\left(\sum_{i=1}^{k} a_i(x)\frac{\partial P}{\partial x_i}\right) + b(x)P = 0 \qquad (*)$$

can be solved by Laplace expansion on G.

JB, Georgoudis, Larsen, Schulze, Zhang (2018).

So for the modules

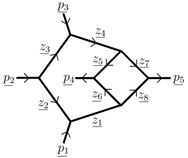
$$M_1 = \langle a(z) \text{ with } (*)\rangle \qquad M_2 = \langle z_i e_i \mid i \leq m\rangle + \langle e_i \mid i > m\rangle$$

calculate $(M_1 \cap M_2)_{\leq d}$ using **non-linear algebra** via Gröbner bases.

Integration-by-Parts Identities for Feynman Integrals

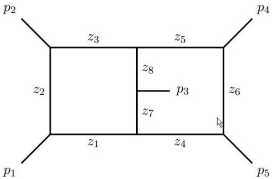
- Full pivoting semi-numeric row reduction over
$$R = \mathbb{Q}(c_1, \ldots, c_r, D)[x_1, \ldots, x_m]$$
using interpolation techniques.

- Full pivoting semi-numeric row reduction over
$$R = \mathbb{Q}(c_1, \ldots, c_r, D)[x_1, \ldots, x_m]$$
using interpolation techniques.
- Handle non-planar hexagon box:



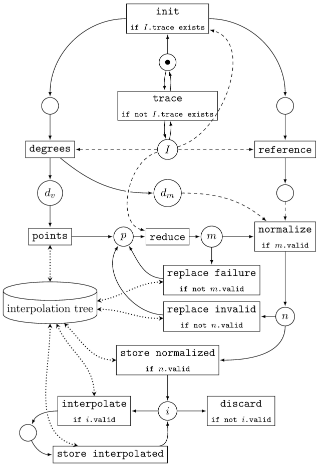
JB, Georgoudis,. Larsen, Schönemann, Zhang (2018).

- Requires massively parallel approach. More difficult:



Reduction with partial interpolation

- find trace with full pivoting
- find degrees by univariate reduction
- dynamic replacement of bad points in reduction
- interpolation tree controls state
- leaves are generic results
- sufficient number of interpolation points triggers interpolation of next parameter

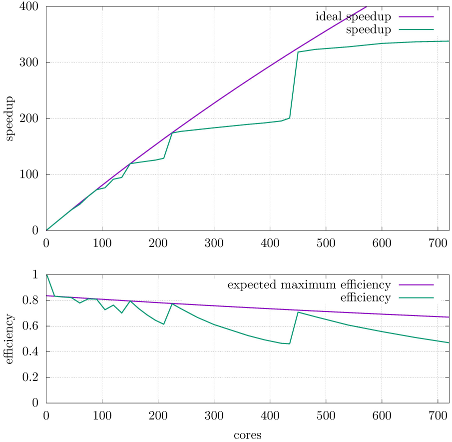


Bendle, JB, Decker, Georgoudis, Rahn, Pfreundt, Wasser, Zhang, arXiv:1908.04301

Reduction with partial interpolation

nodes	cores	runtime	speedup	efficiency
1	1	122857.6	1.000	1.000
1	15	9837.8	12.488	0.832
2	30	4954.8	24.795	0.826
4	60	2625.4	46.794	0.779
8	120	1341.3	91.592	0.763
14	210	952.3	129.011	0.614
15	225	705.6	174.113	0.773
16	240	694.3	176.929	0.737
29	435	611.8	200.810	0.461
30	450	385.4	318.747	0.708
32	480	379.9	323.310	0.673
40	600	367.7	334.109	0.556
48	720	363.2	338.178	0.469

Reduction with partial interpolation



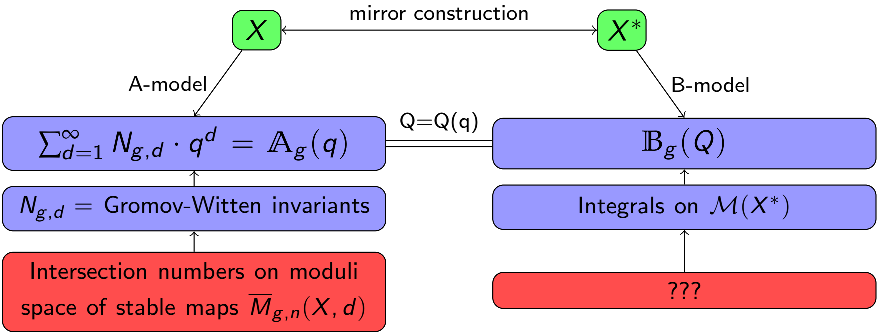
Easiest of 11 cuts of the Feynman diagram. Running time of ≈ 10 minutes on 384 cores. More difficult graphs ≈ 12 hours on 384 cores using interpolation of bidegree up to $(35, 24)$.

Mirror symmetry

For Calabi-Yau variety X (elliptic curve, quintic in $\mathbb{P}^4$,...) and $g \in \mathbb{N}_0$:

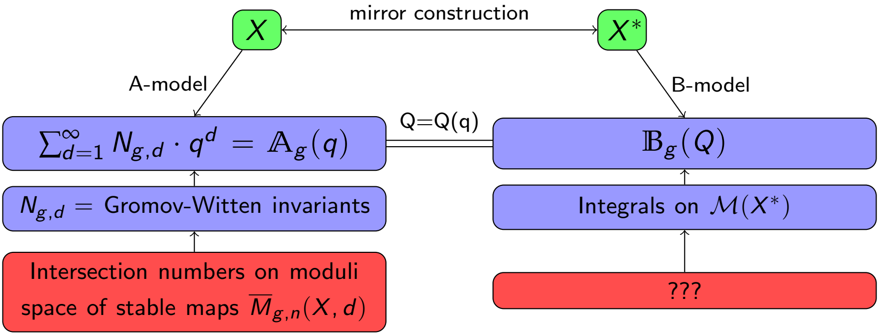
Mirror symmetry

For Calabi-Yau variety X (elliptic curve, quintic in $\mathbb{P}^4$,...) and $g \in \mathbb{N}_0$:



Mirror symmetry

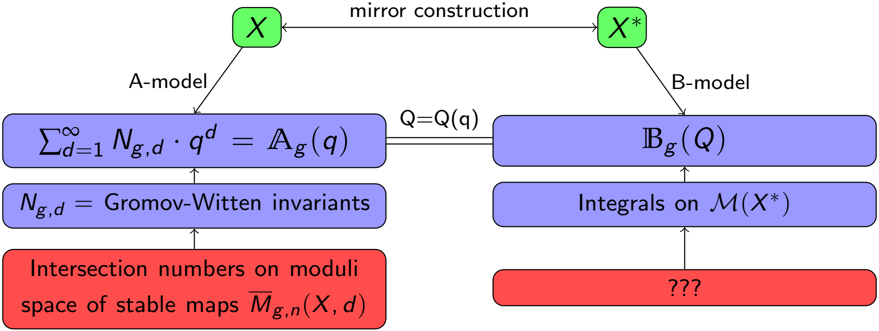
For Calabi-Yau variety X (elliptic curve, quintic in $\mathbb{P}^4$,...) and $g \in \mathbb{N}_0$:



- Mirror constructions: Greene-Plesser '90, Batyrev '93,...

Mirror symmetry

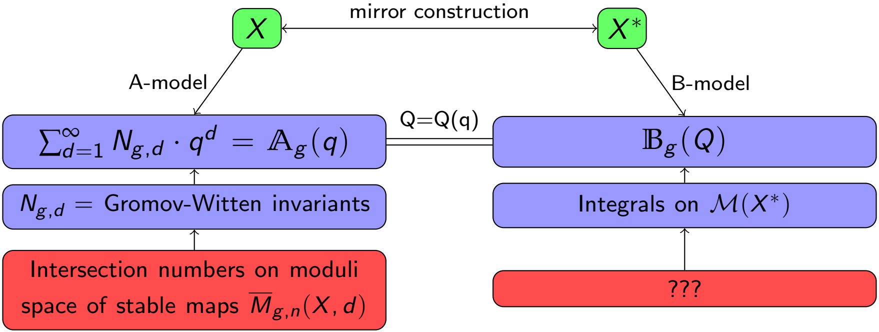
For Calabi-Yau variety X (elliptic curve, quintic in $\mathbb{P}^4$,...) and $g \in \mathbb{N}_0$:



- Mirror constructions: Greene-Plesser '90, Batyrev '93,...
- String theory: Candelas-Horowitz-Strominger-Witten '85, Candelas-de la Ossa-Green-Parkes '91,...

Mirror symmetry

For Calabi-Yau variety X (elliptic curve, quintic in $\mathbb{P}^4$,...) and $g \in \mathbb{N}_0$:



- Mirror constructions: Greene-Plesser '90, Batyrev '93,...
- String theory: Candelas-Horowitz-Strominger-Witten '85, Candelas-de la Ossa-Green-Parkes '91,...
- Algebraic/symplectic geometry: Fulton-Pandharipande '95, Kontsevich '95, Behrend-Fantechi '97,...

Mirror theorems

Theorem (Givental '96, Lian-Liu-Yau '97, Gathmann '03)

$\mathbb{A}_0 = \mathbb{B}_0$ *for quintic hypersurface in* $\mathbb{P}^4$.

Mirror theorems

Theorem (Givental '96, Lian-Liu-Yau '97, Gathmann '03)

$\mathbb{A}_0 = \mathbb{B}_0$ *for quintic hypersurface in* $\mathbb{P}^4$.

$$\Rightarrow \mathbb{A}_0(q) = 23 \cdot 5^3 + \left(4874 \cdot 5^3 + \frac{23 \cdot 5^3}{2^3}\right) \cdot q + \left(2537651 \cdot 5^3 + \frac{23 \cdot 5^3}{3^3}\right) \cdot q^2 + \ldots$$

Mirror theorems

Theorem (Givental '96, Lian-Liu-Yau '97, Gathmann '03)

$\mathbb{A}_0 = \mathbb{B}_0$ *for quintic hypersurface in* $\mathbb{P}^4$.

$$\Rightarrow \mathbb{A}_0(q) = 23 \cdot 5^3 + (4874 \cdot 5^3 + \frac{23 \cdot 5^3}{2^3}) \cdot q + (2537651 \cdot 5^3 + \frac{23 \cdot 5^3}{3^3}) \cdot q^2 + \ldots$$

Is enumerative geometry result on X: number of lines, conics, cubics,...
(number of genus 0 curves on X of degree d, delicate counting).

Mirror theorems

Theorem (Givental '96, Lian-Liu-Yau '97, Gathmann '03)

$\mathbb{A}_0 = \mathbb{B}_0$ *for quintic hypersurface in* $\mathbb{P}^4$.

$$\Rightarrow \mathbb{A}_0(q) = 23 \cdot 5^3 + \left(4874 \cdot 5^3 + \frac{23 \cdot 5^3}{2^3}\right) \cdot q + \left(2537651 \cdot 5^3 + \frac{23 \cdot 5^3}{3^3}\right) \cdot q^2 + \ldots$$

Is enumerative geometry result on X: number of lines, conics, cubics,...
(number of genus 0 curves on X of degree d, delicate counting).

Similar theorems for $g = 0, 1$ in case of degree $n+1$ hypersurfaces in $\mathbb{P}^n$.

Klemm, Pandharipande (2007), Zinger (2007)

Mirror theorems

Theorem (Givental '96, Lian-Liu-Yau '97, Gathmann '03)

$\mathbb{A}_0 = \mathbb{B}_0$ *for quintic hypersurface in* $\mathbb{P}^4$.

$$\Rightarrow \mathbb{A}_0(q) = 23 \cdot 5^3 + \left(4874 \cdot 5^3 + \frac{23 \cdot 5^3}{2^3}\right) \cdot q + \left(2537651 \cdot 5^3 + \frac{23 \cdot 5^3}{3^3}\right) \cdot q^2 + \ldots$$

Is enumerative geometry result on X: number of lines, conics, cubics,... (number of genus 0 curves on X of degree d, delicate counting).

Similar theorems for $g = 0, 1$ in case of degree $n+1$ hypersurfaces in $\mathbb{P}^n$.

Klemm, Pandharipande (2007), Zinger (2007)

Questions:

- Mirror theorems for other Calabi-Yau varieties and $g \geq 2$?

Mirror theorems

Theorem (Givental '96, Lian-Liu-Yau '97, Gathmann '03)

$\mathbb{A}_0 = \mathbb{B}_0$ *for quintic hypersurface in* $\mathbb{P}^4$.

$$\Rightarrow \mathbb{A}_0(q) = 23 \cdot 5^3 + \left(4874 \cdot 5^3 + \frac{23 \cdot 5^3}{2^3}\right) \cdot q + \left(2537651 \cdot 5^3 + \frac{23 \cdot 5^3}{3^3}\right) \cdot q^2 + \ldots$$

Is enumerative geometry result on X: number of lines, conics, cubics,...
(number of genus 0 curves on X of degree d, delicate counting).

Similar theorems for $g = 0, 1$ in case of degree $n+1$ hypersurfaces in $\mathbb{P}^n$.

Klemm, Pandharipande (2007), Zinger (2007)

Questions:

- Mirror theorems for other Calabi-Yau varieties and $g \geq 2$?
- Geometric understanding of mirror theorem beyond combinatorics?

Mirror theorems

Theorem (Givental '96, Lian-Liu-Yau '97, Gathmann '03)

$\mathbb{A}_0 = \mathbb{B}_0$ *for quintic hypersurface in* $\mathbb{P}^4$.

$$\Rightarrow \mathbb{A}_0(q) = 23 \cdot 5^3 + \left(4874 \cdot 5^3 + \frac{23 \cdot 5^3}{2^3}\right) \cdot q + \left(2537651 \cdot 5^3 + \frac{23 \cdot 5^3}{3^3}\right) \cdot q^2 + \ldots$$

Is enumerative geometry result on X: number of lines, conics, cubics,...
(number of genus 0 curves on X of degree d, delicate counting).

Similar theorems for $g = 0, 1$ in case of degree $n+1$ hypersurfaces in $\mathbb{P}^n$.

Klemm, Pandharipande (2007), Zinger (2007)

Questions:

- Mirror theorems for other Calabi-Yau varieties and $g \geq 2$?
- Geometric understanding of mirror theorem beyond combinatorics?
- What are the B-model integrals?

Elliptic curves

Start with easiest Calabi-Yau: elliptic curve E (e.g. smooth plane cubic).
Here, Gromov-Witten numbers are numbers of covers:

Elliptic curves

Start with easiest Calabi-Yau: elliptic curve E (e.g. smooth plane cubic). Here, Gromov-Witten numbers are numbers of covers:

Definition (Hurwitz numbers)

$N_{d,g} = \frac{1}{|\mathrm{Aut}(f)|}$-weighted number of degree d covers $f : C \to E$, where C is smooth of genus g and f has $2g-2$ simple ramifications points.

according to Riemann-Hurwitz formula $2g(C) - 2 = d \cdot (2g(E) - 2) + \sum_{P \in C}(e(P) - 1)$

Elliptic curves

Start with easiest Calabi-Yau: elliptic curve E (e.g. smooth plane cubic). Here, Gromov-Witten numbers are numbers of covers:

Definition (Hurwitz numbers)

$N_{d,g} = \frac{1}{|\mathrm{Aut}(f)|}$-weighted number of degree d covers $f : C \to E$, where C is smooth of genus g and f has $2g-2$ simple ramifications points.

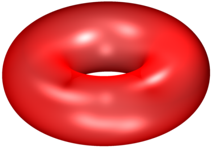
according to Riemann-Hurwitz formula $2g(C) - 2 = d \cdot (2g(E) - 2) + \sum_{P \in C}(e(P) - 1)$

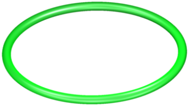
$N_{d,0} = 0$, so have to look at $g \geq 1$ invariants!

Tropical point of view

How to understand *all* $N_{g,d}$? Pass to **tropical geometry**:

$$E \quad \mapsto \quad \mathrm{trop}(E)$$

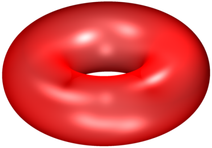


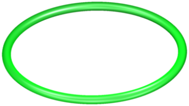


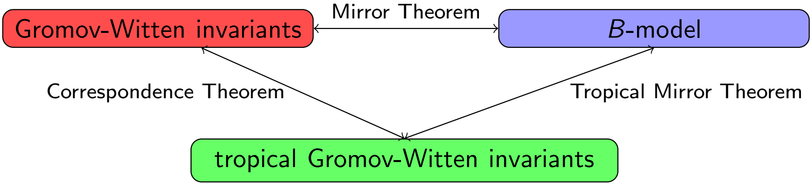
Tropical point of view

How to understand *all* $N_{g,d}$? Pass to **tropical geometry**:

$$E \quad \mapsto \quad \mathrm{trop}(E)$$



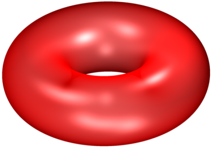


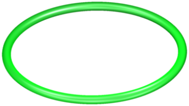


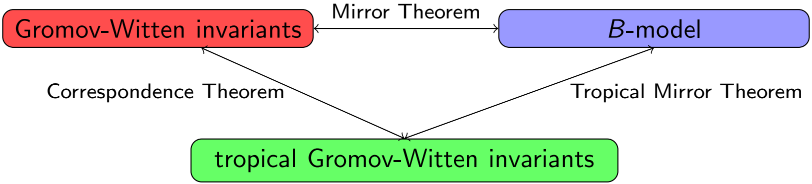
Tropical point of view

How to understand *all* $N_{g,d}$? Pass to **tropical geometry**:

$E \quad \mapsto \quad \mathrm{trop}(E)$



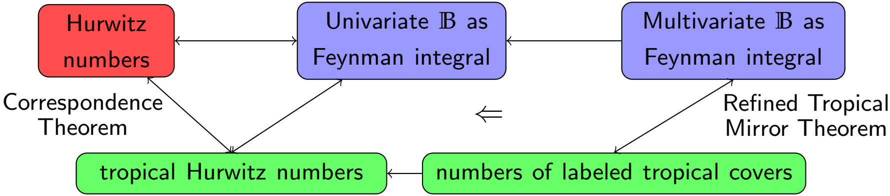




For $X = \mathbb{P}^2$ (building block of C-Y) and $g = 0$ tropical mirror theorem

Gross (2010)

Tropical Mirror Symmetry



- JB, Bringmann, Buchholz, Markwig (2013)
- JB, Goldner, Markwig (2018).

Feynman integrals (B-side)

Definition

A **Feynman graph** is (for us) a 3-valent, connected graph Γ of genus g.

Feynman integrals (B-side)

Definition

A **Feynman graph** is (for us) a 3-valent, connected graph Γ of genus g.

By $g(\Gamma) = 1 - |\text{vert}(\Gamma)| + |\text{edges}(\Gamma)|$ and $3\,|\text{vert}(\Gamma)| = 2\,|\text{edges}(\Gamma)|$

$$|\text{vert}(\Gamma)| = 2g - 2 \qquad |\text{edges}(\Gamma)| = 3g - 3$$

Feynman integrals (B-side)

Definition

A **Feynman graph** is (for us) a 3-valent, connected graph Γ of genus g.

By $g(\Gamma) = 1 - |\mathrm{vert}(\Gamma)| + |\mathrm{edges}(\Gamma)|$ and $3\,|\mathrm{vert}(\Gamma)| = 2\,|\mathrm{edges}(\Gamma)|$

$$|\mathrm{vert}(\Gamma)| = 2g - 2 \qquad |\mathrm{edges}(\Gamma)| = 3g - 3$$

Fix labeling z_i for vertices and q_i for edges.

Feynman integrals (B-side)

Definition

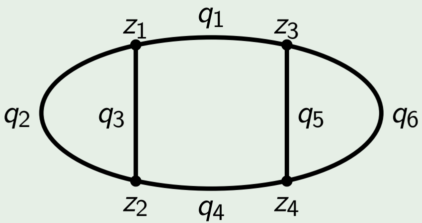
A **Feynman graph** is (for us) a 3-valent, connected graph Γ of genus g.

By $g(\Gamma) = 1 - |\mathrm{vert}(\Gamma)| + |\mathrm{edges}(\Gamma)|$ and $3\,|\mathrm{vert}(\Gamma)| = 2\,|\mathrm{edges}(\Gamma)|$

$$|\mathrm{vert}(\Gamma)| = 2g - 2 \qquad |\mathrm{edges}(\Gamma)| = 3g - 3$$

Fix labeling z_i for vertices and q_i for edges.

Example



Feynman integrals (B-side)

Definition (Propagator)

$$P(z,q) = -\frac{1}{4\pi^2}\wp(z,q) - \frac{1}{12}E_2(q) \qquad \text{for } z \in E = \mathbb{C}/\Lambda$$

Feynman integrals (B-side)

Definition (Propagator)

$$P(z,q) = -\frac{1}{4\pi^2}\wp(z,q) - \frac{1}{12}E_2(q) \qquad \text{for } z \in E = \mathbb{C}/\Lambda$$

with Weierstraß-$\wp$-function $\wp = \frac{1}{z^2} + \ldots$ and the Eisenstein series

$$E_2 = 1 - 24\sum_{d=1}^{\infty}\sigma_1(d)q^{2d} = 1 - 24q^2 - 72q^4 - \ldots \qquad \sigma_1(d) = \sum_{m|d} m$$

Feynman integrals (B-side)

Definition (Propagator)

$$P(z,q) = -\frac{1}{4\pi^2}\wp(z,q) - \frac{1}{12}E_2(q) \qquad \text{for } z \in E = \mathbb{C}/\Lambda$$

with Weierstraß-$\wp$-function $\wp = \frac{1}{z^2} + \ldots$ and the Eisenstein series

$$E_2 = 1 - 24\sum_{d=1}^{\infty}\sigma_1(d)q^{2d} = 1 - 24q^2 - 72q^4 - \ldots \qquad \sigma_1(d) = \sum_{m|d} m$$

Definition (Feynman integral)

For ordering $\Omega \in S_{2g-2}$ of integration paths on E

$$I_{\Gamma,\Omega} = \int_{\gamma_{2g-2}} \ldots \int_{\gamma_1} \left(\prod_{e \in \text{edges}(\Gamma)} P(z_e^+ - z_e^-, q) \right) dz_{\Omega(1)} \ldots dz_{\Omega(2g-2)}$$

Tropical Hurwitz numbers (A-side)

Definition (Tropical Hurwitz number)

$N_{d,g}^{trop}$ = weighted number of tropical covers $\pi : C \to E$ where

Tropical Hurwitz numbers (A-side)

Definition (Tropical Hurwitz number)

$N_{d,g}^{trop}$ = weighted number of tropical covers $\pi : C \to E$ where

- π has degree d,

Tropical Hurwitz numbers (A-side)

Definition (Tropical Hurwitz number)

$N_{d,g}^{trop}$ = weighted number of tropical covers $\pi : C \to E$ where

- π has degree d,
- C is a tropical curve (metric 3-valent graph) of genus g,

Tropical Hurwitz numbers (A-side)

Definition (Tropical Hurwitz number)

$N^{trop}_{d,g}$ = weighted number of tropical covers $\pi : C \to E$ where

- π has degree d,
- C is a tropical curve (metric 3-valent graph) of genus g,
- has $2g-2$ simple ramifications (3-valent vertices) at fixed points $p_1, \ldots, p_{2g-2} \in E$,

Tropical Hurwitz numbers (A-side)

Definition (Tropical Hurwitz number)

$N_{d,g}^{trop}$ = weighted number of tropical covers $\pi : C \to E$ where

- π has degree d,
- C is a tropical curve (metric 3-valent graph) of genus g,
- has $2g - 2$ simple ramifications (3-valent vertices) at fixed points $p_1, \dots, p_{2g-2} \in E$,

with multiplicity $$\mathrm{mult}(\pi) = \frac{1}{|\mathrm{Aut}(\pi)|} \cdot \prod_{e \in \mathrm{edges}(C)} w(e)$$

Tropical Hurwitz numbers (A-side)

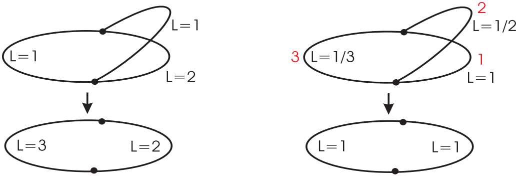
Definition (Tropical Hurwitz number)

$N_{d,g}^{trop}$ = weighted number of tropical covers $\pi : C \to E$ where

- π has degree d,
- C is a tropical curve (metric 3-valent graph) of genus g,
- has $2g-2$ simple ramifications (3-valent vertices) at fixed points $p_1, \dots, p_{2g-2} \in E$,

with multiplicity $$\text{mult}(\pi) = \frac{1}{|\text{Aut}(\pi)|} \cdot \prod_{e \in \text{edges}(C)} w(e)$$

Tropical covers are balanced w.r.t. weights $w(e)$:



Tropical Hurwitz numbers (A-side)

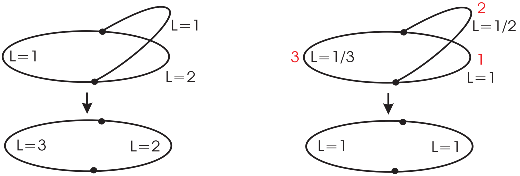
Definition (Tropical Hurwitz number)

$N_{d,g}^{trop}$ = weighted number of tropical covers $\pi : C \to E$ where

- π has degree d,
- C is a tropical curve (metric 3-valent graph) of genus g,
- has $2g-2$ simple ramifications (3-valent vertices) at fixed points $p_1, \ldots, p_{2g-2} \in E$,

with multiplicity $$\mathrm{mult}(\pi) = \frac{1}{|\mathrm{Aut}(\pi)|} \cdot \prod_{e \in \mathrm{edges}(C)} w(e)$$

Tropical covers are balanced w.r.t. weights $w(e)$:



$N_{d,g}^{trop}$ are intersection numbers on tropical moduli space.

Correspondence Theorem

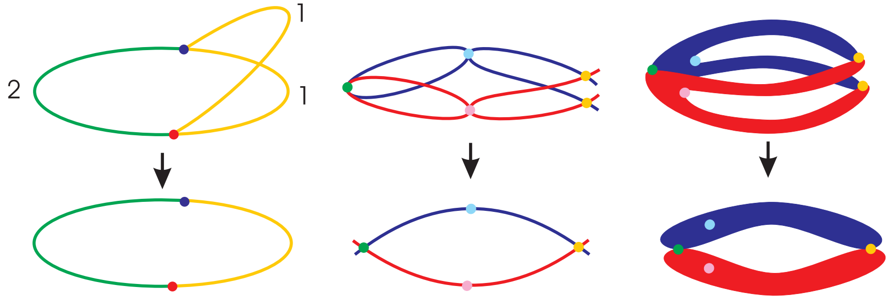
Theorem (BBBM '17)

$N_{d,g} = N_{d,g}^{\text{trop}}$ *by correspondence of tropical and algebraic covers.*

Correspondence Theorem

Theorem (BBBM '17)

$N_{d,g} = N_{d,g}^{\text{trop}}$ *by correspondence of tropical and algebraic covers.*



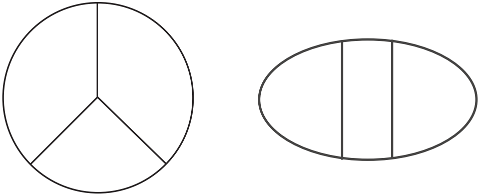
Tropical Hurwitz numbers – Example

$$N_{3,3}^{trop} = ?$$

Tropical Hurwitz numbers – Example

$$N_{3,3}^{trop} = ?$$

Two trivalent, connected combinatorial types (non-metric graphs)



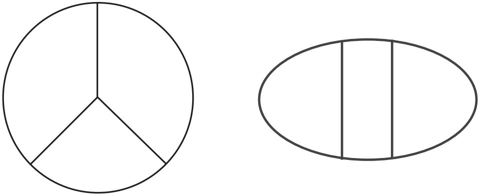
of genus $g = 3$ with

- $2g - 2 = 4$ vertices
- $3g - 3 = 6$ edges
- no bridges

Tropical Hurwitz numbers – Example

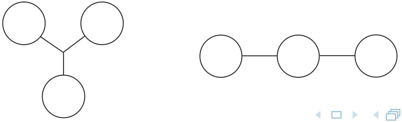
$$N_{3,3}^{trop} = ?$$

Two trivalent, connected combinatorial types (non-metric graphs)



of genus $g = 3$ with

- $2g - 2 = 4$ vertices
- $3g - 3 = 6$ edges
- no bridges (weight 0 edges would be contracted):



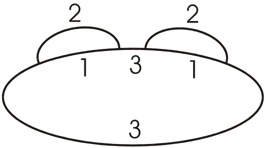
Tropical Hurwitz numbers – Example

$$N_{3,3}^{trop} =$$

Tropical Hurwitz numbers – Example

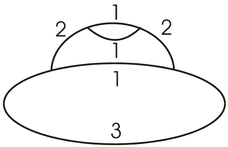
$N^{trop}_{3,3} =$

$2\cdot$



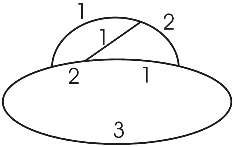
$\mathrm{mult}(\pi) = 2^2 \cdot 3^2 = 36$

$4\cdot$



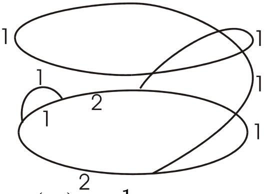
$\mathrm{mult}(\pi) = \frac{1}{2} \cdot 2^2 \cdot 3 = 6$

$4\cdot$



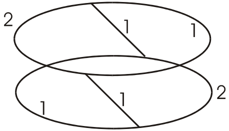
$\mathrm{mult}(\pi) = 2^2 \cdot 3 = 12$

$4\cdot$



$\mathrm{mult}(\pi) = \frac{1}{2} \cdot 2 \cdot 2 = 2$

$2\cdot$

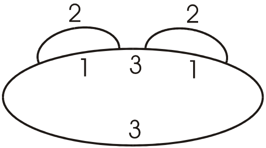


$\mathrm{mult}(\pi) = 2^2 = 4$

Tropical Hurwitz numbers – Example

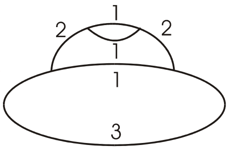
$$N_{3,3}^{trop} = 112 + 48 = 160$$

2·



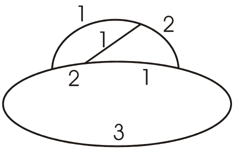
$\mathrm{mult}(\pi) = 2^2 \cdot 3^2 = 36$

4·



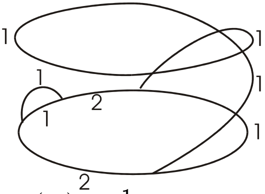
$\mathrm{mult}(\pi) = \frac{1}{2} \cdot 2^2 \cdot 3 = 6$

4·



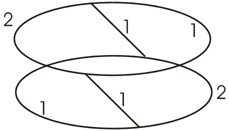
$\mathrm{mult}(\pi) = 2^2 \cdot 3 = 12$

4·



$\mathrm{mult}(\pi) = \frac{1}{2} \cdot 2 \cdot 2 = 2$

2·



$\mathrm{mult}(\pi) = 2^2 = 4$

Multivariate Feynman integrals

Definition (Multivariate Feynman integrals)

$$I_{\Gamma,\Omega}(q_1, \ldots, q_{3g-3}) = \int_{\gamma_{2g-2}} \ldots \int_{\gamma_1} \left(\prod_{k=1}^{3g-3} P(z_k^+ - z_k^-, q_k) \right) dz_{\Omega(1)} \ldots dz_{\Omega(2g-2)}$$

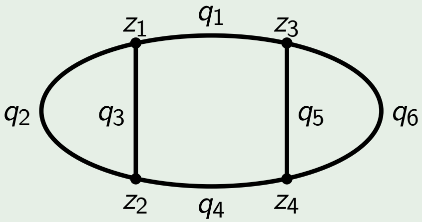
Multivariate Feynman integrals

Definition (Multivariate Feynman integrals)

$$I_{\Gamma,\Omega}(q_1, \dots, q_{3g-3}) = \int_{\gamma_{2g-2}} \dots \int_{\gamma_1} \left(\prod_{k=1}^{3g-3} P(z_k^+ - z_k^-, q_k) \right) dz_{\Omega(1)} \dots dz_{\Omega(2g-2)}$$

Example

For



we have to integrate

$$P(z_1 - z_2, q_1) \cdot P(z_1 - z_2, q_2) \cdot P(z_1 - z_3, q_3) \cdot P(z_2 - z_4, q_4) \cdot P(z_3 - z_4, q_5) \cdot P(z_3 - z_4, q_6)$$

Tropical mirror theorem

Theorem (Multivariate tropical mirror theorem, BBBM '17)

If Γ is any trivalent Feynman graph, then

$$\sum_{\underline{a}} N^{trop}_{\underline{a},\Gamma,\Omega}\, q^{2\underline{a}} = I_{\Gamma,\Omega}(q_1, \ldots, q_{3g-3})$$

Tropical mirror theorem

Theorem (Multivariate tropical mirror theorem, BBBM '17)

If Γ is any trivalent Feynman graph, then

$$\sum_{\underline{a}} N^{trop}_{\underline{a},\Gamma,\Omega} \; q^{2\underline{a}} = I_{\Gamma,\Omega}(q_1, \dots, q_{3g-3})$$

Setting $q_i = q$ we get (using the action of $\mathrm{Aut}(\Gamma)$ on labeled covers):

Corollary (Tropical mirror theorem)

$$\sum_{d} N^{trop}_{d,g} \; q^{2d} = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\Omega} I_{\Gamma,\Omega}(q)$$

Tropical mirror theorem

Theorem (Multivariate tropical mirror theorem, BBBM '17)

If Γ is any trivalent Feynman graph, then

$$\sum_{\underline{a}} N^{trop}_{\underline{a},\Gamma,\Omega}\ q^{2\underline{a}} = I_{\Gamma,\Omega}(q_1, \ldots, q_{3g-3})$$

Setting $q_i = q$ we get (using the action of $\mathrm{Aut}(\Gamma)$ on labeled covers):

Corollary (Tropical mirror theorem)

$$\sum_{d} N^{trop}_{d,g}\ q^{2d} = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\Omega} I_{\Gamma,\Omega}(q)$$

Together with the correspondence theorem this proves:

Corollary (Mirror symmetry for elliptic curves)

For elliptic curves $\mathbb{A}_g = \mathbb{B}_g$ for all g.

Computing Feynman integrals

By coordinate change $x_k = \exp(i\pi z_k)$,

Computing Feynman integrals

By coordinate change $x_k = \exp(i\pi z_k)$, path γ_k becomes circle around 0,

Computing Feynman integrals

By coordinate change $x_k = \exp(i\pi z_k)$, path γ_k becomes circle around 0, factor $\frac{1}{x_k}$,

Computing Feynman integrals

By coordinate change $x_k = \exp(i\pi z_k)$, path γ_k becomes circle around 0, factor $\frac{1}{x_k}$, integral becomes residue,

Computing Feynman integrals

By coordinate change $x_k = \exp(i\pi z_k)$, path γ_k becomes circle around 0, factor $\frac{1}{x_k}$, integral becomes residue, difference becomes quotient.

Proposition (BBBM '17)

$$P(x,q) = \frac{x^2}{(x^2-1)^2} + \sum_{a=1}^{\infty}\sum_{w|a} w(x^{2w}+x^{-2w})q^{2a}$$

Computing Feynman integrals

By coordinate change $x_k = \exp(i\pi z_k)$, path γ_k becomes circle around 0, factor $\frac{1}{x_k}$, integral becomes residue, difference becomes quotient.

Proposition (BBBM '17)

$$P(x,q) = \frac{x^2}{(x^2-1)^2} + \sum_{a=1}^{\infty} \sum_{w|a} w(x^{2w} + x^{-2w}) q^{2a}$$

$$P_a(x,y) := \begin{cases} \frac{x^2y^2}{(x^2-y^2)^2} & \text{for } a = 0 \\ \sum_{w|a} w \frac{x^{4w}+y^{4w}}{(xy)^{2w}} & \text{for } a > 0 \end{cases}$$

Computing Feynman integrals

By coordinate change $x_k = \exp(i\pi z_k)$, path γ_k becomes circle around 0, factor $\frac{1}{x_k}$, integral becomes residue, difference becomes quotient.

Proposition (BBBM '17)

$$P(x,q) = \frac{x^2}{(x^2-1)^2} + \sum_{a=1}^{\infty} \sum_{w|a} w(x^{2w} + x^{-2w}) q^{2a}$$

$$P_a(x,y) := \begin{cases} \frac{x^2y^2}{(x^2-y^2)^2} & \text{for } a = 0 \\ \sum_{w|a} w \frac{x^{4w}+y^{4w}}{(xy)^{2w}} & \text{for } a > 0 \end{cases}$$

Theorem (BBBM '17)

$$N^{trop}_{\underline{a},\Gamma,\Omega} = \text{const}_{x_{\Omega(2g-2)}} \dots \text{const}_{x_{\Omega(1)}} \prod_{k=1}^{3g-3} P_{a_k}(x_k^+, x_k^-)$$

Quasi-modularity

Corollary (BBBM 2017, Goujard-Möller 2016)

For all Feynman graphs Γ *of genus* g *and all orders* Ω *the function* $I_{\Gamma,\Omega}$ *is a quasi-modular form* ($I_{\Gamma,\Omega} \in \mathbb{Q}[E_2, E_4, E_6]$) *of uniform weight* $6g-6$.

Quasi-modularity

Corollary (BBBM 2017, Goujard-Möller 2016)

For all Feynman graphs Γ of genus g and all orders Ω the function $I_{\Gamma,\Omega}$ is a quasi-modular form ($I_{\Gamma,\Omega} \in \mathbb{Q}[E_2, E_4, E_6]$) of uniform weight $6g-6$.

Eisenstein series $E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n}$ $\sigma_{2k-1}(n) = \sum_{m|n} m^{2k-1}$

Quasi-modularity

Corollary (BBBM 2017, Goujard-Möller 2016)

For all Feynman graphs Γ of genus g and all orders Ω the function $I_{\Gamma,\Omega}$ is a quasi-modular form ($I_{\Gamma,\Omega} \in \mathbb{Q}[E_2, E_4, E_6]$) of uniform weight $6g-6$.

Eisenstein series $E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n}$ $\qquad \sigma_{2k-1}(n) = \sum_{m|n} m^{2k-1}$

$$E_2 = 1 - 24q^2 - 72q^4 - \dots$$
$$E_4 = 1 + 240q^2 + 2160q^4 + \dots$$
$$E_6 = 1 - 504q^2 - 16632q^4 - \dots$$

Quasi-modularity

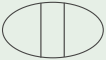
Corollary (BBBM 2017, Goujard-Möller 2016)

For all Feynman graphs Γ of genus g and all orders Ω the function $I_{\Gamma,\Omega}$ is a quasi-modular form ($I_{\Gamma,\Omega} \in \mathbb{Q}[E_2, E_4, E_6]$) of uniform weight $6g-6$.

Eisenstein series $E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n}$ $\qquad \sigma_{2k-1}(n) = \sum_{m|n} m^{2k-1}$

$$E_2 = 1 - 24q^2 - 72q^4 - \dots$$
$$E_4 = 1 + 240q^2 + 2160q^4 + \dots$$
$$E_6 = 1 - 504q^2 - 16632q^4 - \dots$$

Example

$I_\Gamma = \sum_\Omega I_{\Gamma,\Omega}$ for $\Gamma =$  is a linear combination of the weight 12 monomials

$$E_6^2,\ E_4^3,\ E_2E_4E_6,\ E_2^2E_4^2,\ E_2^3E_6,\ E_2^4E_4,\ E_2^6$$

Quasi-modularity

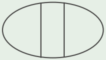
Corollary (BBBM 2017, Goujard-Möller 2016)

For all Feynman graphs Γ of genus g and all orders Ω the function $I_{\Gamma,\Omega}$ is a quasi-modular form ($I_{\Gamma,\Omega} \in \mathbb{Q}[E_2, E_4, E_6]$) of uniform weight $6g - 6$.

Eisenstein series $E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^{2n}$ $\quad \sigma_{2k-1}(n) = \sum_{m|n} m^{2k-1}$

$$E_2 = 1 - 24q^2 - 72q^4 - \dots$$
$$E_4 = 1 + 240q^2 + 2160q^4 + \dots$$
$$E_6 = 1 - 504q^2 - 16632q^4 - \dots$$

Example

$I_\Gamma = \sum_\Omega I_{\Gamma,\Omega}$ for $\Gamma =$  is a linear combination of the weight 12 monomials

$$E_6^2,\ E_4^3,\ E_2 E_4 E_6,\ E_2^2 E_4^2,\ E_2^3 E_6,\ E_2^4 E_4,\ E_2^6$$

Via Feynman integral compute

$$I_\Gamma = 32q^4 + 1792q^6 + 25344q^8 + 182272q^{10} + 886656q^{12} + O(q^{14})$$

Quasi-modularity

Example (cont.)

Solving a linear system of equations yields

$$I_\Gamma = \frac{16}{1492992}\left(4E_6^2 + 4E_4^3 - 12E_2E_4E_6 - 3E_2^2E_4^2 + 4E_2^3E_6 + 6E_2^4E_4 - 3E_2^6\right).$$

Quasi-modularity

Example (cont.)

Solving a linear system of equations yields

$$I_\Gamma = \frac{16}{1492992}\left(4E_6^2 + 4E_4^3 - 12E_2E_4E_6 - 3E_2^2E_4^2 + 4E_2^3E_6 + 6E_2^4E_4 - 3E_2^6\right).$$

$\Rightarrow$ Can compute $I_\Gamma(q)$ fast up to arbitrary high order:

$$\begin{aligned} I_\Gamma = {} & 32q^4 + 1792q^6 + 25344q^8 + 182272q^{10} + 886656q^{12} \\ & + 3294720q^{14} + 10246144q^{16} + 27353088q^{18} \\ & + 66497472q^{20} + 145337600q^{22} + \ldots \end{aligned}$$

References

J. Boehm, W. Decker, A. Frühbis-Krüger, F.-J. Pfreundt, M. Rahn, L. Ristau. *Towards Massively Parallel Computations in Algebraic Geometry*. arXiv:1808.09727.

J. Boehm, A. Georgoudis, K. J. Larsen, M. Schulze, Y. Zhang. *Complete sets of logarithmic vector fields for integration-by-parts identities of Feynman integrals.* Phys. Rev. D 98 (2018).

J. Boehm, A. Georgoudis, K. J. Larsen, H. Schönemann, Y. Zhang. *Complete integration-by-parts reductions of the non-planar hexagon-box via module intersections.* J. High Energ. Phys. 09 (2018).

D. Bendle, J. Boehm, W. Decker, A. Georgoudis, M. Rahn, F.-J. Pfreundt, Y. Zhang. *Integration-by-parts reductions of Feynman integrals using Singular and GPI-Space.* arXiv:1908.04301

J. Boehm, K. Bringmann, A. Buchholz, H. Markwig, *Tropical mirror symmetry for elliptic curves*, J. Reine Angew. Math. 732 (2017).

J. Boehm, Ch. Goldner, H. Markwig, *Tropical mirror symmetry in dimension one*. arXiv:1809.10659