Characteristic classes in Intersection theory

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Intersection Theory & Feynman Integrals.
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1. Intersection theory in algebraic geometry

2. Characteristic classes of singular/noncompact algebraic varieties

3. c_{SM} classes of graph hypersurfaces

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Obvious task: Establish a direct relation with ‘Intersection theory’ as meant in this workshop.

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- Fast forward: *"Schubert calculus"*, ($\sim$1880) aka intersection theory in Grassmannians and beyond,
  $\rightsquigarrow$ enumerative geometry

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In particular: every algebraic scheme has a *Chow group.* (Think: homology.) The Chow group of a nonsingular algebraic varieties has a well-defined intersection product, making it into a ring. (Think: cohomology.)

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All expected properties of intersection theory (and more) can be proven from this definition.

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  - Manin ('94): *Generating functions in algebraic geometry...*: e.g., Betti numbers for $\overline{\mathcal{M}}_{0,n}$.
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- Dually (up to sign): obstruction to defining linearly independent global differential forms.

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Chow group / Homology of noncompact varieties: losing too much information; e.g., can't recover χ.

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(E.g., V not compact $\rightsquigarrow$?)

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Fact (not difficult): $(f \circ g)_* = f_* \circ g_*$.

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Fact (not difficult): $(f \circ g)_* = f_* \circ g_*$.

Remark: homology, H_* (or Chow, A_*) is *also* a covariant functor {varieties} to {abelian groups}

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Content:

> *There is a* functorial *theory of 'Chern classes' for arbitrarily singular projective varieties, satisfying the same combinatorial properties of the topological Euler characteristic.*

Fix a projective V (e.g., $V = \mathbb{P}^n$); $X \subseteq V$: locally closed subset.

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'Singular/noncompact Poincaré-Hopf'

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Rest of the talk: *'Graph hypersurfaces'* (— -Marcolli, ∼'10).

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G: graph (think: Feynman);
edges $e_1, \dots, e_n \leftrightarrow$ variables t_e: $t_1, \dots, t_n$.

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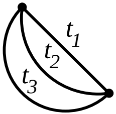

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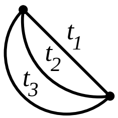

$\Psi_G(\underline{t}) = t_2 t_3 + t_1 t_3 + t_1 t_2.$

$\Psi_G(\underline{t})$ is homogeneous, so it defines a hypersurface in $\mathbb{P}^{n-1}$.

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Fact (Belkale-Brosnan; Brown; ...): not true!

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Disproved!

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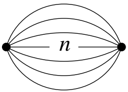
Can $c_{\rm SM}$ be used to construct an algebro-geometric Feynman rule?

(joint with Matilde Marcolli)



Example: $G = \Gamma_n$, 'banana graph', n edges. Then

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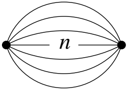

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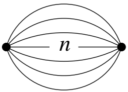

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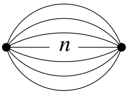

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One moral is that the work needed to compute the motive of Γ_n can often be retooled to say something about c_{SM}.

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Definition

$C_G(t)$ determined by

$$c_{\mathrm{SM}}(\mathbb{P}^{n-1} \smallsetminus X_G) = (H^n C_G(1/H)) \cap [\mathbb{P}^{n-1}]$$

This satisfies $C_{G_1 \amalg G_2}(t) = C_{G_1}(t) C_{G_2}(t)$.
The corresponding 'propagator' is $1 + t$.

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- G *not* 1-particle irreducible $\implies C_G(-1) = 0$

Theorem (—)

e edges of G, satisfying unfortunate technical conditions. Then

$$C_{G_{2e}}(t) = (2t-1)C_G(t) - t(t-1)C_{G\smallsetminus e}(t) + C_{G/e}(t)$$

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Question: are the 'unfortunate technical conditions' necessary?

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- 'Twisted' algebro-geometric Feynman rules?
- Do relations among c_{SM} classes of relevant loci imply relations of corresponding Feynman amplitudes?
- $\cdots$

Thank you for your attention!