Scattering forms and the CHY representation

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I. **Scattering amplitudes**

II. **Review of recent developments**

III. **Geometric interpretation of tree scattering amplitudes**

Detailed outline

I. Scattering amplitudes

- The zeroth copy: Bi-adjoint scalar theory
- The single copy: Yang-Mills theory
- The double copy: Gravity

II. Review of recent developments

- Jacobi-like relations (BCJ numerators)
- The scattering equations (CHY representation)
- KLT relations
- Positive geometries and canonical forms
- Intersection theory

III. Geometric interpretation of tree scattering amplitudes

Part I

Scattering amplitudes

Amplitudes

In this talk we are interested in amplitudes of the following theories:

The zeroth copy: Bi-adjoint scalar theory

The single copy: Yang-Mills theory

The double copy: Gravity

We consider tree amplitudes with an arbitrary number of external particles n.

The single copy: Yang-Mills theory

The Lagrangian of a non-Abelian gauge theory:

$$\mathscr{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

Decompose the tree amplitudes $\mathscr{A}_n(p,\varepsilon)$ into group-theoretical factors and cyclic-ordered amplitudes $A_n(\sigma, p, \varepsilon)$:

$$\mathscr{A}_n(p,\varepsilon) = g^{n-2} \sum_{\sigma \in S_n/\mathbb{Z}_n} 2\,\mathrm{Tr}\left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}\right) A_n(\sigma, p, \varepsilon)$$

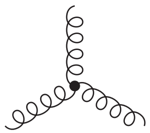
with

$p = (p_1, \ldots, p_n)$	momenta
$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$	polarisations
$\sigma = (\sigma_1, \ldots, \sigma_n)$	cyclic order

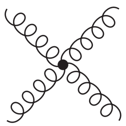
Primitive amplitudes

The primitive amplitudes are gauge-invariant and each primitive amplitude has a fixed cyclic order of the external legs.
The primitive amplitudes are calculated from cyclic-ordered Feynman rules:

$$= -\frac{i g_{\mu\nu}}{p^2}$$



$$= i \left[g^{\mu_1\mu_2} \left(p_1^{\mu_3} - p_2^{\mu_3} \right) + g^{\mu_2\mu_3} \left(p_2^{\mu_1} - p_3^{\mu_1} \right) + g^{\mu_3\mu_1} \left(p_3^{\mu_2} - p_1^{\mu_2} \right) \right]$$



$$= i \left[2 g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3} \right]$$

The zeroth copy: Bi-adjoint scalar theory

A scalar field in the adjoint representation of two gauge-groups $G \times \tilde{G}$ with Lagrange density

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi^{ab}\right)\left(\partial^\mu \phi^{ab}\right) - \frac{\lambda}{3!} f^{a_1 a_2 a_3} \tilde{f}^{b_1 b_2 b_3} \phi^{a_1 b_1} \phi^{a_2 b_2} \phi^{a_3 b_3}$$

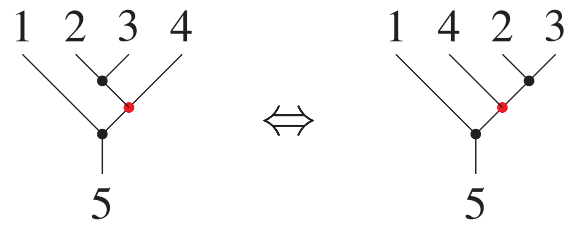
Decompose the tree amplitudes $m_n(p)$ into group-theoretical factors and double-ordered amplitudes $m_n(\sigma, \tilde{\sigma}, p)$:

$$m_n(p) = \lambda^{n-2} \sum_{\sigma \in S_n / \mathbb{Z}_n} \sum_{\tilde{\sigma} \in S_n / \mathbb{Z}_n} 2\,\mathrm{Tr}\left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}\right)\; 2\,\mathrm{Tr}\left(\tilde{T}^{b_{\tilde{\sigma}(1)}} \ldots \tilde{T}^{b_{\tilde{\sigma}(n)}}\right)\; m_n(\sigma, \tilde{\sigma}, p)$$

The permutations σ and $\tilde{\sigma}$ denote two cyclic orders.

Double-ordered amplitudes

Flip: exchange two branches at a vertex.

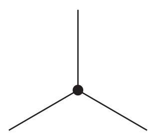


Two diagrams with different external orders are equivalent, if we can transform one diagram into the other by a sequence of flips.

The double-ordered amplitude $m_n(\sigma, \tilde{\sigma}, p)$ is computed from the Feynman diagrams compatible with the cyclic orders σ and $\tilde{\sigma}$.

Feynman rules:

$$= \frac{i}{p^2}$$



$$= i$$

The double copy: Gravity

Let us consider (small) fluctuations around the flat Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu},$$

with $\kappa = \sqrt{32\pi G}$ and consider an effective theory defined by the Einstein-Hilbert Lagrangian

$$\mathscr{L}_{\mathrm{EH}} = -\frac{2}{\kappa^2}\sqrt{-g}R.$$

The field $h_{\mu\nu}$ describes a graviton.

The inverse metric $g^{\mu\nu}$ and $\sqrt{-g}$ are infinite series in $h_{\mu\nu}$, therefore

$$\mathscr{L}_{\mathrm{EH}} + \mathscr{L}_{\mathrm{GF}} = \sum_{n=2}^{\infty} \mathscr{L}^{(n)},$$

where $\mathscr{L}^{(n)}$ contains exactly n fields $h_{\mu\nu}$.
Thus the Feynman rules will give an infinite tower of vertices.

Feynman rules for gravity

External edge:

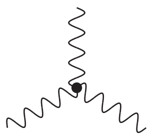
μ_1, μ_2  $= \quad \varepsilon_{\mu_1}(k)\varepsilon_{\mu_2}(k)$

Internal edge:

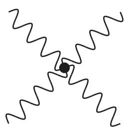
μ_1, μ_2  ν_1, ν_2

$$= \quad \frac{1}{2}\left(\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} + \eta_{\mu_1\nu_2}\eta_{\mu_2\nu_1} - \frac{2}{D-2}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\right)\frac{i}{k^2}$$

Vertices:



$=$ long expression



$=$ even longer expression

plus Feynman rules for 5-graviton vertex, 6-graviton vertex, etc.

Graviton amplitudes

The graviton amplitudes are un-ordered, we simply factor out the coupling:

$$\mathcal{M}_n(p, \varepsilon, \tilde{\varepsilon}) = \left(\frac{\kappa}{4}\right)^{n-2} M_n(p, \varepsilon, \tilde{\varepsilon})$$

$p = (p_1, ..., p_n)$ momenta
$\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$ first set of spin-1 polarisation vectors
$\tilde{\varepsilon} = (\tilde{\varepsilon}_1, ..., \tilde{\varepsilon}_n)$ second set of spin-1 polarisation vectors

$\varepsilon_j^+ \tilde{\varepsilon}_j^+$ and $\varepsilon_j^- \tilde{\varepsilon}_j^-$ describe the two polarisation states of the spin-2 graviton with index j.

Amplitudes

We consider the double ordered bi-adjoint scalar amplitudes $m_n(\sigma, \tilde{\sigma}, p)$, the single ordered Yang-Mills amplitudes $A_n(\sigma, p, \varepsilon)$ and the un-ordered graviton amplitudes $M_n(p, \varepsilon, \tilde{\varepsilon})$.

All these amplitudes can be computed from Feynman diagrams.

$$m_n(\sigma, \tilde{\sigma}, p) = i(-1)^{n-3+n_{\text{flip}}(\sigma,\tilde{\sigma})} \sum_{\substack{\text{trivalent graphs } G \\ \text{compatible with } \sigma \text{ and } \tilde{\sigma}}} \frac{1}{D(G)}, \qquad D(G) = \prod_{\text{edges } e} s_e,$$

$$A_n(\sigma, p, \varepsilon) = \text{long expression},$$

$$M_n(p, \varepsilon, \tilde{\varepsilon}) = \text{even longer expression}.$$

Part II

Review of recent developments

1. Jacobi-like relations (BCJ numerators)

2. The scattering equations (CHY representation)

3. KLT relations

4. Positive geometries and canonical forms

5. Intersection theory

Part II.1

Jacobi-like relations

Jacobi relation

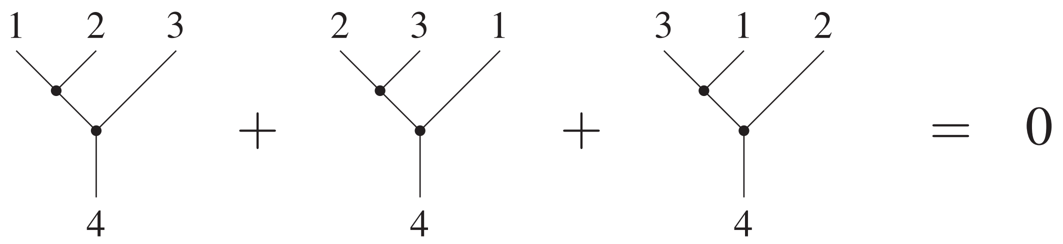
Jacobi relation:

$$\left[\left[T^a, T^b\right], T^c\right] + \left[\left[T^b, T^c\right], T^a\right] + \left[\left[T^c, T^a\right], T^b\right] \quad = \quad 0,$$

In terms of structure constants:

$$\left(i f^{abe}\right)\left(i f^{ecd}\right) + \left(i f^{bce}\right)\left(i f^{ead}\right) + \left(i f^{cae}\right)\left(i f^{ebd}\right) \quad = \quad 0.$$

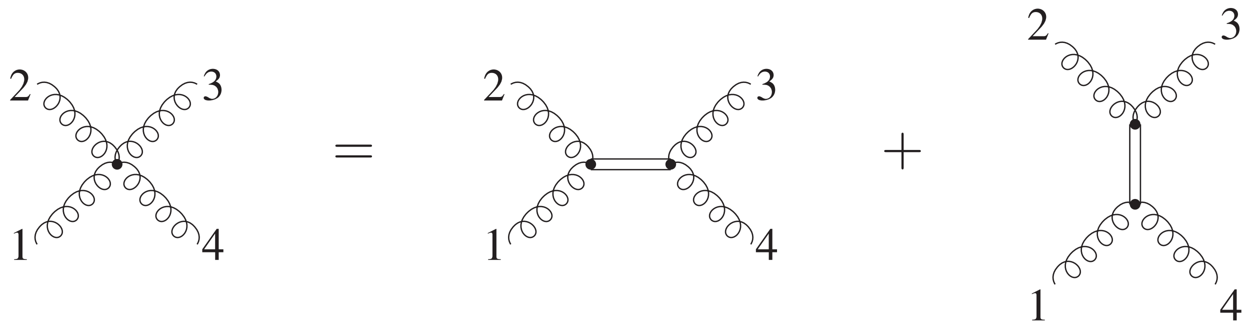
Graphically:



Expansion in graphs with three-valent vertices only

In Yang-Mills theory we have a three-valent and a four-valent vertex.

We may always re-write a four-valent vertex in terms of two three-valent vertices:



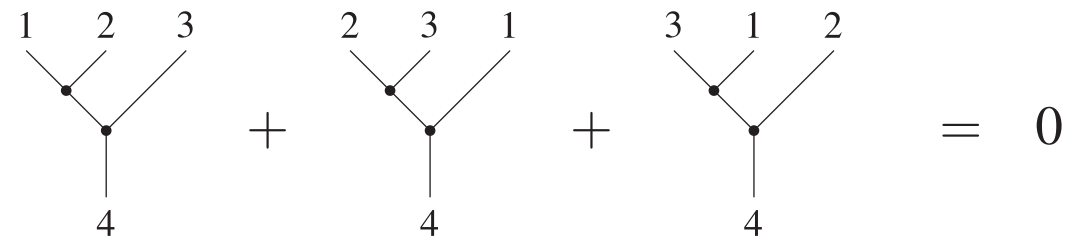
This is not unique!

BCJ numerators

We may write the Yang-Mills amplitude in a form

$$A_n(\sigma, p, \varepsilon) \quad = \quad i(-1)^{n-3} \sum_{\substack{\text{trivalent graphs } G \\ \text{with order } \sigma}} \frac{N(G)}{D(G)},$$

with numerators $N(G)$ satisfying anti-symmetry relations and Jacobi relations:

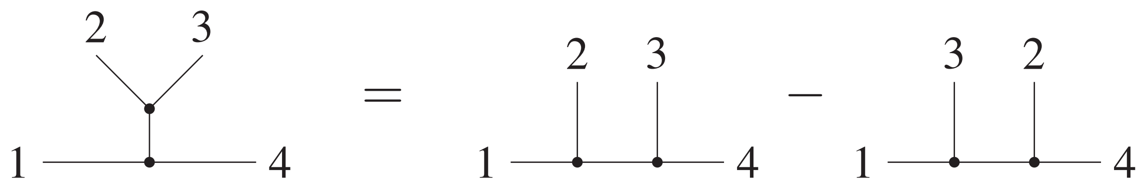


$$N(G_1) \quad + \quad N(G_2) \quad + \quad N(G_3) \quad = \quad 0$$

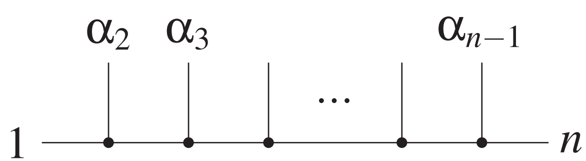
Bern, Carrasco, Johansson, ’10

Multi-peripheral graphs

Combining the anti-symmetry of the vertices and the Jacobi identity one has



We may express all BCJ-numerators in terms of the BCJ-numerators of multi-peripheral graphs (or comb graphs):



Double copy and colour-kinematics duality

If the Yang-Mills amplitude is written in terms of BCJ-numerators $N(G)$ and group-theoretical factors $C(G)$

$$\mathcal{A}_n(p,\varepsilon) = i(-1)^{n-3} g^{n-2} \sum_{\text{trivalent graphs } G} \frac{C(G)N(G)}{D(G)},$$

then

$$\mathcal{M}_n(p,\varepsilon,\tilde{\varepsilon}) = i(-1)^{n-3} \left(\frac{\kappa}{4}\right)^{n-2} \sum_{\text{trivalent graphs } G} \frac{N(G)\tilde{N}(G)}{D(G)},$$

and of course

$$m_n(p) = i(-1)^{n-3} \lambda^{n-2} \sum_{\text{trivalent graphs } G} \frac{C(G)\tilde{C}(G)}{D(G)}.$$

Bern, Carrasco, Johansson, '10

Effective Lagrangian

We may construct an effective Lagrangian, which gives directly BCJ-numerators

$$\mathscr{L}_{\mathrm{YM}} + \mathscr{L}_{\mathrm{GF}} = \sum_{n=2}^{\infty} \mathscr{L}^{(n)},$$

$\mathscr{L}^{(2)}$, $\mathscr{L}^{(3)}$ and $\mathscr{L}^{(4)}$ agree with the standard terms and $\mathscr{L}^{(n\geq 5)}$ are a complicated zero.

The effective Lagrangian is not unique.

Tolotti, S.W, ’13

Part II.2

The scattering equations

The Riemann sphere

The Riemann sphere is the complex plane plus the point at infinity:

$$\hat{\mathbb{C}} \quad = \quad \mathbb{C} \cup \{\infty\}$$

Each $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{C})$ acts on $z \in \hat{\mathbb{C}}$ through a Möbius transformation:

$$g \cdot z \quad = \quad \frac{az+b}{cz+d}.$$

Mark n distinct points $(z_1, \ldots, z_n)$ on $\hat{\mathbb{C}}$.

The moduli space of genus 0 curves with n distinct marked points is denoted by

$$\mathcal{M}_{0,n} \quad = \quad \left\{ z \in \hat{\mathbb{C}}^n : z_i \neq z_j \right\} / \mathrm{PSL}\left(2, \mathbb{C}\right).$$

$\mathcal{M}_{0,n}$ is an affine algebraic variety of dimension $(n-3)$.

The scattering equations

Set

$$f_i(z,p) = \sum_{j=1, j\neq i}^{n} \frac{2 p_i \cdot p_j}{z_i - z_j}.$$

The scattering equations:

$$f_i(z,p) = 0, \qquad 1 \leq i \leq n.$$

Only $(n-3)$ equations of the n equations are independent.

Two solutions which are related by a Möbius-transformation are called equivalent solutions.

There are $(n-3)!$ inequivalent solutions not related by a Möbius-transformation.

The CHY representation

There exists two functions $C(\sigma, z)$ and $E(p, \varepsilon, z)$ on $\hat{\mathbb{C}}^n$ such that

$$
\begin{aligned}
m_n(\sigma, \tilde{\sigma}, p) &= i \oint\limits_{\mathcal{C}} d\Omega_{\text{CHY}}\, C(\sigma, z)\, C(\tilde{\sigma}, z), \\
A_n(\sigma, p, \varepsilon) &= i \oint\limits_{\mathcal{C}} d\Omega_{\text{CHY}}\, C(\sigma, z)\, E(p, \varepsilon, z), \\
M_n(p, \varepsilon, \tilde{\varepsilon}) &= i \oint\limits_{\mathcal{C}} d\Omega_{\text{CHY}}\, E(p, \varepsilon, z)\, E(p, \tilde{\varepsilon}, z).
\end{aligned}
$$

Details on the definition of the measure $d\Omega_{\text{CHY}}$:

$$
d\Omega_{\text{CHY}} = \frac{1}{(2\pi i)^{n-3}} \frac{d^n z}{d\omega} \prod{}' \frac{1}{f_a(z,p)}, \qquad \prod{}' \frac{1}{f_a(z,p)} = (-1)^{i+j+k} (z_i - z_j)(z_j - z_k)(z_k - z_i) \prod_{a \neq i,j,k} \frac{1}{f_a(z,p)},
$$

$$
d\omega = (-1)^{p+q+r} \frac{dz_p dz_q dz_r}{(z_p - z_q)(z_q - z_r)(z_r - z_q)}.
$$

Cachazo, He and Yuan, '13

Global residue

The $(n-3)$-independent scattering equations $f_i(z,p)=0$ may be re-written as a system of $(n-3)$ polynomial equations $h_i(z,p)=0$.

Dolan, Goddard, '14

The contour integrals are global residues:

$$A_n(\sigma,p,\varepsilon) \quad = \quad i\,\mathrm{Res}_{h'_2,\ldots,h'_{n-2}}\left(R \right),$$

where the prime denotes gauge-fixed quantities ($z_1=0, z_{n-1}=1, z_n=\infty$).

The rational function R is given by

$$R \quad = \quad -z_n^4 \left(\prod_{i<j<n} z_{ij} \right) C(\sigma,z)\; E(z,p,\varepsilon) \Bigg|_{z_1=0, z_{n-1}=1, z_n=\infty}$$

M. Søgaard and Y. Zhang, '16

The cyclic factor

The cyclic factor (or Parke-Taylor factor) is given by

$$C(\sigma, z) = \frac{1}{(z_{\sigma_1} - z_{\sigma_2})(z_{\sigma_2} - z_{\sigma_3}) \dots (z_{\sigma_n} - z_{\sigma_1})}.$$

The cyclic factor encodes the information on the cyclic order.

The polarisation factor

The polarisation factor $E(p,\varepsilon,z)$ encodes the information on the helicities of the external particles.

One possibility to define this factor is through a reduced Pfaffian.

(All definitions have to agree on the solutions of the scattering equations, but may differ away from this zero-dimensional sub-variety.)

The reduced Pfaffian

Define a $(2n) \times (2n)$ antisymmetric matrix $\Psi(z,p,\varepsilon)$ through

$$\Psi(z,p,\varepsilon) = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

with

$$A_{ab} = \begin{cases} \frac{2p_a \cdot p_b}{z_a - z_b} & a \neq b, \\ 0 & a = b, \end{cases} \qquad B_{ab} = \begin{cases} \frac{2\varepsilon_a \cdot \varepsilon_b}{z_a - z_b} & a \neq b, \\ 0 & a = b, \end{cases} \qquad C_{ab} = \begin{cases} \frac{2\varepsilon_a \cdot p_b}{z_a - z_b} & a \neq b, \\ -\sum\limits_{j=1, j \neq a}^{n} \frac{2\varepsilon_a \cdot p_j}{z_a - z_j} & a = b. \end{cases}$$

Denote by Ψ_{ij}^{ij} the $(2n-2) \times (2n-2)$-matrix, where rows and columns i and j have been deleted $(1 \leq i < j \leq n)$.

The reduced Pfaffian $E^{\mathrm{Pfaff}}(z,p,\varepsilon)$ is defined by

$$E^{\mathrm{Pfaff}}(z,p,\varepsilon) = \frac{(-1)^{i+j}}{2(z_i - z_j)} \mathrm{Pf}\, \Psi_{ij}^{ij}(z,p,\varepsilon).$$

Cachazo, He and Yuan, '13

Part II.3

KLT relations

Independent primitive amplitudes

How many independent primitive amplitudes $A_n(\sigma, p, \varepsilon)$ are there for fixed momenta p and polarisations ε ?

- There are $n!$ external orderings.

- Cyclic invariance reduce the number to $(n-1)!$.

- Anti-symmetry of the vertices reduce the number to $(n-2)!$.

  Kleiss, Kuijf, 1989

- Jacobi relations reduce the number to $(n-3)!$.

  Bern, Carrasco, Johansson, 2008

Basis B of independent amplitudes consists of $(n-3)!$ elements.

KLT relations

Define $(n-3)! \times (n-3)!$-dimensional matrix $m_{\sigma\tilde{\sigma}}$ for $\sigma, \tilde{\sigma} \in B$ by

$$m_{\sigma\tilde{\sigma}} = m_n(\sigma, \tilde{\sigma}, p).$$

The matrix m is invertible.

Define the KLT-matrix as the inverse of the matrix m:

$$S = m^{-1}$$

Kawai, Lewellen, Tye, 1986,

Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove, 2010,

Cachazo, He and Yuan, 2013,

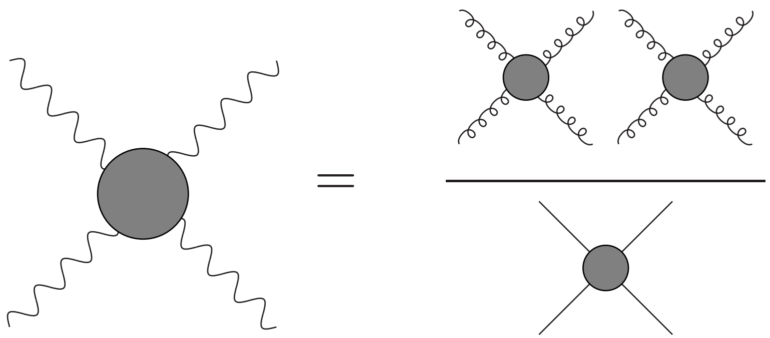
de la Cruz, Kniss, S.W., 2016

KLT relations

The KLT relations express the graviton amplitude $M_n(p,\varepsilon,\tilde{\varepsilon})$ through products of Yang-Mills amplitudes $A_n(\sigma,p,\varepsilon)$ and the KLT-matrix S:

$$M_n(p,\varepsilon,\tilde{\varepsilon}) = \sum_{\sigma,\tilde{\sigma}\in B} A_n(\sigma,p,\varepsilon)\, S_{\sigma\tilde{\sigma}}\, A_n(\tilde{\sigma},p,\tilde{\varepsilon})$$

Graphically:



Part II.4

Positive geometries and canonical forms

Multivariate residues of differential forms

Let X be a m-dimensional variety and Y a co-dimension one sub-variety.
Let us choose a coordinate system such that Y is given locally by $z_1 = 0$.
Assume that Ω has a pole of order 1 on Y:

$$\Omega = \frac{dz_1}{z_1} \wedge \psi + \theta.$$

The residue of Ω at Y is defined by

$$\mathrm{Res}_Y(\Omega) = \psi|_Y .$$

A pole of order 1 on Y is called a logarithmic singularity on Y.

Positive geometries and canonical forms

Let X be a m-dimensional (complex) variety and $X_{\geq 0}$ the positive part.
A m-form Ω is called a canonical form if

1. For $m = 0$ one has $\Omega = \pm 1$.

2. The only singularities of Ω are on the boundary of $X_{\geq 0}$.

3. The singularities are logarithmic.

4. The residue of Ω on a boundary component is again the canonical form of a $(m-1)$-dimensional positive geometry.

Arkani-Hamed, Bai, Lam, ’17,

Abreu, Britto, Duhr, Gardi, Matthew, ’19

Salvatori, Stanojevic, ’19

Part II.5

Intersection theory

The CHY representation

The CHY half-integrands $C(\sigma, z)$ and $E(p, \varepsilon, z)$ transform under $\mathrm{PSL}(2,\mathbb{C})$-transformations as

$$F(g \cdot z) \quad = \quad \left(\prod_{j=1}^{n} (cz_j + d)^2 \right) F(z)$$

Therefore, the $(n-3)$-forms

$$\Omega^{\mathrm{cyclic}}(\sigma, z) \; = \; C(\sigma, z) \frac{d^n z}{d\omega}, \qquad \Omega^{\mathrm{pol}}(p, \varepsilon, z) \; = \; E(p, \varepsilon, z) \frac{d^n z}{d\omega}.$$

are $\mathrm{PSL}(2,\mathbb{C})$-invariant.

Remark: We may add to $C(\sigma, z)$ and $E(p, \varepsilon, z)$ terms which vanish on the solutions of the scattering equations.

Intersection theory

Consider a space X of dimension m, equipped with a connection $\nabla = d + \eta$.
The connection one-form η is called the twist.

Elements of

$$H^m(X, \nabla) = \{ \varphi \mid \nabla\varphi = 0 \} / \{ \nabla\xi \}$$

are called twisted co-cycles.

The intersection number of two twisted co-cycles is defined by

$$(\varphi_1, \varphi_2) = \frac{1}{(2\pi i)^m} \int\limits_X \iota(\varphi_1) \wedge \varphi_2,$$

where ι maps φ_1 to a twisted co-cycle in the same cohomology class but with compact support.

Cho, Matsumoto, ’95; Aomoto, Kita, ’94 (jap.), ’11 (engl.); Yoshida, ’97

Intersection theory

Apply this to $X = \mathcal{M}_{0,n}$ and take

$$\eta = \sum_{i=1}^{n} f_i(z,p)\, dz_i.$$

Then

$$\begin{aligned} m_n(\sigma, \tilde{\sigma}, p) &= i\left(\Omega^{\text{cyclic}}(\sigma, z), \Omega^{\text{cyclic}}(\tilde{\sigma}, z)\right), \\ A_n(\sigma, p, \varepsilon) &= i\left(\Omega^{\text{cyclic}}(\sigma, z), \Omega^{\text{pol}}(p, \varepsilon, z)\right), \\ M_n(p, \varepsilon, \tilde{\varepsilon}) &= i\left(\Omega^{\text{pol}}(p, \varepsilon, z), \Omega^{\text{pol}}(p, \tilde{\varepsilon}, z)\right). \end{aligned}$$

(Mizera, '17)

Remark: We may still add to Ω^{cyclic} and Ω^{pol} terms which vanish on the solutions of the scattering equations.

Part III

Geometric interpretation of scattering amplitudes

Geometric interpretation of tree amplitudes

There exist two $(n-3)$-forms $\Omega^{\text{cyclic}}(\sigma, z)$ and $\Omega^{\text{pol}}(p, \varepsilon, z)$ on the compactified moduli space $\overline{\mathcal{M}}_{0,n}$ such that

1. The twisted intersection numbers give the amplitudes for the bi-adjoint scalar theory (cyclic, cyclic), Yang-Mills theory (cyclic,polarisation) and gravity (polarisation,polarisation).

2. The only singularities of the scattering forms are on the divisor $\overline{\mathcal{M}}_{0,n} \backslash \mathcal{M}_{0,n}$.

3. The singularities are logarithmic.

4. The residues at the singularities factorise into two scattering forms of lower points.

L. de la Cruz, A. Kniss, S.W., '17

The scattering forms

The cyclic scattering form is defined by

$$\Omega^{\text{cyclic}}(\sigma, z) = C(\sigma, z) \frac{d^n z}{d\omega}, \qquad C(\sigma, z) = \frac{1}{(z_{\sigma_1} - z_{\sigma_2})(z_{\sigma_2} - z_{\sigma_3}) \dots (z_{\sigma_n} - z_{\sigma_1})}.$$

The polarisation scattering form is defined by

$$\Omega^{\text{pol}}(p, \varepsilon, z) = E(p, \varepsilon, z) \frac{d^n z}{d\omega}, \qquad E(p, \varepsilon, z) = \sum_{\kappa \in S_{n-2}^{(1,n)}} C(\kappa, z)\, N_{\text{comb}}(\kappa),$$

where the sum is now over all permutations keeping $\kappa_1 = 1$ and $\kappa_n = n$ fixed.

L. de la Cruz, A. Kniss, S.W., '17

Wrap-up

The n-graviton amplitude is given by

$$M_n(p,\varepsilon,\tilde{\varepsilon}) = i(-1)^{n-3} \sum_{\text{trivalent graphs } G} \frac{N(G)\tilde{N}(G)}{D(G)}$$ colour-kinematics duality

$$= i \oint_{\mathscr{C}} d\Omega_{\text{CHY}}\, E(p,\varepsilon,z)\, E(p,\tilde{\varepsilon},z)$$ CHY representation

$$= \sum_{\sigma,\tilde{\sigma}\in B} A_n(\sigma,p,\varepsilon)\, S_{\sigma\tilde{\sigma}}\, A_n(\tilde{\sigma},p,\tilde{\varepsilon})$$ KLT relation

$$= i\left(\Omega^{\text{pol}}(p,\varepsilon,z), \Omega^{\text{pol}}(p,\tilde{\varepsilon},z)\right)$$ intersection number

Conclusions

- Clear geometric picture of tree-level amplitudes within the bi-adjoint scalar theory, Yang-Mills theory and gravity for any number of external particles n.

- Scattering amplituds are given as intersection numbers of two scattering forms. Are the scattering forms more fundamental?

- Relations between bi-adjoint scalar theory, Yang-Mills theory and gravity are not manifest in the action as a coordinate space integral over a Lagrange density.

  Should we not find and work with a formulation, which makes these structures manifest from the beginning?

References

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