Computing cohomology intersection numbers of GKZ hypergeometric systems

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Notation

GKZ hypergeometric integral is a function in z of the following form:

$$f_\Gamma(z) = \int_\Gamma \prod_{l=1}^{k} h_l(x)^{-\gamma_l} x^c \varphi.$$

$h_l(x) = h_{l,z^{(l)}}(x) = \sum_{j=1}^{N_l} z_j^{(l)} x^{\mathbf{a}^{(l)}(j)}$: Laurent polynomials

$x = (x_1, \ldots, x_n)$: a coordinate of $(\mathbb{G}_m)^n = (\mathbb{C}^*)^n$

$\gamma_l \in \mathbb{C}$, $c \in \mathbb{C}^{n\times 1}$: parameters

$\varphi \in \Omega^n_{(\mathbb{G}_m)^n}(*\{\prod_l h_l(x) = 0\})$: algebraic n-form

Γ: a "cycle"

$z = (z_j^{(l)})_{j,l}$: generic variables

Twisted de Rham machinery

$D := \{\prod_l h_l(x) = 0\}$, $X_z := (\mathbb{G}_m)^n \setminus D$, $\Phi := \prod_{l=1}^k h_l(x)^{-\gamma_l} x^c$.

$\nabla_\Phi := \Phi^{-1} \circ d_x \circ \Phi = d_x - \sum_{l=1}^k \gamma_l d_x \log h_l(x) \wedge + \sum_{i=1}^n c_i d \log x_i \wedge$: integrable connection

$\mathrm{H}^n_{dR}(X_z, \nabla_\Phi) := \mathrm{H}^n \left(\overset{\nabla_\Phi}{\to} \Omega^{\bullet}_{(\mathbb{G}_m)^n}(*D) \overset{\nabla_\Phi}{\to} \right)$: algebraic de Rham cohomology group

$\mathcal{L}^\vee := \mathrm{Ker}\left(\nabla^{an}_\Phi : \mathcal{O}_{X_z^{an}} \to \Omega^1_{X_z^{an}} \right) = \mathbb{C}\Phi^{-1}$: local system

Twisted period paring and the cohomology intersection number

The twisted period pairing

$$\begin{array}{ccc} \mathrm{H}^n_{dR}(X_z, \nabla_\Phi) \times \mathrm{H}_n(X_z^{an}, \mathcal{L}) & \to & \mathbb{C} \\ \cup & & \cup \\ (\varphi, \Gamma) & \mapsto & \int_\Gamma \Phi\varphi \end{array}$$

is perfect, and some properties of hypergeometric functions can be recaptured from this viewpoint (青本 (Aomoto)).
We call the pairing

$$\langle \bullet, \bullet \rangle_{ch} : \begin{array}{ccc} \mathrm{H}^n_{dR}(X_z, \nabla_\Phi) \times \mathrm{H}^n_c(X_z^{an}, \mathcal{L}) & \to & \mathbb{C} \\ \cup & & \cup \\ (\varphi, \psi) & \mapsto & \int_{X_z^{an}} \varphi \wedge \psi \end{array}$$

the cohomology intersection pairing.

Our motivation

Theorem (趙-松本 (Cho-Matsumoto))

Quadratic relation is a consequence of the twisted version of Riemann-Hodge bilinear relation (twisted period relation).

$$\begin{aligned}&(1-\gamma+\alpha)(1-\gamma+\beta)_2F_1\left({\alpha,\beta \atop \gamma};z\right)\,_2F_1\left({-\alpha,-\beta \atop 2-\gamma};z\right)\\&-\alpha\beta_2F_1\left({\gamma-\alpha-1,\gamma-\beta-1 \atop \gamma};z\right)\,_2F_1\left({1-\gamma+\alpha,1-\gamma+\beta \atop 2-\gamma};z\right)\\=&(1-\gamma+\alpha+\beta)(1-\gamma).\end{aligned}$$

$$_2F_1\left({\alpha,\beta \atop \gamma};z\right)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n}z^n$$

趙 and 松本 observed that the number $(1-\gamma+\alpha+\beta)(1-\gamma)$ comes from the cohomology intersection number.

The study of cohomology intersection numbers

Gauß, Appell-Lauricella's F_D	趙 (Cho)、松本 (Matsumoto)
↓	
Graßman hypergeometric system (integral)	松本、岩崎 (Iwasaki)、喜多 (Kita)、吉田 (Yoshida) and many others
↓	
GKZ hypergeometric system (integral)	Today (by a new method)

Twisted period relation

$$\{\varphi_i\}_i \subset \mathrm{H}^n_{dR}\left(X_z, \nabla_\Phi\right), \quad \{\Gamma_i\}_i \subset \mathrm{H}_n\left(X_z^{an}; \mathcal{L}\right),$$

$\{\psi_i\}_i \subset \mathrm{H}^n_c\left(X_z^{an}, \mathcal{L}\right), \quad \{\delta_i^\vee\}_i \subset \mathrm{H}^{lf}_n\left(X_z^{an}; \mathcal{L}^\vee\right)$: bases

$$P := \left(\int_{\Gamma_j} \Phi\varphi_i\right), \quad P^\vee := \left(\int_{\delta_j^\vee} \Phi^{-1}\psi_i\right),$$

$$I_{ch} := \left(\langle\varphi_i, \psi_j\rangle_{ch}\right), \quad I_h := \left(\langle\Gamma_i, \delta_j^\vee\rangle_h\right)$$

Proposition (Twisted period relation)

$$I_h = {}^tP\, {}^tI_{ch}^{-1} P^\vee$$

Regularization

$\{\varphi_i\}_i \subset \mathrm{H}^n_{dR}(X_z, \nabla_\Phi)$: computable by means of Gröbner basis (日比-西山-高山 (Hibi-Nishiyama-Takayama))

$\{\psi_i\}_i \subset \mathrm{H}^n_c(X_z^{an}, \mathcal{L})$: transcendental

Theorem (Regularization)

$$\mathrm{H}^n_c(X_z^{an}, \mathcal{L}) \tilde{\to} \mathrm{H}^n_{dR}(X_z, \nabla_{\Phi^{-1}})$$

is true for non-resonant γ_l and c (Gelfand-Kapranov-Zelevinsky).

Under regularization, we may assume $\{\psi_i\}_i$ consists of rational forms and computable.

The secondary equation

There exist matrices $\Omega, \Omega^\vee$ whose entries are rational 1-forms such that

$$d_z P = \Omega P, \quad d_z P^\vee = \Omega^\vee P^\vee \quad \text{(Gauß-Manin connection)}$$

This is again, computable.

$$I_h = {}^tP\, {}^tI_{ch}^{-1} P^\vee, \quad I = {}^tI_{ch}^{-1}$$

$$\overset{d_z}{\rightsquigarrow} \quad 0 = d_z I + {}^t\Omega I + I \Omega^\vee \quad \text{(the secondary equation).}$$

Characterization of the c.i.n.

Theorem (M.-H.-高山 (Takayama) ArXiv1904.01253)

Under regularization condition, the secondary equation is regular. Moreover, one has an equality

$$\{\textit{rational solutions of the secondary equation}\} = \mathbb{C}\,{}^tI_{ch}^{-1}.$$

The left-hand side is computable.

Example

We consider

$$f(z) = \int_\Gamma \Phi\varphi$$

$$\Phi = (z_1x^3 + z_2x^2y + z_3x^2y^{-1} + z_4x^2 + z_5x)^{-c_1}x^{c_2}y^{c_3}$$

Interesting case: $c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$ and $z_1 = z_2 = z_3 = z_5 = 1$

$\rightsquigarrow$Period of a family of K3 surfaces (成宮-志賀 (Narumiya-Shiga)).

$\rightsquigarrow$ resonant and non-generic.

We set $c = \begin{pmatrix} 1/2 \\ 1+\varepsilon \\ \varepsilon \end{pmatrix}$ and $z_1 = z_2 = z_3 = 1$.

$$\left\{ \omega = \frac{dx \wedge dy}{xy}, \frac{\partial \log \Phi}{\partial z_5}\omega, \frac{\partial \log \Phi}{\partial z_4}\omega, \frac{\partial^2 \Phi}{\partial z_5}/\Phi\omega \right\}$$

is a basis of the twisted cohomology groups for generic parameters.

We can find a solution I of the secondary equation

$$I = \begin{pmatrix} 1 & \frac{-z_5}{\varepsilon} & \frac{-z_5 - 1/4z_4^2 + 1}{\varepsilon z_4} & 0 \\ \frac{z_5}{\varepsilon} & \frac{-8z_5^2}{4\varepsilon^2 - \varepsilon} & c_{23} & 0 \\ \frac{z_5 + 1/4z_4^2 - 1}{\varepsilon z_4} & c_{32} & \frac{-2z_4^2 + 8}{4\varepsilon^2 - \varepsilon} & c_{34} \\ 0 & 0 & c_{43} & 0 \end{pmatrix}$$

where c_{ij} are rational functions in z_4 and z_5.

$$c_{23} = \frac{(16\varepsilon+8)z_5^2+((-8\varepsilon-10)z_4^2-32\varepsilon-24)z_5+(\varepsilon+1)z_4^4+(-8\varepsilon-8)z_4^2+16\varepsilon+16}{(4\varepsilon^2-\varepsilon)z_4}$$

$$c_{32} = \frac{(16\varepsilon-24)z_5^2+((-8\varepsilon+6)z_4^2-32\varepsilon+40)z_5+(\varepsilon-1)z_4^4+(-8\varepsilon+8)z_4^2+16\varepsilon-16}{(4\varepsilon^2-\varepsilon)z_4}$$

$$c_{34} = \frac{-16z_5^3+(8z_4^2+32)z_5^2+(-z_4^4+8z_4^2-16)z_5}{(4\varepsilon^2-\varepsilon)z_4}$$

$$c_{43} = \frac{16z_5^3+(-8z_4^2-32)z_5^2+(z_4^4-8z_4^2+16)z_5}{(4\varepsilon^2-\varepsilon)z_4}$$

Towards fine structure of the c.i.n.

Problem: we still have the ambiguity of constant multiplication.

We want to extract more explicit information of I_{ch}.

$$I_h = {}^tP^tI_{ch}^{-1}P^\vee \iff I_{ch} = P^tI_h^{-1t}P^\vee$$

P and $P^\vee$ are solutions of GKZ system

$\rightsquigarrow$ Combinatorial structure of the secondary fan

Newton polytope and normal fan

$I_{ch} = \frac{1}{f(z)}\tilde{I}$, $f(z) \in \mathbb{C}[z]$, $\tilde{I}$: polynomial matrix.
$f(z) = \sum_\alpha f_\alpha z^\alpha, \qquad z = (z_1, \dots, z_N)$.
$\rightsquigarrow \mathrm{New}(f) :=$ convex hull of $\{\alpha \mid f_\alpha \neq 0\}$.

$\rightsquigarrow N(f)$: normal fan of $\mathrm{New}(f)$.

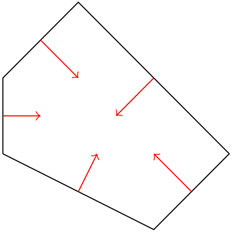


Figure: $\mathrm{New}(f)$

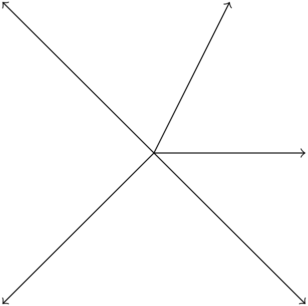


Figure: The normal fan $N(f)$

Normal fan and Laurent expansion

$f \rightsquigarrow \mathrm{New}(f) \rightsquigarrow N(f) \rightsquigarrow X(N(f))$

$X(N(f))$: (partial) toric compactification of $(\mathbb{C}^*)^N$

α_0: vertex of $\mathrm{New}(f)$
$\rightsquigarrow C_0$: cone of $N(f)$
$\rightsquigarrow$ torus fixed point z_0 of $X(N(f))$
$\rightsquigarrow$

$$\frac{1}{f(z)} = \frac{1}{f_{\alpha_0} z^{\alpha_0} \left(1 + \sum_{\alpha \neq \alpha_0} f_{\alpha_0}^{-1} f_\alpha z^{\alpha - \alpha_0}\right)}$$

$$= \{\text{Laurent expansion converging at } z_0\}$$

Expansion theorem

Theorem (M.-H. ArXiv1904.00565, M.-H.-後藤 (Goto) in progress)

The secondary fan F is a refinement of $N(f)$. Moreover, at each torus fixed point z_0 of $X(F)$,

$$\exists(\text{an explicit formula of Laurent expansion of } I_{ch} \text{ around } z_0).$$

F is computable (in the worst case, partially) while $N(f)$ is abstract.

F has a rich combinatorial structure.

Example revisited

$$f(z) = \int_\Gamma \Phi\varphi$$

$$\Phi = (z_1x^3 + z_2x^2y + z_3x^2y^{-1} + z_4x^2 + z_5x)^{-c_1}x^{c_2}y^{c_3}$$

$$c = \begin{pmatrix} 1/2 \\ 1+\varepsilon \\ \varepsilon \end{pmatrix} \text{ and } z_1 = z_2 = z_3 = 1.$$

A solution of the secondary equation is

$$I = \begin{pmatrix} 1 & \frac{-z_5}{\varepsilon} & \frac{-z_5-1/4z_4^2+1}{\varepsilon z_4} & 0 \\ \frac{z_5}{\varepsilon} & \frac{-8z_5^2}{4\varepsilon^2-\varepsilon} & c_{23} & 0 \\ \frac{z_5+1/4z_4^2-1}{\varepsilon z_4} & c_{32} & \frac{-2z_4^2+8}{4\varepsilon^2-\varepsilon} & c_{34} \\ 0 & 0 & c_{43} & 0 \end{pmatrix}.$$

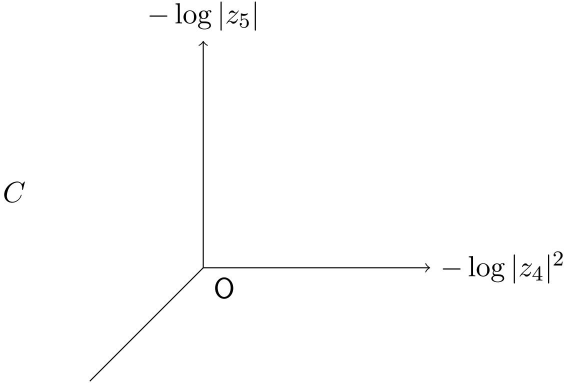
Determining the constant

There exists a constant α such that $I_{ch} = \alpha\, {}^{t}I^{-1}$.

We can show that the $(1,1)$ entry of the matrix ${}^{t}I^{-1}$ is $\frac{8\varepsilon}{4\varepsilon+1}$.

α is determined by $\langle \frac{dx\wedge dy}{xy}, \frac{dx\wedge dy}{xy} \rangle_{ch} = \alpha \frac{8\varepsilon}{4\varepsilon+1}$.

The secondary fan



$\rightsquigarrow$ The (partial) compactification is $(\mathbb{C}^*)^3_{z_1,z_2,z_3} \times \mathbb{P}^2_{z_4^2,z_5}$

$$C \leftrightarrow |z_4^{-2}| << 1 \text{ and } |z_4^{-2} z_5| << 1$$

By the general expansion formula, we get

$$\begin{aligned}
&\frac{1}{(2\pi\sqrt{-1})^2}\langle\frac{dx\wedge dy}{xy},\frac{dx\wedge dy}{xy}\rangle_{ch}\Big|_{z_1=z_2=z_3=z_5=1}\\
=&\frac{1}{2}\Bigg\{\frac{\pi^3}{\sin^2\pi\varepsilon\cos\pi(2\varepsilon)}\varphi_1(z_4;\varepsilon)\varphi_1^\vee(z_4;\varepsilon)\\
&-\frac{2\pi^3}{\sin^2\pi\varepsilon}\varphi_2(z_4;\varepsilon)\varphi_2^\vee(z_4;\varepsilon)\\
&+\frac{\pi^3}{\sin^2\pi\varepsilon\cos\pi(2\varepsilon)}\varphi_3(z_4;\varepsilon)\varphi_3^\vee(z_4;\varepsilon)\Bigg\}. \qquad (1)
\end{aligned}$$

$$\varphi_1(z_4;\varepsilon)$$
$$=z_4^{-\frac{1}{2}-2\varepsilon}\sum_{m,n\geq 0}\frac{z_4^{-2m-2n}}{\Gamma(1+\varepsilon+n)\Gamma(\frac{1}{2}-2\varepsilon-2m-2n)\Gamma(1+\varepsilon+m)m!n!}$$

$$\varphi_1^{\vee}(z_4;\varepsilon)$$
$$=z_4^{\frac{1}{2}+2\varepsilon}\sum_{m,n\geq 0}\frac{z_4^{-2m-2n}}{\Gamma(1-\varepsilon+n)\Gamma(\frac{3}{2}+2\varepsilon-2m-2n)\Gamma(1-\varepsilon+m)m!n!}$$

$$\varphi_2(z_4;\varepsilon)$$
$$=z_4^{-\frac{1}{2}}\sum_{m,n\geq 0}\frac{z_4^{-2m-2n}}{\Gamma(1-\varepsilon+n)\Gamma(\frac{1}{2}-2m-2n)\Gamma(1+\varepsilon+m)m!n!}$$

$$\varphi_2^{\vee}(z_4;\varepsilon)$$
$$=z_4^{\frac{1}{2}}\sum_{m,n\geq 0}\frac{z_4^{-2m-2n}}{\Gamma(1+\varepsilon+n)\Gamma(\frac{3}{2}-2m-2n)\Gamma(1-\varepsilon+m)m!n!}$$

$$\varphi_3(z_4;\varepsilon)$$
$$= z_4^{-\frac{1}{2}+2\varepsilon} \sum_{m,n\geq 0} \frac{z_4^{-2m-2n}}{\Gamma(1-\varepsilon+n)\Gamma(\frac{1}{2}+2\varepsilon-2m-2n)\Gamma(1-\varepsilon+m)m!n!}.$$

$$\varphi_3^{\vee}(z_4;\varepsilon)$$
$$= z_4^{\frac{1}{2}-2\varepsilon} \sum_{m,n\geq 0} \frac{z_4^{-2m-2n}}{\Gamma(1+\varepsilon+n)\Gamma(\frac{3}{2}-2\varepsilon-2m-2n)\Gamma(1+\varepsilon+m)m!n!}.$$

By expansion formula, we have

$$\frac{1}{(2\pi\sqrt{-1})^2}\langle\frac{dx\wedge dy}{xy},\frac{dx\wedge dy}{xy}\rangle_{ch}\ |_{z_1=z_2=z_3=z_5=1,z_4=\infty}=\frac{32}{1-16\varepsilon^2}.$$

Since $\langle\frac{dx\wedge dy}{xy},\frac{dx\wedge dy}{xy}\rangle_{ch}$ is a priori a constant, we have

$$\frac{1}{(2\pi\sqrt{-1})^2}\langle\frac{dx\wedge dy}{xy},\frac{dx\wedge dy}{xy}\rangle_{ch}\ =\frac{32}{1-16\varepsilon^2}.$$

In other words, the cohomology intersection matrix is equal to

$$I_{ch}=(2\pi\sqrt{-1})^2\frac{4}{\varepsilon(1-4\varepsilon)}{}^tI^{-1}.$$

The limit

Taking the limit of (1), we get

$$\tilde{\Phi}\begin{pmatrix} 4\pi^3 & 0 & 0 & \pi \\ 0 & 0 & \pi & 0 \\ 0 & \pi & 0 & 0 \\ \pi & 0 & 0 & 0 \end{pmatrix} {}^t\tilde{\Phi}^\vee = 64. \tag{2}$$

$$\tilde{\Phi} = (\tilde{\Phi}_1, \tilde{\Phi}_2, \tilde{\Phi}_3, \tilde{\Phi}_4)$$

$$\tilde{\Phi}^\vee = (\tilde{\Phi}_1^\vee, \tilde{\Phi}_2^\vee, \tilde{\Phi}_3^\vee, \tilde{\Phi}_4^\vee)$$

The elements of the vector $\tilde{\Phi}$ are as follows.

$$\tilde{\Phi}_1 = \frac{1}{\sqrt{\pi}\sqrt{z_4}} \left(1 + \frac{3}{2z_4^2} + O\left(\left(\frac{1}{z_4}\right)^4\right)\right)$$

$$\tilde{\Phi}_2 = \tilde{\Phi}_3 = \frac{1}{\sqrt{\pi}\sqrt{z_4}}(\phi'_{20} + (\log z_4)\phi'_{21})$$

$$\tilde{\Phi}_4 = \frac{1}{\sqrt{\pi}\sqrt{z_4}}(\phi'_{40} + (\log z_4)\phi'_{41} + (\log z_4)^2\phi'_{42})$$

$$\tilde{\Phi}_2 = \tilde{\Phi}_3 = \frac{1}{\sqrt{\pi}\sqrt{z_4}}(\phi'_{20} + (\log z_4)\phi'_{21})$$

$$\begin{aligned} \phi'_{20} &= \left(\frac{364288}{45045} - 2\gamma - 2\psi^{(0)}\left(-\frac{15}{2}\right)\right) + \\ &\quad \frac{\frac{169093}{30030} - 3\gamma - 3\psi^{(0)}\left(-\frac{15}{2}\right)}{z_4^2} + O\left(\left(\frac{1}{z_4}\right)^4\right) \\ \phi'_{21} &= 2 + \frac{3}{z_4^2} + O\left(\left(\frac{1}{z_4}\right)^4\right) \end{aligned}$$

$$\tilde{\Phi}_4 = \frac{1}{\sqrt{\pi}\sqrt{z_4}}(\phi'_{40} + (\log z_4)\phi'_{41} + (\log z_4)^2\phi'_{42})$$

$$\begin{aligned}
\phi_{40} &= 2\big(66352873472 - 32818705920\gamma + 4058104050\gamma^2 \\
&\quad -2029052025\pi^2 - 32818705920\psi^{(0)}\left(-\frac{15}{2}\right) \\
&\quad +8116208100\gamma\psi^{(0)}\left(-\frac{15}{2}\right) \\
&\quad +4058104050\psi^{(0)}\left(-\frac{15}{2}\right)^2\Bigg)/2029052025 + O\left(\frac{1}{z_4^2}\right) \\
\phi'_{41} &= \left(\frac{1457152}{45045} - 8\gamma - 8\psi^{(0)}\left(-\frac{15}{2}\right)\right) + +O\left(\frac{1}{z_4^2}\right) \\
\phi'_{42} &= 4 + \frac{6}{z_4^2} + O\left(\left(\frac{1}{z_4}\right)^4\right)
\end{aligned}$$

The elements of the vector $\tilde{\Phi}^\vee$ are as follows.

$$\tilde{\Phi}_1^\vee = \frac{2\sqrt{z_4}}{\sqrt{\pi}}\left(1 - \frac{1}{2z_4^2} + O\left(\left(\frac{1}{z_4}\right)^4\right)\right)$$

$$\tilde{\Phi}_2^\vee \quad = \quad \tilde{\Phi}_3^\vee = \frac{2\sqrt{z_4}}{\sqrt{\pi}}(\phi_{20} + (\log z_4)\phi_{21})$$

$$\tilde{\Phi}_4^\vee \quad = \quad \frac{2\sqrt{z_4}}{\sqrt{\pi}}(\phi_{40} + (\log z_4)\phi_{41} + (\log z_4)^2\phi_{42}).$$

Here, ϕ_{ij} and ϕ'_{ij} are power series in z_4^{-2}.

As for $\tilde{\Phi}_1$, it can be related to Thomae's and Gauß' hypergeometric series by a simple transformation

$$\pi^{1/2} z_4^{1/2} \tilde{\Phi}_1(z_4) = {}_3F_2\left({1/4,2/4,3/4 \atop 1,1}; 16/z_4^2\right) = \left({}_2F_1\left({1/8,3/8 \atop 1}; 16/z_4^2\right)\right)^2 .$$

The last identity is the Clausen's identity.

THANK YOU VERY MUCH FOR YOUR ATTENTION!

Regular triangulation

Each cone of the secondary fan F has a combinatorial interpretation.

$$f_\Gamma(z) = \int_\Gamma \prod_{l=1}^{k} h_l(x)^{-\gamma_l} x^c \varphi = \int_\Gamma h(x)^{-\gamma} x^c \varphi.$$

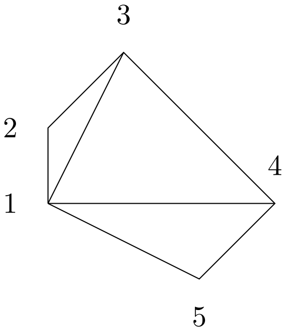
$h_l(x) = h_{l,z^{(l)}}(x) = \sum_{j=1}^{N_l} z_j^{(l)} x^{\mathbf{a}^{(l)}(j)}$

$\rightsquigarrow A_l = \left(\mathbf{a}^{(l)}(1)|\cdots|\mathbf{a}^{(l)}(N_l)\right)$

$$\rightsquigarrow A = \left(\begin{array}{ccc|ccc|c|ccc} 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \hline & \vdots & & & \vdots & & \ddots & & \vdots & \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \\ \hline & A_1 & & & A_2 & & \cdots & & A_k & \end{array}\right)$$

$$= (\mathbf{a}(1)|\cdots|\mathbf{a}(N))$$

For a cone C of the secondary fan F, we can assign a (regular) polyhedral triangulation of the convex body $\Delta_A =$ convex hull of $\{\mathbf{a}(1), \ldots, \mathbf{a}(N)\}$.



$= \{123, 134, 145\}$

Hypergeometric series at the torus fixed point

$\sigma \subset \{1, \dots, N\}$: an $(n+k)$-dimensional simplex, i.e., the square matrix $A_\sigma = (\mathbf{a}(j))_{j\in\sigma}$ is invertible.

$$d := \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_k \\ c \end{pmatrix}$$

$$\varphi_\sigma(z) := z_\sigma^{-A_\sigma^{-1}d} \sum_{\mathbf{m}\in\mathbb{Z}_{\geq 0}^{\bar{\sigma}}} \frac{(z_\sigma^{-A_\sigma^{-1}A_{\bar{\sigma}}} z_{\bar{\sigma}})^{\mathbf{m}}}{\Gamma(\mathbf{1}_\sigma - A_\sigma^{-1}(d + A_{\bar{\sigma}}\mathbf{m}))\mathbf{m}!}$$

T is said to be unimodular if $\det A_\sigma = \pm 1$ for any simplex $\sigma \in T$.

Theorem (M.-H. ArXiv1904.00565)

Suppose that four vectors $\mathbf{a}, \mathbf{a}' \in \mathbb{Z}^{n\times 1}, \mathbf{b}, \mathbf{b}' \in \mathbb{Z}^{k\times 1}$ *and a unimodular regular triangulation T are given. If the parameter* d *is generic, one has an identity*

$$\frac{\langle x^{\mathbf{a}} h^{\mathbf{b}} \frac{dx}{x}, x^{\mathbf{a}'} h^{\mathbf{b}'} \frac{dx}{x} \rangle_{ch}}{(2\pi\sqrt{-1})^n}$$

$$=(-1)^{|\mathbf{b}|+|\mathbf{b}'|}\gamma_1 \cdots \gamma_k (\gamma - \mathbf{b})_{\mathbf{b}} (-\gamma - \mathbf{b}')_{\mathbf{b}'} \times$$

$$\sum_{\sigma\in T} \frac{\pi^{n+k}}{\sin \pi A_\sigma^{-1} d} \varphi_\sigma \left(z; \begin{pmatrix} \gamma - \mathbf{b} \\ c + \mathbf{a} \end{pmatrix} \right) \varphi_\sigma \left(z; \begin{pmatrix} -\gamma - \mathbf{b}' \\ -c + \mathbf{a}' \end{pmatrix} \right)$$

near the torus fixed point corresponding to T. *Here,* $\frac{dx}{x} = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$, $(\gamma - \mathbf{b})_{\mathbf{b}} = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \mathbf{b})}$, *and* $(-\gamma - \mathbf{b}')_{\mathbf{b}'} = \frac{\Gamma(-\gamma)}{\Gamma(-\gamma-\mathbf{b}')}$

What is behind the proof?

The key is the construction of a good basis of $\mathrm{H}_n(X_z;\mathcal{L})$.

Standard way of computing is the method of stationary phase (Lefschetz thimbles)
$\Rightarrow$ I_h is an identity matrix, P and $P^\vee$ are computed through stationary phase approximation (青本, Mizera).

Combinatorial method based on multidimensional Pochhammer cycles
$\Rightarrow$ orthogonal decomposition of the twisted homology groups
$\mathrm{H}_n(X_z;\mathcal{L})=\bigoplus_{\sigma\in T}\mathrm{H}_{n,\sigma}$, $\mathrm{H}_n(X_z;\mathcal{L}^\vee)=\bigoplus_{\sigma\in T}\mathrm{H}^\vee_{n,\sigma}$.
$\Rightarrow$ I_h is not an identity matrix but still computable. P and $P^\vee$ are series solutions.

THANK YOU VERY MUCH FOR YOUR ATTENTION!