Closed string amplitudes from single-valued correlation functions

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Derivative expansion in string theory

String theory low-energy effective action (α' is the inverse tension of the string) in $D \leqslant 10$ dimensions

$$\mathcal{L} = \int d^D x \, |-g|^{\frac{1}{2}} \left(\frac{e^{-2\phi}}{\alpha'^4} \mathcal{R} + \sum_{k \geqslant 0} f_k(\vec{\varphi}) (\alpha' \partial^2)^k \alpha'^3 \mathcal{R}^4 + \cdots \right)$$

The coefficients of the higher derivative terms are automorphic functions of the moduli $\vec{\varphi} \in E_d(\mathbb{R})/K_d$

$$f_k(\gamma \cdot \vec{\varphi}) = f_k(\varphi) \qquad \gamma \in E_d(\mathbb{Z})$$

with $E_d(\mathbb{R})$ the duality group

Supersymmetry leads to differential equations on $f_k(\vec{\varphi})$

This is extremely important for

- UV divergences of maximal supergravity [Green, Russo, Vanhove; Pioline]
  In $D = 4 + 6/L$ with $L = 1, 2, 3$ the counter-term of the UV divergences of maximal supergravity have been extracted from the constant Fourier mode of the automorphic couplings. As well as the $L = 2$ $D = 8$ divergence with counter-term D^6R^4 [Green,Russo,Vanhove; Pioline]
- flux compactification of type II and M-theory and the and de Sitter space vacua
- Flat space limit $AdS_5 \times S^5$ and (discontinuities) of $\mathcal{N} = 4$ correlators [Alday, Bissi, Perlmutter]
- Flat space limit $AdS_4 \times S^7/\mathbb{Z}_k$ and ABJM amplitude [Binder, Chester, Pufu]
- ...

Type IIB superstring in $D = 10$

The vacuum parametrized by $\Omega = C^{(0)} + ie^{-\phi}$ in the duality group coset $SL(2, \mathbb{R})/SO(2)$

The Einstein frame effective action $\ell_P^8 = \alpha'^4 \exp(2\phi)$

$$\mathcal{L}_{IIB} = \int d^{10}x \, |-g|^{\frac{1}{2}} \left(\frac{1}{\ell_P^8} \mathcal{R} + \sum_{k \geqslant 0} f_k^0(\Omega) \, (\ell_P^2 \partial^2)^k \ell_P^6 \, \mathcal{R}^4 + \cdots \right)$$

The coefficients are modular forms for $SL(2, \mathbb{Z})$ with a $U(1)$ weight w set by the R-symmetry charge of the coupling

$$f_k^w \left(\frac{a\Omega + b}{c\Omega + d} \right) = \left(\frac{c\Omega + d}{c\bar{\Omega} + d} \right)^w f_k(\Omega) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Supersymmetry protected couplings

The $\frac{1}{2}$- and $\frac{1}{4}$-BPS protected terms satisfy [Green, Vanhove; Green, Sethi; Sinha]

$$4(\Im m\Omega)^2\partial_\Omega\bar{\partial}_{\bar{\Omega}} f^w_k(\Omega) = \lambda^w_k\, f^w_k(\Omega); \quad k = 0, 2$$

Given by Eisenstein series $f^w_k(\Omega) = E^w_{\frac{3}{2}+k}(\Omega)$

$$E^w_{\frac{3}{2}+k}(\Omega) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{(\Im m\Omega)^{\frac{3+k}{2}}}{(m\Omega+n)^{\frac{3+k}{2}+w}(m\bar{\Omega}+n)^{\frac{3+k}{2}-w}}$$

Supersymmetry protected couplings

- $\mathcal{R}^4$: $\frac{1}{2}$-BPS term one-loop exact

$$\Omega_2^{\frac{1}{2}} \mathrm{E}^0_{\frac{3}{2}}(\Omega) = 2\zeta(3)\Omega_2^2 + 4\zeta(2) + \text{non-pert.}$$

- $\mathrm{D}^4\mathcal{R}^4$: $\frac{1}{4}$-BPS term two-loop exact

$$\Omega_2^{-\frac{1}{2}} \mathrm{E}^0_{\frac{5}{2}}(\Omega) = 2\zeta(5)\Omega_2^2 + 0 + \frac{8}{3}\zeta(4)\Omega_2^{-2} + \text{non-pert.}$$

Only a finite number of perturbative term:

tree-level, one-loop, two-loop

Supersymmetry protected couplings

The $\frac{1}{8}$-BPS is not an Eisenstein series [Green, Vanhove; Green, Miller, Vanhove]

$$4\Im\mathrm{m}(\Omega)^2\partial_\Omega\bar{\partial}_{\bar{\Omega}}f_3^0(\Omega) = 12f_3^0(\Omega) - 6(f_0^0(\Omega))^2$$

with $f_0^0(\Omega) = E^0_{\frac{3}{2}}(\Omega)$ the $\mathcal{R}^4$ coupling.

- $D^6\mathcal{R}^4$: $\frac{1}{8}$-BPS three-loop exact

$$\Omega_2^{-1}f_3^0(\Omega) = \frac{2}{3}\zeta(3)^2\Omega_2^2 + \frac{4\zeta(2)\zeta(3)}{3} + 4\zeta(4)\Omega_2^{-2} + \frac{4\zeta(6)}{27}\Omega_2^{-4} + \text{non-pert.}$$

Only a finite number of perturbative term:

tree-level, one-loop, two-loop, three-loop

Weak coupling expansion

The weak coupling expansion reads

$$\Omega_2^{-\alpha_k} f_k^w(\Omega) = \sum_{g \geqslant 0} a_g^k \Omega_2^{2-2g} + \text{non-pert.}$$

The power behaved terms are the perturbative contributions given by the analytic contribution from genus-g four gravitons amplitudes in string theory

- The tree level coefficients a_0^k are polynomial in odd zeta values
- At genus one $a_1^k/\zeta(2)$ are polynomial in odd zeta values. $\zeta(2)$ is an overall normalisation of the genus one amplitude
- At genus two $a_2^k/\zeta(4)$ are polynomial in odd zeta values.

Tree-level closed string amplitudes

Any closed string tree-level amplitudes can be decomposed

$$M_{N+3}(\mathbf{s},\boldsymbol{\epsilon}) = \sum_r c_r(\mathbf{s},\boldsymbol{\epsilon}) M_{N+3}(\mathbf{s},\mathbf{n}^r,\bar{\mathbf{n}}^r)$$

$c_r(\mathbf{s},\boldsymbol{\epsilon})$ rational functions of kinematic invariant, polarisation tensors, and colour factors

$$M_{N+3}(\mathbf{s},\mathbf{n}^r,\bar{\mathbf{n}}^r) := \int_{(\mathbb{P}^1_{\mathbb{C}})^N} \prod_{i=1}^N d^2 w_i \times$$

$$\times \prod_{1\leqslant i<j\leqslant N} |w_i - w_j|^{2\alpha' s_{ij}} \prod_{i=1}^N |w_i|^{2\alpha' s_{N+1\,i}} |w_i - 1|^{2\alpha' s_{N+2\,i}}$$

$$\prod_{i=1}^N w_i^{n_i} \bar{w}_i^{\bar{n}_i} (1-w_i)^{m_i} (1-\bar{w}_i)^{\bar{m}_i} \prod_{1\leqslant i<j\leqslant N} (w_i - w_j)^{u_{ij}} (\bar{w}_i - \bar{w}_j)^{\bar{u}_{ij}}$$

with n_i, $\bar{n}_i$, m_i, $\bar{m}_i$, u_{ij}, $\bar{u}_{ij}$ integers

Four points closed string amplitude

The partial amplitudes of four points tree-level closed string amplitudes are (n_{ij}, $\bar{n}_{ij}$ in $\mathbb{Z}$)

$$M_4(\mathbf{s},\mathbf{n},\bar{\mathbf{n}}) = \int_{\mathbb{P}^1_{\mathbb{C}}} d^2w |w|^{2\alpha' k_1\cdot k_2} |1-w|^{2\alpha' k_2\cdot k_3} \times w^{n_{12}} \bar{w}^{\bar{n}_{12}} (1-w)^{n_{23}} (1-\bar{w})^{\bar{n}_{23}} .$$

E.g. For $n_{12} = n_{23} = \bar{n}_{12} = \bar{n}_{23} = -1$ we have

$$\int_{\mathbb{P}^1_{\mathbb{C}}} |z|^{2\alpha-2} |1-z|^{2\beta-2} \frac{dz d\bar{z}}{(-2\pi i)} =$$

$$= \frac{(\alpha+\beta)}{\alpha\beta} \exp\left(-2 \sum_{n\geqslant 1} \frac{\zeta(2n+1)}{(2n+1)} (\alpha^{2n+1} + \beta^{2n+1} - (\alpha+\beta)^{2n+1})\right)$$

This expression has only odd zeta values

Four points closed string amplitude

One can evaluate the integral by holomorphic factorisation

$$\int_{\mathbb{P}^1_{\mathbb{C}}} d^2w |w|^{2\alpha' k_1\cdot k_2} |1-w|^{2\alpha' k_2\cdot k_3}$$
$$\times\, w^{n_{12}} \bar{w}^{\bar{n}_{12}} (1-w)^{n_{23}} (1-\bar{w})^{\bar{n}_{23}} = \sin(\alpha'\pi k_2\cdot k_3)$$
$$\times \int_0^1 dw\, w^{\alpha' k_1\cdot k_2} (1-w)^{\alpha' k_2\cdot k_3} w^{n_{12}} (1-w)^{n_{23}}$$
$$\times \int_0^1 dw\, w^{\alpha' k_1\cdot k_2} (1-w)^{\alpha' k_2\cdot k_4} w^{\bar{n}_{12}} (1-w)^{\bar{n}_{23}}$$

Four points open string amplitude

We have the product of standard hypergeometric functions, e.g.

$$\int_{[0,1]} x^{\alpha-1}(1-x)^{\beta-1}dx =$$

$$= \frac{\alpha+\beta}{\alpha\beta}\,\exp\left(\sum_{n\geqslant 2}\frac{(-1)^n\zeta(n)}{n}\left(\alpha^n+\beta^n-(\alpha+\beta)^n\right)\right)$$

These function contain all the ζ-values. But in expansion of the holomorphic factorisation the even ζ-values have disappeared.

This is a manifestation of the single-valued projection introduced by F. Brown

Hyperlogarithms

Definition

For $\sigma_0 := 0, \sigma_1 := 1, \sigma_2, \ldots, \sigma_n$ distinct points of $\mathbb{P}^1_{\mathbb{C}}$ we consider the formal alphabet $X = \{\hat{\sigma}_0, \ldots, \hat{\sigma}_n\}$, with words $w = \hat{\sigma}_{i_1} \cdots \hat{\sigma}_{i_n} \in X^*$.

We define recursively **hyperlogarithms** (w.r.t. X) by setting for all $r \geqslant 0$ $L_{\hat{\sigma}_0^r}(z) := \log^r(z)/r!$ and for all $w = \hat{\sigma}_{i_1} \cdots \hat{\sigma}_{i_n} \in X^*$

$$L_w(z) = \int_0^z \frac{dz'}{z' - \sigma_{i_1}} L_{\hat{\sigma}_{i_2} \cdots \hat{\sigma}_{i_n}}(z').$$

- Holomorphic, multi-valued on $\mathbb{P}^1_{\mathbb{C}} \setminus X$.
- If $X = \{0, 1\}$ we obtain **multiple polylogarithms**

$$\mathrm{Li}_{k_1,\ldots,k_r}(z) = \sum_{0<\nu_1<\cdots<\nu_r} \frac{z^{\nu_r}}{\nu_1^{k_1} \cdots \nu_r^{k_r}}.$$

Special values

Proposition/Definition

For all $\sigma_i \in X$ and all $w \in X^*$, there exists locally $K_i \in \mathbb{Z}_{\geqslant 0}$ s.t.

$$L_w(z) = \sum_{k=0}^{K_i} \sum_{j \geqslant 0} c_{k,j}^{(i)}(w)(z - \sigma_i)^j \log^k(z - \sigma_i).$$

We define $L_w(\sigma_i) := c_{0,0}^{(i)}(w)$, and $\mathcal{S}_X := \mathbb{Q}[\{L_w(\sigma_i)\}_{\sigma_i \in X}]$.

- $L_w(0) = 0$ for any $w \in X^*$.
- $\mathcal{S}_{\{0,1\}}$ is just the ring generated by **multiple zeta values** (MZVs)

$$\zeta(k_1, \ldots, k_r) = \mathrm{Li}_{k_1,\ldots,k_r}(1) = \sum_{0<\nu_1<\cdots<\nu_r} \frac{1}{\nu_1^{k_1} \ldots \nu_r^{k_r}}.$$

The KZ-equation

Theorem

The formal series $L_X(z) := \sum_{w \in X^*} L_w(z)\, w \in \mathbb{C}\langle\langle X^* \rangle\rangle$ is the unique holomorphic multi-valued solution of the **KZ-equation**

$$\frac{\partial}{\partial z} F(z) = \sum_{i=0}^{n} \frac{\hat{\sigma}_i}{z - \sigma_i} F(z)$$

such that $F(z) \sim \exp\left(\hat{\sigma}_0 \log(z)\right)$ as $z \to 0$.

In particular, $L_{\{0,1\}}(1)$ is the Drinfel'd associator.

Theorem (F. Brown)

There is a unique real-analytic single-valued solution $\mathcal{L}_X(z) \in \mathbb{C}\langle\langle X^* \rangle\rangle$ of the **KZ-equation** s.t. $\mathcal{L}_X(z) \sim \exp\left(\hat{\sigma}_0 \log |z|^2\right)$ as $z \to 0$.

Single-valued hyperlogarithms

Definition

If we write $\mathcal{L}_X(z) = \sum_{w \in X^*} \mathcal{L}_w(z)\, w$, we call $\mathcal{L}_w(z)$ **single-valued hyperlogarithms**.

- Single-valued hyperlogarithms are given by $\mathbb{C}$-linear combinations of products $L_{w_1}(z)\overline{L_{w_2}(z)}$.
- $\mathcal{L}_0(z) = L_0(z) + \overline{L_0(z)} = \log(z) + \overline{\log(z)} = \log|z|^2$.

Proposition/Definition

For all $\sigma_i \in X$ and all $w \in X^*$, there exists locally $K_i \in \mathbb{Z}_{\geqslant 0}$ s.t.

$$\mathcal{L}_w(z) = \sum_{k=0}^{K_i} \sum_{m,n \geqslant 0} d^{(i)}_{k,m,n}(w)(z-\sigma_i)^m(\bar{z}-\bar{\sigma}_i)^n \log^k|z-\sigma_i|^2.$$

We define $\mathcal{L}_w(\sigma_i) := d^{(i)}_{0,0,0}(w)$, and $\mathcal{S}^{sv}_X := \mathbb{Q}[\{\mathcal{L}_w(\sigma_i)\}_{\sigma_i \in X}]$.

Single-valued multiple zeta values

- $\mathcal{L}_w(0) = 0$ for all $w \in X^*$.
- If $X = \{0, 1\}$ we call $\mathcal{L}_w(1)$ **single-valued MZVs**. They generate the ring $\mathcal{S}^{\mathsf{sv}}_{\{0,1\}}$.
- We have $\mathcal{S}^{\mathsf{sv}}_{\{0,1\}} \subset \mathcal{S}_{\{0,1\}}$. One can define a map

$$\mathsf{sv} : \zeta(k_1, \ldots, k_r) \longmapsto \zeta^{\mathsf{sv}}(k_1, \ldots, k_r).$$

- $\zeta^{\mathsf{sv}}(2k) = 0$
- $\zeta^{\mathsf{sv}}(2k+1) = 2\zeta(2k+1)$
- $\zeta^{\mathsf{sv}}(3,5) = -10\zeta(3)\zeta(5)$
- $\zeta^{\mathsf{sv}}(3,5,3) = 2\zeta(3,5,3) - 2\zeta(3,5)\zeta(3) - 10\zeta(3)^2\zeta(5)$.

Closed string amplitudes from correlators

Introduce

$$\mathcal{G}_N(z,\bar{z}) :=$$

$$\left\langle \widetilde{V}_1(0)\prod_{r=2}^{N+1}\int d^2z_i\widetilde{\mathcal{V}}(w_r,\bar{w}_r)\widetilde{V}_{N+2}(1)\widetilde{V}_{N+3}(\infty) : e^{ik_*\cdot X(z,\bar{z})} : \right\rangle$$

$$= \int_{\mathbb{C}^N}\prod_{1\leqslant i<j\leqslant N}(w_i-w_j)^{g_{ij}}(\bar{w}_i-\bar{w}_j)^{\bar{g}_{ij}}\prod_{i=1}^{N}d^2w_i$$

$$\prod_{i=1}^{N}w_i^{a_i}(w_i-1)^{b_i}(w_i-z)^{c_i}\bar{w}_i^{\bar{a}_i}(\bar{w}_i-1)^{\bar{b}_i}(\bar{w}_i-\bar{z})^{\bar{c}_i}$$

- By construction we have that $M_{N+3}(\mathbf{s},\mathbf{n},\bar{\mathbf{n}}) = \mathcal{G}_N(1,1)$
- As a CFT correlator $\mathcal{G}_N(z,\bar{z})$ is single-valued function of z
- The difference between the holomorphic and anti-holomorphic exponent are integer (spins) $\mathbf{a}-\bar{\mathbf{a}}, \mathbf{b}-\bar{\mathbf{b}}, \mathbf{c}-\bar{\mathbf{c}}, \mathbf{g}-\bar{\mathbf{g}} \in \mathbb{Z}$

Four points case

For the four points case the holomorphic factorisation reads

$$\mathcal{G}_1(z,\bar{z}) := \int_{\mathbb{C}} w^{a_1}(w-1)^{b_1}(w-z)^{c_1}\bar{w}^{\bar{a}_1}(\bar{w}-1)^{\bar{b}_1}(\bar{w}-\bar{z})^{\bar{c}_1}\, d^2w$$

One can perform the holomorphic factorisation

$$\mathcal{G}_1(z,\bar{z}) = \begin{pmatrix} I_1(\bar{a}_1,\bar{b}_1,\bar{c}_1,\bar{z}) & I_2(\bar{a}_1,\bar{b}_1,\bar{c}_1,\bar{z}) \end{pmatrix} G \begin{pmatrix} I_1(a_1,b_1,c_1,z) \\ I_2(a_1,b_1,c_1,z) \end{pmatrix}$$

with

$$I_1(a_1,b_1,c_1,z) = \int_1^{+\infty} w^{a_1}(w-1)^{b_1}(w-z)^{c_1}\, dw,$$
$$I_2(a_1,b_1,c_1,z) = \int_0^{z} w^{a_1}(1-w)^{b_1}(z-w)^{c_1}\, dw,$$

Holomorphic factorisation

CFT correlator decompose on conformal blocks

$$\mathcal{G}_N(z,\bar z) = \sum_{r,s=1}^{(N+1)!} G_{r,s} I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) I_s(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar z)$$

The conformal block are the ordered integrals

$$I_{(\sigma,\rho)}(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) = \int_{\Delta(\sigma,\rho)} \prod_{j=1}^{N} dw_j$$

$$\prod_{m,n} |w_m - w_n|^{g_{mn}} \prod_m w_m^{a_m} (w_m - 1)^{b_m} (w_m - z)^{c_m} ,$$

integrated along the real line

$$\Delta(\sigma,\rho) := \{0 \leqslant w_{\rho(1)} \leqslant \cdots \leqslant w_{\rho(s)} \leqslant z \leqslant 1 \leqslant w_{\sigma(1)} \leqslant \cdots \leqslant w_{\sigma(r)}\}$$

Aomoto-Gel'fand integrals

$$I_{(\sigma,\rho)}(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) = \int_{\Delta(\sigma,\rho)} \prod_{j=1}^{N} dw_j$$

$$\prod_{m,n} |w_m - w_n|^{g_{mn}} \prod_m w_m^{a_m} (w_m - 1)^{b_m} (w_m - z)^{c_m} \,,$$

integrated along the real line

$$\Delta(\sigma,\rho) := \{0 \leqslant w_{\rho(1)} \leqslant \cdots \leqslant w_{\rho(s)} \leqslant z \leqslant 1 \leqslant w_{\sigma(1)} \leqslant \cdots \leqslant w_{\sigma(r)}\}$$

The integrals $I_{(\sigma,\rho)}(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z)$ are

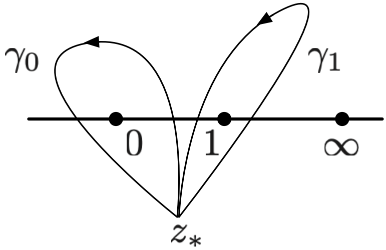
- Aomoto-Gel'fand (hypergeometric) integrals
- multivalued solution of the Knizhnik-Zamolodchikov equations
- conformal blocks for the CFT

Monodromies

As CFT correlator $\mathcal{G}_N(z,\bar{z})$ is single-valued in $\mathbb{C}$

$$\mathcal{G}_N(z,\bar{z}) = \sum_{r,s=1}^{(N+1)!} G_{r,s} I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) I_s(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$$

The integrals $I_r(\cdots;z)$ have monodromies



$$I_r(\cdots;z) \xrightarrow{\gamma_0} \sum_s (g_0)_r{}^s I_s(\cdots;z)$$

$$I_r(\cdots;z) \xrightarrow{\gamma_1} \sum_s (g_1)_r{}^s I_s(\cdots;z)$$

The monodromy matrices g_0 and g_1 are the *same* for $I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z)$ and $I_r(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$ because $\mathbf{a}-\bar{\mathbf{a}} \in \mathbb{Z}^N$, $\mathbf{b}-\bar{\mathbf{b}} \in \mathbb{Z}^N$, $\mathbf{c}-\bar{\mathbf{c}} \in \mathbb{Z}^N$, $\mathbf{g}-\bar{\mathbf{g}} \in \mathbb{Z}^{\frac{N(N+1)}{2}}$

Monodromies around $z = 0$

$I_r(\cdots; z)$ have diagonal monodromies around $z = 0$

$$\mathcal{G}_N(z,\bar{z}) = \sum_{r,s=1}^{(N+1)!} G_{r,s} I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) I_s(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$$

This imposes that the matrix G_{rs} has the bloc diagonal form

$$G_N = \begin{pmatrix} G_N^{(1)} & 0 & 0 \\ 0 & G_N^{(2)} & 0 \\ 0 & 0 & G_N^{(3)} \end{pmatrix}$$

- $G_N^{(i)}$ with $i = 1, 3$ are real square matrices of size $N!$
- $G_N^{(2)}$ are diagonal matrix of size $(N-1)\,N!$

Monodromies around $z = 1$

The monodromies of $I_r(\cdots;z)$ around $z = 1$ are not diagonal but $J_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) := I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};1-z)$ have diagonal monodromies around $z = 1$

$$\mathcal{G}_N(z,\bar{z}) = \sum_{r,s=1}^{(N+1)!} \hat{G}_{r,s} J_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) J_s(\bar{\mathbf{a}},\bar{\mathbf{b}},\bar{\mathbf{c}};\bar{\mathbf{g}};\bar{z})$$

therefore

$$\hat{G}_N = \begin{pmatrix} \hat{G}_N^{(1)} & 0 & 0 \\ 0 & \hat{G}_N^{(2)} & 0 \\ 0 & 0 & \hat{G}_N^{(3)} \end{pmatrix}$$

- $\hat{G}_N^{(i)}$ with $i = 1,3$ are real square matrices of size $N!$
- $\hat{G}_N^{(2)}$ are diagonal matrix of size $(N-1)\,N!$

Monodromies constraints

The two sets of integral are related by linear relations derived using the contour deformation method of [Bjerrum-Bohr, Damgaard, Vanhove]

$$I_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z) = \sum_{r=1}^{(N+1)!} S(\mathbf{A},\mathbf{B},\mathbf{C};\mathbf{G})_r{}^s J_r(\mathbf{a},\mathbf{b},\mathbf{c};\mathbf{g};z)$$

We need to solve the linear system

$$\hat{G}_N = S(\mathbf{A},\mathbf{B},\mathbf{C};\mathbf{G}) \begin{pmatrix} G_N^{(1)} & 0 & 0 \\ 0 & G_N^{(2)} & 0 \\ 0 & 0 & G_N^{(3)} \end{pmatrix} S(\mathbf{A},\mathbf{B},\mathbf{C};\mathbf{G})$$

must have the above block diagonal form of $\hat{G}_N$

Monodromies constraints

- The linear system has *unique solution* up to an scale
- Matching the closed string partial amplitude *determines* the scale factor, *therefore there is no ambiguities*
- The coefficients of the matrices G_N and $\hat{G}_N$ are rational functions $\sin(\pi\alpha' x)$ where x are linear combination of kinematic invariants. This is a non-local version of the momentum kernel
- The small α' expansion of the $I_r(\cdots;z)$ and $J_r(\cdots;z)$ are on multiple polylogarithm with coefficients polynomials of MZV's and $2\pi i$.
- The proof is constructive as it is for any CFT minimal models [Dotsenko, Fateev]

Matching closed string amplitudes

At $z = 1$ we get the colour-ordered open string amplitudes

$$J_{(\sigma,\emptyset)}(\mathfrak{a},\mathfrak{b},\mathfrak{c};\mathfrak{g};1) = A_{N+3}(\sigma(1,\dots,N+1),1,N+2,N+3;\mathfrak{n})$$
$$J_{(\sigma,\rho)}(\mathfrak{a},\mathfrak{b},\mathfrak{c};\mathfrak{g};1) = 0$$

$$M_{N+3}(\mathfrak{s},\mathfrak{n},\bar{\mathfrak{n}}) = \sum_{\sigma,\rho\in\mathfrak{S}_N} \hat{G}_{\sigma,\rho} \times A_{N+3}(\sigma(2,\dots,N+1),1,N+2,N+3;\mathfrak{n}) \times \bar{A}_{N+3}(\rho(2,\dots,N+1),1,N+2,N+3;\bar{\mathfrak{n}}),$$

The α' has only single-valued multiple zeta values as the valuation at $z = 1$ of combination of single-valued multiple polylogarithms

Back to the four-point amplitude

$$\mathcal{G}_1(z,\bar z) = \begin{pmatrix} J_{((1),\emptyset)}(\bar a,\bar b,\bar c;\bar z) & J_{(\emptyset,(1))}(\bar a,\bar b,\bar c;\bar z)\end{pmatrix} \hat G_1 \begin{pmatrix} J_{((1),\emptyset)}(a,b,c;z) \\ J_{(\emptyset,(1))}(a,b,c;z)\end{pmatrix}$$

with

$$\hat G_1 = \begin{pmatrix} -\frac{\sin(\pi(A_1+B_1+C_1))\sin(\pi A_1)}{\sin(\pi(B_1+C_1))} & 0 \\ 0 & -\frac{\sin(\pi C_1)\sin(\pi B_1)}{\sin(\pi(B_1+C_1))}\end{pmatrix}$$

The J_r integrals map at $z=1$ to the open string amplitudes

$$J_{((1),\emptyset)}(a,b,c;1) = A_4(2,1,3,4;\mathfrak{n});$$
$$J_{(\emptyset,(1))}(a,b,c;1) = 0\,.$$

The value at $z=1$ gives $M_4(\mathfrak{s},\mathfrak{n},\bar{\mathfrak{n}}) = \mathcal{G}_1(1,1)$ gives the non-local version of the KLT relations given in [Bjerrum-Bohr, Damgaard, Vanhove]

$$M_4(\mathfrak{s},\mathfrak{n},\bar{\mathfrak{n}}) = \frac{\sin(2\pi\alpha' k_1\cdot k_2)\sin(2\pi\alpha' k_2\cdot k_4)}{\sin(2\pi\alpha' k_2\cdot k_3)}|A_4(2,1,3,4;\mathfrak{n})|^2$$

Remarks

$$\mathcal{M}_{N+3}(\mathbf{s},\boldsymbol{\epsilon}) = \sum_r c_r(\mathbf{s},\boldsymbol{\epsilon})\mathcal{M}_{N+3}(\mathbf{s},\mathbf{n}^r,\bar{\mathbf{n}}^r)$$

with $\mathcal{M}_{N+3}(\mathbf{s},\mathbf{n},\bar{\mathbf{n}}) = \mathcal{G}_N(1,1)$

- It is not necessary that the total amplitude is given by the special value at $z=1$ of a single-valued correlation function. It is enough that each partial amplitude arises this way
- a given order in the α'-expansion can mix single-valued multiple zeta values of different way (due to tachyonic pole in the kinematic coefficients $c_r(\mathbf{s},\boldsymbol{\epsilon})$ for heterotic-string amplitudes)

Yet another look at the four points amplitude

We look back at the closed string four point amplitude

$$\int_{\mathbb{P}^1_{\mathbb{C}}} |z|^{2\alpha-2}|1-z|^{2\beta-2}\frac{dz d\bar{z}}{(-2\pi i)}$$

$$= \int_{\mathbb{P}^1_{\mathbb{C}}} \sum_{m,n\geqslant 0} \frac{\alpha^m \beta^n}{m!n!}\frac{(\log|z|^2)^m(\log|1-z|^2)^n}{|z|^2|1-z|^2}\frac{dz d\bar{z}}{(-2\pi i)}$$

$$= \int_{\mathbb{P}^1_{\mathbb{C}}} \sum_{m,n\geqslant 0} \alpha^m \beta^n \frac{\mathcal{L}_{0^m}(z)\mathcal{L}_{1^n}(z)}{|z|^2|1-z|^2}\frac{dz d\bar{z}}{(-2\pi i)}$$

$$= \frac{\alpha+\beta}{\alpha\beta}\sum_{m,n\geqslant 1} \alpha^m \beta^n \int_{\mathbb{P}^1_{\mathbb{C}}} \frac{\mathcal{L}_{0^m}(z)\mathcal{L}_{1^n}(z)}{|z|^2|1-z|^2}\frac{dz d\bar{z}}{(-2\pi i)}$$

We have the integral single-valued multiple-polylogarithm.

Integration of single-valued hyperlogarithms

Theorem (O. Schnetz)

Let $\mathcal{A}_X^{sv} := \mathbb{C}\left[z, \frac{1}{z-\sigma_i}, \bar{z}, \frac{1}{\bar{z}-\bar{\sigma}_i}, \{\mathcal{L}_w(z)\}_{w \in X^*}\right]$.
If $f(z), F(z) \in \mathcal{A}_{X,\mathbb{C}}^{sv}$, $\int_{\mathbb{P}^1_\mathbb{C}} f(z) dz d\bar{z} < \infty$ and $\frac{\partial}{\partial z} F(z) = f(z)$ then

$$\int_{\mathbb{P}^1_\mathbb{C}} f(z) \frac{dz d\bar{z}}{(-2\pi i)} = \mathrm{Res}_{\bar{z}=\infty} F(z) - \sum_{i=0}^{n} \mathrm{Res}_{\bar{z}=\bar{\sigma}_i} F(z)$$

Theorem (F. Brown)

For all $f(z) \in \mathcal{A}_X^{sv}$ there exists $F(z) \in \mathcal{A}_X^{sv}$ such that $\frac{\partial}{\partial z} F(z) = f(z)$.

Back to the four-point amplitude

For all $m, n \geqslant 1$ we have that

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \frac{\mathcal{L}_{0^m}(z)\mathcal{L}_{1^n}(z)}{|z|^2|1-z|^2} dz d\bar{z} < \infty.$$

Moreover,

$$F(z) = \sum_{w=0^m 1^n} \frac{\mathcal{L}_{0w}(z) - \mathcal{L}_{1w}(z)}{\bar{z}(1-\bar{z})}$$

satisfies

$$\frac{\partial}{\partial z} F(z) = \frac{\mathcal{L}_{0^m}(z)\mathcal{L}_{1^n}(z)}{|z|^2|1-z|^2}.$$

Therefore, by Schnetz's theorem

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \frac{\mathcal{L}_{0^m}(z)\mathcal{L}_{1^n}(z)}{|z|^2|1-z|^2} \frac{dz d\bar{z}}{(-2\pi i)} = \sum_{w=0^m 1^n} \mathcal{L}_{0w}(1) - \mathcal{L}_{1w}(1) \in \mathcal{S}^{\mathrm{sv}}_{\{0,1\}}$$

The five-point amplitude

For all $m, n, p, q, r \geqslant 1$ we have using Schnetz's theorem

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^2} \frac{\mathcal{L}_{0^m}(z)\mathcal{L}_{1^n}(z)\mathcal{L}_{0^p}(u)\mathcal{L}_{1^q}(u)\mathcal{L}_{z^r}(u)}{|z|^2|1-z|^2|u|^2|1-u|^2} \frac{dz d\bar{z} du d\bar{u}}{(-2\pi i)^2} < \infty$$

$$= \int_{\mathbb{P}^1_{\mathbb{C}}} \sum_{w=0^p 1^q z^r} \frac{(\mathcal{L}_{0w}(1) - \mathcal{L}_{1w}(1))\mathcal{L}_{0^m}(z)\mathcal{L}_{1^n}(z)}{|z|^2|1-z|^2} \frac{dz d\bar{z}}{(-2\pi i)}$$

Problem: the dependence on z is also in the alphabet!

Theorem (V., Zerbini)

If $\hat{\sigma}_i \in X$, we define X_i to be $X \setminus \{\hat{\sigma}_i\}$. For any $w \in X^*$, any $2 \leqslant i \leqslant n$ and any $0 \leqslant j \leqslant n$

$$\mathcal{L}_w(\sigma_j) = \sum_u c_u \mathcal{L}_u(\sigma_i),$$

where $c_u \in \mathcal{S}^{\mathsf{sv}}_{X_i,\mathbb{Q}}$ and the sum over words $u \in X_i^*$ is finite.

Higher-point amplitudes

Let $k = n - 3$, $\sigma, \rho \in \mathfrak{S}_k$ a permutation of k letters

Open superstring integrals:

$$Z^{(k)}_{\sigma,\rho}(s_{i,j}) =$$

$$\int_{0\leqslant x_{\sigma(1)}\leqslant\cdots\leqslant x_{\sigma(k)}\leqslant 1} \frac{\prod_{i=1}^{k} x_i^{s_{0,i}}(1-x_i)^{s_{i,k+1}}\prod_{1\leqslant i<j\leqslant k}(x_i-x_j)^{s_{i,j}}}{x_{\rho(1)}(1-x_{\rho(k)})\prod_{i=1}^{k}(x_{\rho(i)}-x_{\rho(i-1)})}.$$

Closed superstring integrals:

$$(-2\pi i)^k J^{(k)}_{\sigma,\rho}(s_{i,j}) =$$

$$\int_{(\mathbb{P}^1_{\mathbb{C}})^k} \frac{\prod_{i=1}^{k}|z_i|^{2s_{0,i}}|1-z_i|^{2s_{i,k+1}}\prod_{1\leqslant i<j\leqslant k}|z_i-z_j|^{2s_{i,j}}}{z_{\rho(1)}\bar{z}_{\sigma(1)}(1-z_{\rho(k)})(1-\bar{z}_{\sigma(k)})\prod_{i=1}^{k}(z_{\rho(i)}-z_{\rho(i-1)})(\bar{z}_{\sigma(i)}}$$

Old results

Theorem (Broedel, Schlotterer, Stieberger, Terasoma)

All $Z^{(k)}_{\sigma,\rho}(s_{i,j})$ can be expanded as Laurent series as the $s_{i,j}$'s tend to zero. The coefficients of these series belong to $\mathcal{S}_{\{0,1\}}$ (MZVs).

Before 2018:

Conjecture (Schlotterer, Stieberger)

All $J^{(k)}_{\sigma,\rho}(s_{i,j})$ can be expanded as Laurent series as the $s_{i,j}$'s tend to zero. The coefficients of these series belong to $\mathcal{S}^{\mathsf{sv}}_{\{0,1\}}$ (single-valued MZVs).

Conjecture (Stieberger)

$$\mathsf{sv}(Z^{(k)}_{\sigma,\rho}(s_{i,j})) = J^{(k)}_{\sigma,\rho}(s_{i,j}).$$

New results

Theorem (Broedel, Schlotterer, Stieberger, Terasoma)

All $Z^{(k)}_{\sigma,\rho}(s_{i,j})$ can be expanded as Laurent series as the $s_{i,j}$'s tend to zero. The coefficients of these series belong to $\mathcal{S}_{\{0,1\}}$ (MZVs).

After 2018:

Theorem (Brown, Dupont – V., Zerbini)

All $J^{(k)}_{\sigma,\rho}(s_{i,j})$ can be expanded as Laurent series as the $s_{i,j}$'s tend to zero. The coefficients of this series belong to $\mathcal{S}^{\mathsf{sv}}_{\{0,1\}}$ (single-valued MZVs).

Theorem (Brown, Dupont)

$$\mathsf{sv}(Z^{(k)}_{\sigma,\rho}(s_{i,j})) = J^{(k)}_{\sigma,\rho}(s_{i,j}).$$

Conclusion

- The holomorphic factorisation construction clarifies the role of the momentum kernel in the single-valued projection $\mathcal{S}_{\alpha'}$ is one block of $G_N S(\boldsymbol{A}, \mathbf{B}, \mathbf{C}; \mathbf{G})$
- Notice that the α' expansion does not need to have uniform weight: tree-level heterotic string from the tachyonic pole, or genus two type II expansion [Green, Vanhove]
- Closed string amplitude are special value single-valued CFT correlators, and open strings are multivalued conformal block extended to higher genus
- The low-energy expansion of genus one closed string amplitudes has single-valued modular graph functions [D'Hoker, Green, Gurdogan, Vanhove; Zerbini; Brown; Gerken, Kleinschmidt, Schlotterer]
- Single-valued modular graph functions in degeneration limits of genus-two amplitudes [D'Hoker, Green, Pioline]