Iterated Eisenstein integrals and analytic continuation of Feynman integrals

Claude Duhr

based on work in in collaboration with
J. Brödel, F. Dulat, Robin Marzucca, B. Penante, L. Tancredi

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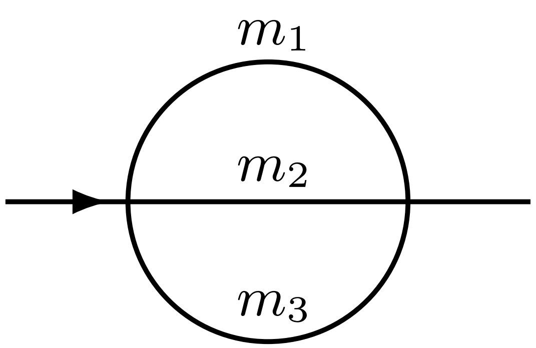
Feynman integrals

- Feynman integrals that evaluate to multiple polylogarithms (MPLs) are well understood.

- MPLs are not the end of the story.
  - ➡ Prime example: the massive sunrise.



[Sabry; Broadhurst; Bauberger, Berends, Bohm, Buza; Caffo, Czyz, Laporta, Remiddi; Laporta Remiddi; Bloch, Vanhove; Remiddi, Tancredi; Adams, Bogner, Weinzierl, Schweitzer; Broedel, CD, Dulat, Penante, Tancredi; Hidding, Moriello]

- Goal of this talk:
  - ➡ Review (some) functions related to elliptic and modular curves that show up in Feynman integrals.
  - ➡ Focus on families of hypergeometric functions to illustrate ideas.
  - ➡ All concepts also show up for Feynman integrals.

Hypergeometric functions

- Consider the family of integrals: $n_i \in \mathbb{Z} \quad a,b,c \in \mathbb{C}$

$$T(n_1,n_2,n_3;\lambda) = \int_0^1 dx\, x^{n_1+a\epsilon}\,(1-x)^{n_2+b\epsilon}\,(1-\lambda x)^{n_3+c\epsilon}$$

  ➡ For simplicity: $a=b=c=1$

- There are two 'master integrals':

$$T_1(\lambda) = T(-1,0,0;\lambda) \qquad T_2(\lambda) = T(0,0,-1;\lambda)$$

- Goal: Compute the first few terms in the expansion in ϵ:

$$T_i(\lambda) = \sum_{k\geq k_0} t_{i,k}(\lambda)\,\epsilon^k$$

- Q1: How can we compute the $t_{i,k}(\lambda)$?
- Q2: What are the properties of the $t_{i,k}(\lambda)$?

Direct integration

- If the integral is finite as $\epsilon \to 0$, expand under the integral sign:

$$T_2(\lambda) = \int_0^1 dx\, x^\epsilon\, (1-x)^\epsilon\, (1-\lambda x)^{-1+\epsilon}$$

$$= \int_0^1 \frac{dx}{1-\lambda x} \left[1 + \epsilon\, (\log x + \log(1-x) + \log(1-\lambda x)) + \mathcal{O}(\epsilon^2)\right]$$

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- Then integrate back in terms of MPLs:

[Poincaré; Kummer; … ; Goncharov; Brown]

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1}\,G(a_2,\ldots,a_n;t) \qquad G(a_1;z) = \log\left(1-\frac{z}{a_1}\right)$$

$$T_2(\lambda) = -\frac{1}{\lambda}\,G(1;\lambda) + \frac{2\epsilon}{\lambda}\,[G(0,1;\lambda) - G(1,1;\lambda)] + \mathcal{O}(\epsilon^2)$$

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$$T_2(\lambda) = -\frac{1}{\lambda}\, G(1;\lambda) + \frac{2\epsilon}{\lambda}\, [G(0,1;\lambda) - G(1,1;\lambda)] + \mathcal{O}(\epsilon^2)$$

- $T_1(\lambda)$ diverges, but can still be done with slight modification:

$$T_1(\lambda) = \frac{1}{\epsilon} + \epsilon\, [G(0,1;\lambda) - \zeta_2] + \mathcal{O}(\epsilon^2)$$

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  - ➡ 1st de Rham cohomology of punctured Riemann sphere is generated by [the classes] of

$$d\log(x - x_i) \qquad\qquad x_i \in \{0, 1, 1/\lambda\}$$

  - ➡ Rational fct. in $x \xrightarrow{\int dx} \log(x - x_i) \xrightarrow{\int dx} \ldots \xrightarrow{\int dx}$ MPLs

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- Numerical evaluation: We have fast and general numerical codes to evaluate MPLs: [GiNaC; ...]
  - ➡ Need to fix a branch for the log, e.g., for $\lambda > 1$

$$G(1;\lambda) = \log(1-\lambda) = \log(1 - 1/\lambda) + \log\lambda + i\pi$$

Differential equations

- We obtain a basis of pure functions: [Arkani-Hamed, Bourjaily, Cachazo, Trnka]

$$\begin{pmatrix} T_1(\lambda) \\ T_2(\lambda) \end{pmatrix} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon\,\lambda \end{pmatrix} \begin{pmatrix} P_1(\lambda) \\ P_2(\lambda) \end{pmatrix}$$

$$P_1(\lambda) = 1 + \epsilon^2\,[G(0,1;\lambda) - \zeta_2] + \mathcal{O}(\epsilon^3)$$

$$P_2(\lambda) = -\epsilon\,G(1;z) + 2\epsilon^2\,[G(0,1;\lambda) - G(1,1;\lambda)] + \mathcal{O}(\epsilon^3)$$

➡ Differentiation lowers the weight.

➡ Only log-singularities.

- Pure functions satisfy nice differential equations: [Henn]

$$\partial_\lambda \begin{pmatrix} P_1(\lambda) \\ P_2(\lambda) \end{pmatrix} = \epsilon\left(\frac{A_0}{\lambda} + \frac{A_1}{1-\lambda}\right) \begin{pmatrix} P_1(\lambda) \\ P_2(\lambda) \end{pmatrix}$$

$$A_0 = \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$$

Properties

	Non-elliptic 2F1	Elliptic 2F1
Direct integration	✓ [MPLs]	
Pure basis	✓ [Uniform weight]	
Canonical DE	✓ [dlogs]	
Numerical Evaluation	✓	

Elliptic 2F1

- Consider the family of integrals: $n_i \in \mathbb{Z}$ $\quad a, b, c \in \mathbb{C}$

$$T(n_1, n_2, n_3; \lambda) = \int_0^1 dx\, x^{-1/2+n_1+a\epsilon}\,(1-x)^{-1/2+n_2+b\epsilon}\,(1-\lambda x)^{-1/2+n_3+c\epsilon}$$

  ➡ For simplicity: $a = b = c = 1$

  ➡ 'Elliptic ${}_2F_1$': $y^2 = x(x-1)(x-1/\lambda)$ defines a family of elliptic curves.

- There are two 'master integrals':

$$T_1(\lambda) = T(0,0,0;\lambda) \qquad T_2(\lambda) = T(1,0,0;\lambda)$$

$$T_1(\lambda)_{|\epsilon=0} = \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}} = 2\,\mathrm{K}(\lambda)$$

- Goal: illustrate how the concepts known from previous example generalise to elliptic case.

Elliptic 2F1

- Proceed in the same way as in the non-elliptic case:

$$T_1(\lambda) = \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{dx}{y} \left[1 + \epsilon \left(\log x + \log(1-x) + \log(1-\lambda x)\right) + \mathcal{O}(\epsilon^2)\right]$$

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- 1st de Rham cohomology of (punctured) elliptic curve generated by

$$\frac{dx}{y} \qquad \frac{x\,dx}{y} \qquad \frac{dx}{x-x_i} \qquad \frac{dx}{y(x-x_i)}$$

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- Build iterated integrals [from kernels with log-singularities]: elliptic MPLs [Brown, Levin]

$$\mathcal{E}_3\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; \chi\right) = \int_0^\chi dx\, \varphi_{n_1}(x,c)\, \mathcal{E}_3\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; x\right)$$

$$\varphi_0(x,0) = \frac{dx}{2\sqrt{\lambda}\,\mathrm{K}(\lambda)\, y}$$

$$\varphi_1(x,c) = \frac{dx}{x-c}$$

$$T_1(\lambda) = 2\mathrm{K}(\lambda) \left[1 + 2\epsilon \left(\mathcal{E}_3\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}; 1\right) + \mathcal{E}_3\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}; 1\right) + \mathcal{E}_3\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1/\lambda \end{smallmatrix}; 1\right)\right) + \mathcal{O}(\epsilon^2)\right]$$

Elliptic polylogarithms

- eMPLs also appear in string amplitudes — Relation?

- Definition of eMPLs in appearing in string amplitudes:

Genus 1: $\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k \end{smallmatrix}; z, \tau\right) = \int_0^z dz'\, g^{(n_1)}(z' - z_1, \tau)\, \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ z_2 & \dots & z_k \end{smallmatrix}; z', \tau\right)$ $\qquad n_i \in \mathbb{N}$, $z_i \in \mathbb{C}$

[~ Brown, Levin; Brödel, Mafra, Matthes, Schlotterer]

- Eisenstein-Kronecker series:

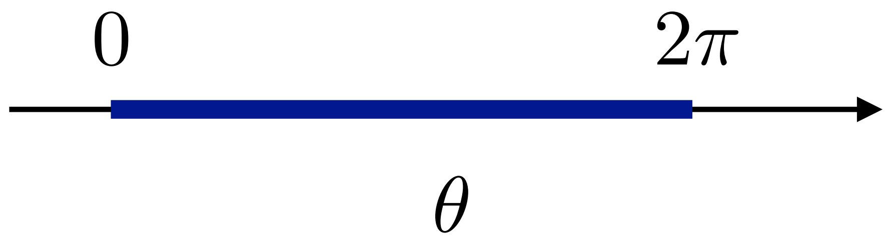
$$F(z, \alpha, \tau) = \frac{1}{\alpha} \sum_{n \geq 0} g^{(n)}(z, \tau)\, \alpha^n = \frac{\theta_1'(0, \tau)\, \theta_1(z + \alpha, \tau)}{\theta_1(z, \tau)\, \theta_1(\alpha, \tau)}$$

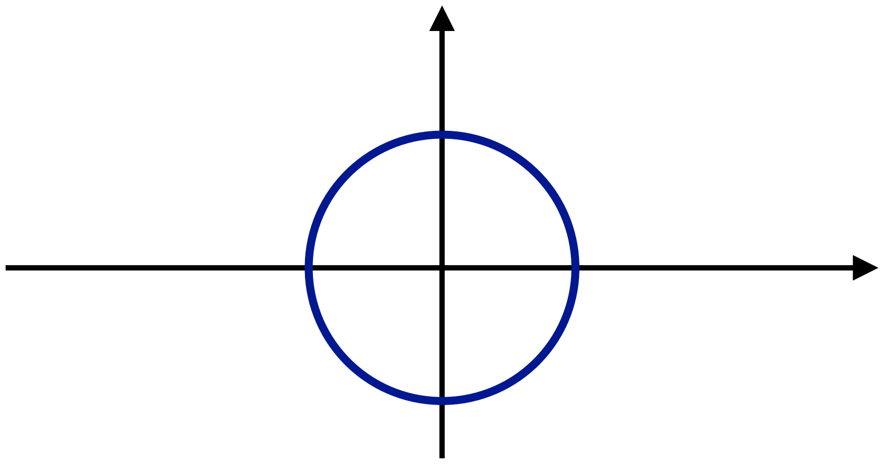
➡ Each $g^{(n)}$ has (at most) simple poles at $z = m + n\tau$, $m, n \in \mathbb{Z}$.

- Relation between $\mathcal{E}_3$ and $\tilde{\Gamma}$?

The circle

- How to describe a circle?





(x, y) with $y^2 = 1 - x^2$

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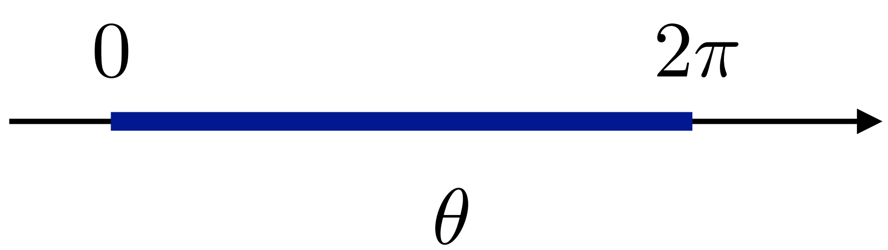
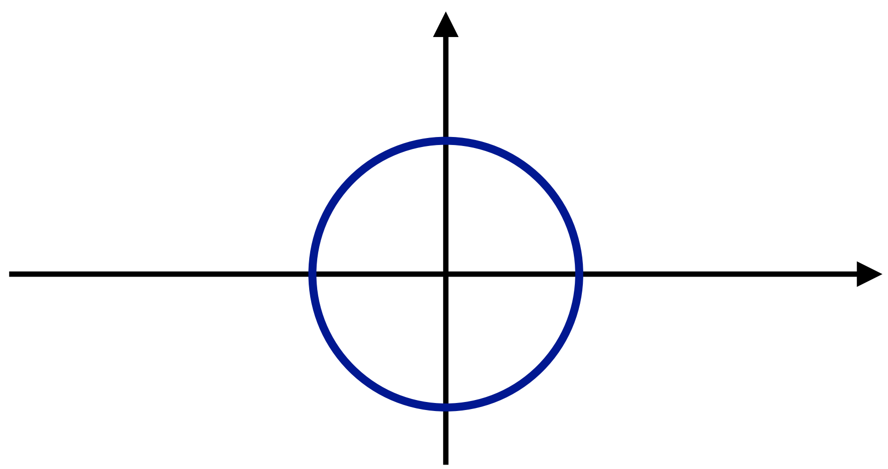




(x, y) with $y^2 = 1 - x^2$

➡ Can rescale 'circumference' to 1.

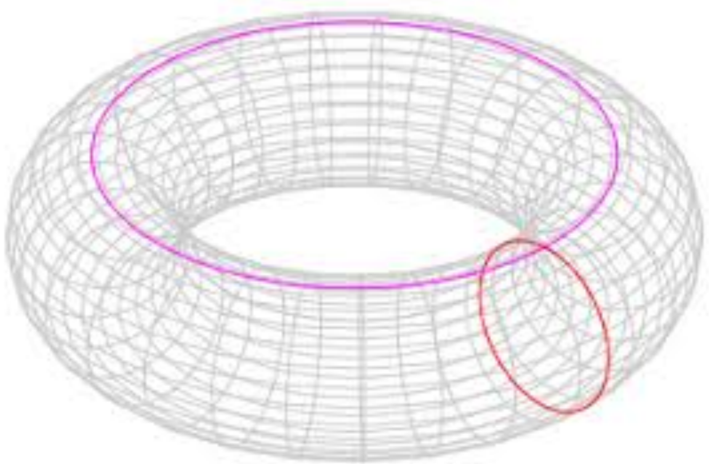
➡ Trigonometric function: $\cos\theta$

$$(\cos'\theta)^2 = 1 - (\cos\theta)^2 \qquad \cos(\theta + 2\pi) = \cos\theta$$

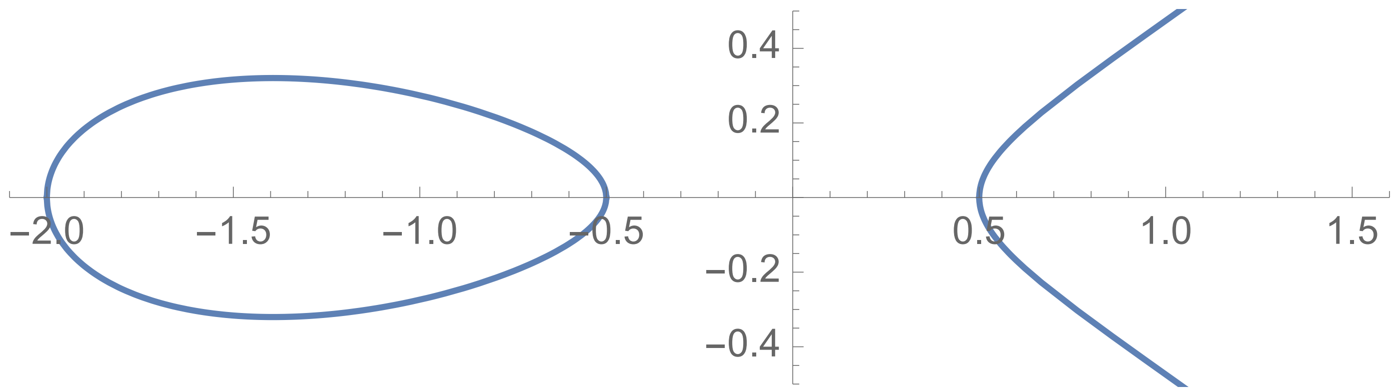
➡ Inverse map: $\theta = -\int_0^x \frac{dx'}{\sqrt{1 - x'^2}}$

Elliptic curves

- Elliptic curves are the same as tori!



z



$[x, y, 1]$ with $\quad y^2 = 4x^3 - g_2 x - g_3$

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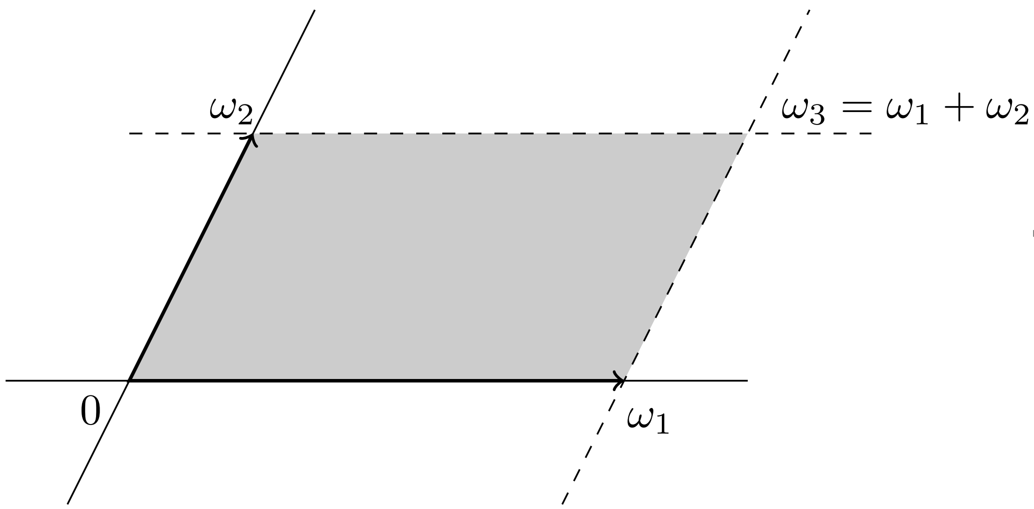

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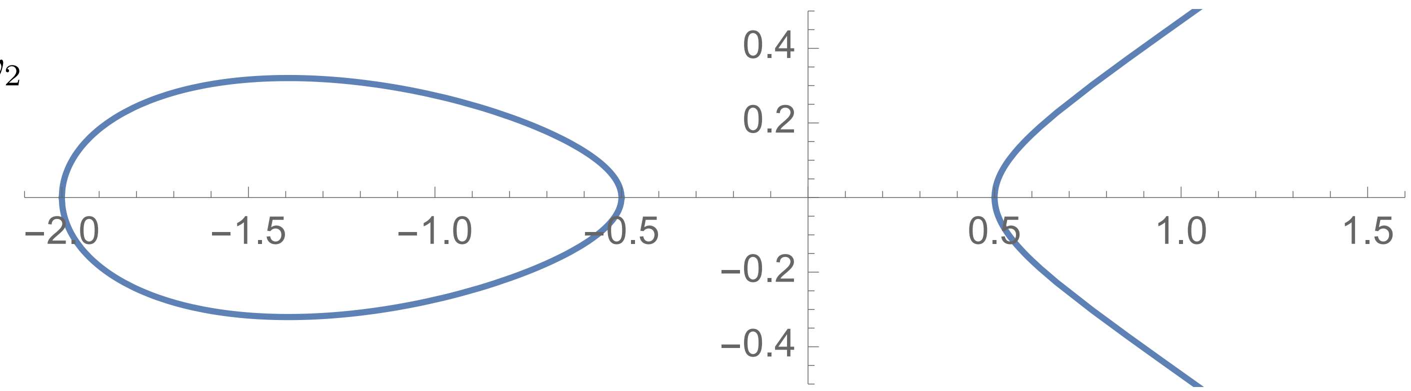

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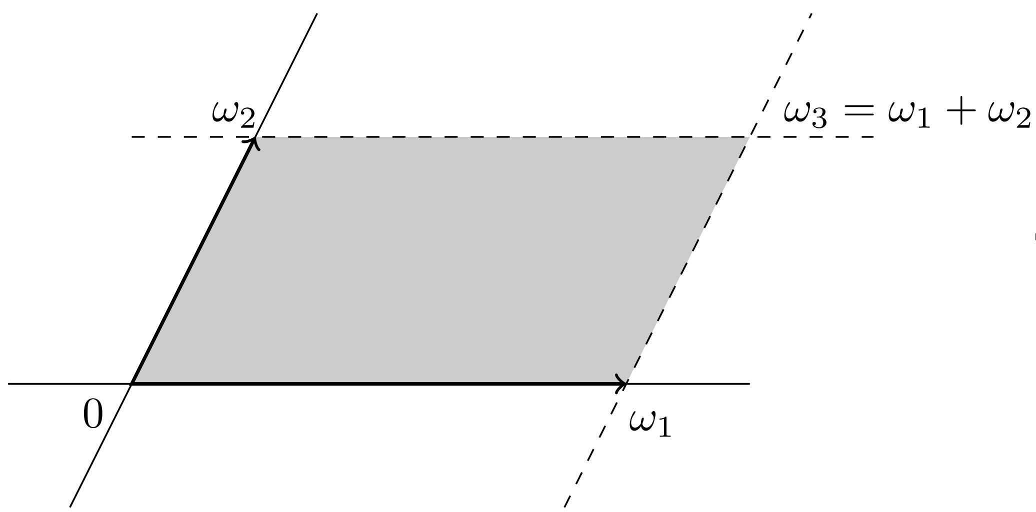
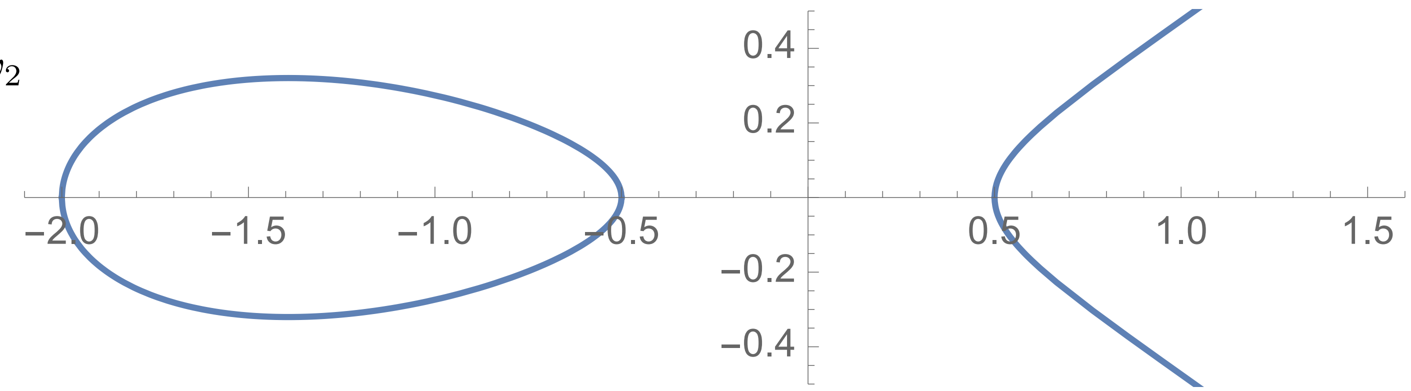


z



$[x, y, 1]$ with $y^2 = 4x^3 - g_2 x - g_3$

➡ Can always rescale one 'radius' to 1: $\tau = \omega_2/\omega_1 \quad \text{Im}\,\tau > 0$

➡ Weierstrass $\wp$-function:

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z + m\,\omega_1 + n\,\omega_2)^2} - \frac{1}{(m\,\omega_1 + n\,\omega_2)^2} \right)$$

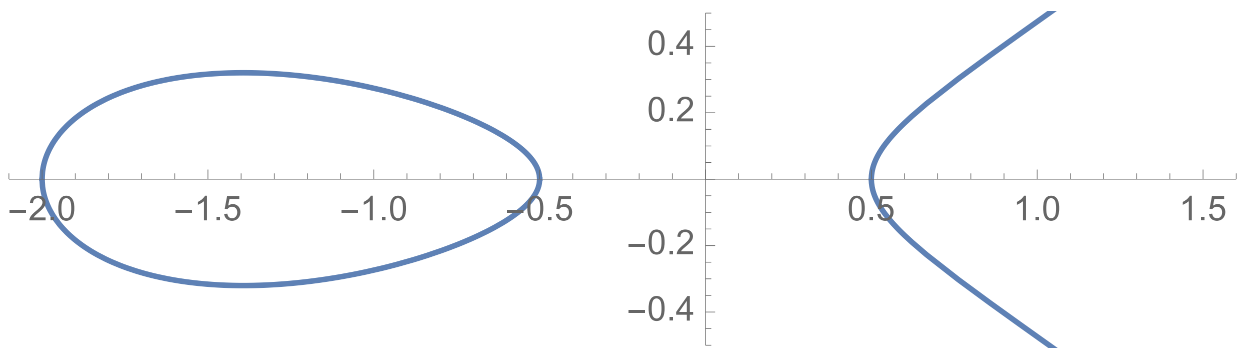
$$\wp'^2 = 4\wp^3 - g_2\,\wp - g_3 \qquad \wp(z+\omega_i;\omega_1,\omega_2) = \wp(z;\omega_1,\omega_2)$$

➡ Inverse map: $$z = \int_\infty^x \frac{dx'}{\sqrt{4x'^3 - g_2 x' - g_3}}$$

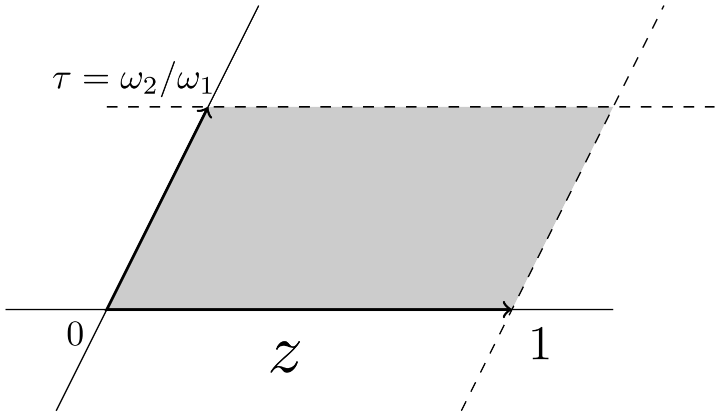
Elliptic polylogarithms

- Relation to integrals like $\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}} \log(1-\lambda x)$?

➡ They are the same thing! [Brödel, CD, Dulat, Tancredi]



$[x, y, 1]$



$$dx\, \varphi_0(x,0) = \frac{dx}{4\sqrt{\lambda}\,\mathrm{K}(\lambda)\, y} = dz$$

$$dx\, \varphi_1(x,c) = \frac{dx}{x-c} = dz\, \left[g^{(1)}(z-z_c,\tau) \pm g^{(1)}(z+z_c,\tau) - 2\, g^{(1)}(z,\tau)\right]$$

$$dx\, \varphi_{\pm n}(x,c) = dz\, \left[g^{(n)}(z-z_c,\tau) \pm g^{(n)}(z+z_c,\tau) - 2\, \delta_{\pm n,1}\, g^{(1)}(z,\tau)\right]$$

Elliptic 2F1

- Final result:

$$\partial_\lambda \begin{pmatrix} U_1(\lambda) \\ U_2(\lambda) \end{pmatrix} = \begin{pmatrix} 2\mathrm{K}(\lambda) & 0 \\ \frac{2(1+2\,\epsilon\,(1+\lambda))\mathrm{K}(\lambda)-\mathrm{E}(\lambda)}{\lambda(1+6\epsilon)} & \frac{i\pi\,\epsilon}{\lambda(1+6\epsilon)\mathrm{K}(\lambda)} \end{pmatrix} \begin{pmatrix} U_1(\lambda) \\ U_2(\lambda) \end{pmatrix}$$

$$U_1(\lambda) = 1 + 2\epsilon\,\left(\mathcal{E}_3\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix};1\right) + \mathcal{E}_3\left(\begin{smallmatrix}0 & 1\\ 0 & 1\end{smallmatrix};1\right) + \mathcal{E}_3\left(\begin{smallmatrix}0 & 1\\ 0 & 1/\lambda\end{smallmatrix};1\right)\right) + \mathcal{O}(\epsilon^2)$$

$$U_2(\lambda) = 2\pi i - \mathcal{E}_3\left(\begin{smallmatrix}-1\\ 0\end{smallmatrix};1\right) - \mathcal{E}_3\left(\begin{smallmatrix}-1\\ 1\end{smallmatrix};1\right) + \mathcal{E}_3\left(\begin{smallmatrix}1\\ 1/\lambda\end{smallmatrix};1\right) +$$

$$+\frac{1}{2\pi i}\left(2\,\mathcal{E}_3\left(\begin{smallmatrix}2\\ 0\end{smallmatrix};1\right) + 2\,\mathcal{E}_3\left(\begin{smallmatrix}2\\ 1\end{smallmatrix};1\right) + 2\,\mathcal{E}_3\left(\begin{smallmatrix}2\\ 1/\lambda\end{smallmatrix};1\right) - 9\,\mathcal{E}_3\left(\begin{smallmatrix}2\\ \infty\end{smallmatrix};1\right)\right) + \mathcal{O}(\epsilon)$$

➡ Very reminiscent of non-elliptic case!

➡ $U_i(\lambda)$ Are pure functions of uniform weight:

$$\mathcal{E}_3\left(\begin{smallmatrix}n_1 & \dots & n_k\\ c_1 & \dots & c_k\end{smallmatrix};\chi\right)$$

Weight = $\sum_{i=1}^{k} |n_i|$

Length = k

Elliptic Feynman integrals

	Non-elliptic 2F1	Elliptic 2F1
Direct integration	✓ [MPLs]	✓ [eMPLs]
Pure basis	✓ [Uniform weight]	✓ [Uniform weight]
Canonical DE	✓ [dlogs]	
Numerical Evaluation	✓	

Differential equations

- Differential equation $(T_1(\lambda), T_2(\lambda))$ is not in canonical form:

$$\partial_\lambda \begin{pmatrix} T_1(\lambda) \\ T_2(\lambda) \end{pmatrix} = (A + \epsilon\, B) \begin{pmatrix} T_1(\lambda) \\ T_2(\lambda) \end{pmatrix}$$

$$A = \frac{1}{\lambda} \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & -1 \end{pmatrix} + \frac{1}{\lambda - 1} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad B = \frac{1}{\lambda} \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix} + \frac{1}{\lambda - 1} \begin{pmatrix} -1 & 3 \\ -1 & 3 \end{pmatrix}$$

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- Differential equation $(U_1(\lambda), U_2(\lambda))$ takes the form:

$$\partial_\lambda \begin{pmatrix} U_1(\lambda) \\ U_2(\lambda) \end{pmatrix} = \epsilon\, \Omega \begin{pmatrix} U_1(\lambda) \\ U_2(\lambda) \end{pmatrix} \qquad \Omega = \begin{pmatrix} \frac{1}{(\lambda-1)\lambda} & \frac{i\pi}{4(\lambda-1)\lambda \mathrm{K}(\lambda)^2} \\ \frac{4(\lambda^2-\lambda+1)\mathrm{K}(\lambda)^2}{i\pi(\lambda-1)\lambda} & \frac{1}{(\lambda-1)\lambda} \end{pmatrix}$$

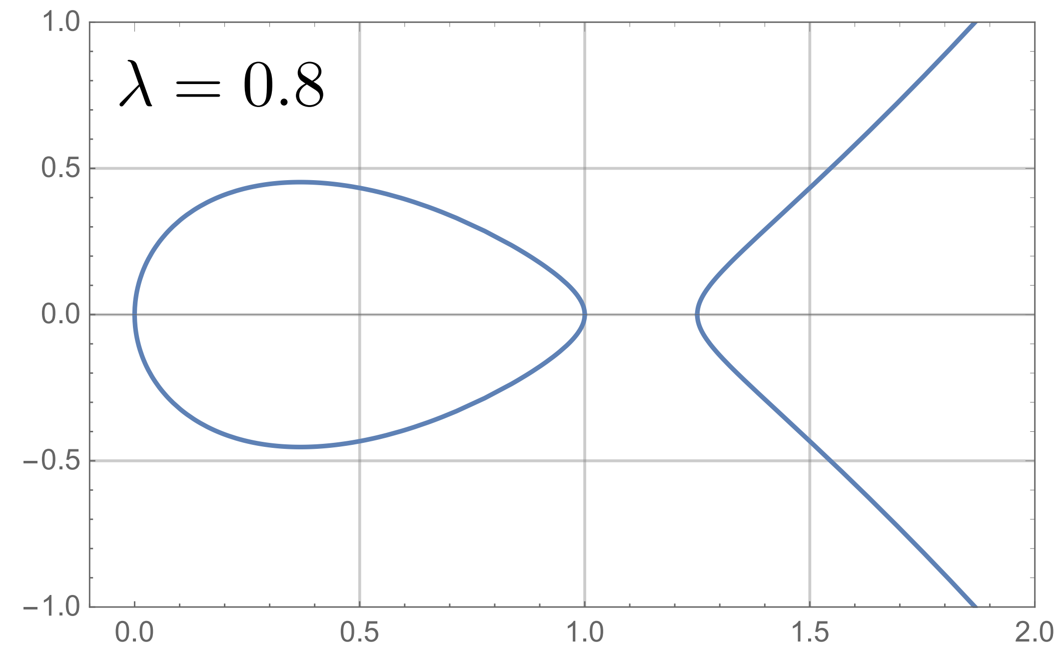
➡ Equation in ϵ-form, but matrix involves elliptic integrals.

➡ Looks very different from integration kernels for eMPLs?!?

From dynamics to geometry

- $y^2 = x(x-1)(x-1/\lambda)$ defines a family of elliptic curves.
  - ➡ Different values of λ correspond to elliptic curves of 'different shapes'.

$$\tau = i\frac{\mathrm{K}(1-\lambda)}{\mathrm{K}(\lambda)} = i\,0.735\ldots$$



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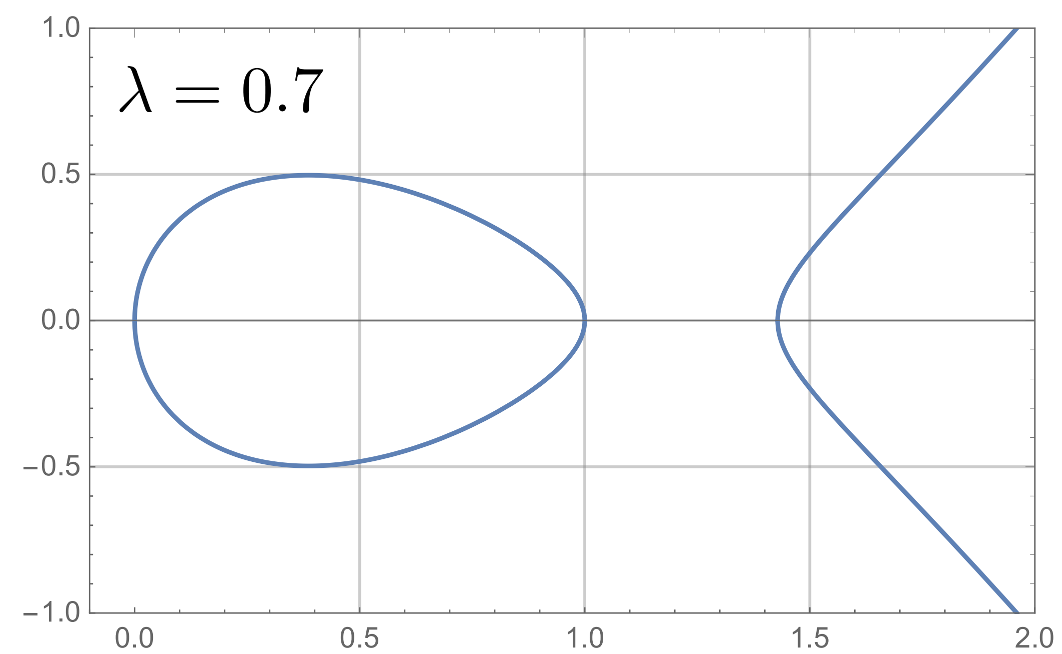
$$\tau = i\frac{\mathrm{K}(1-\lambda)}{\mathrm{K}(\lambda)} = i\,0.825\ldots$$



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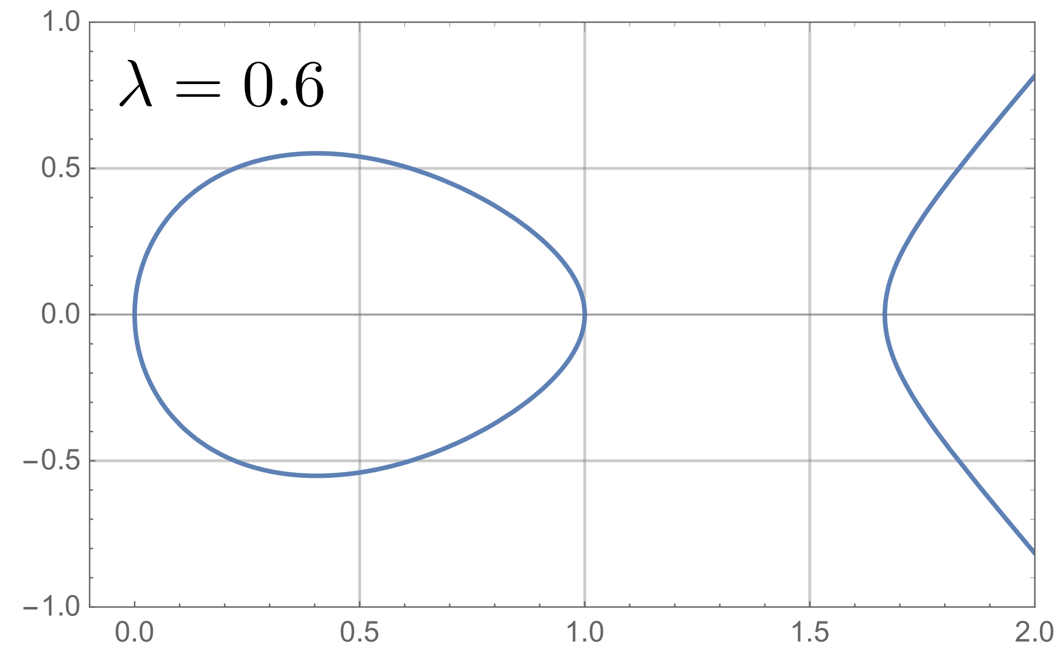
$$\tau = i\frac{\mathrm{K}(1-\lambda)}{\mathrm{K}(\lambda)} = i\,0.911\ldots$$



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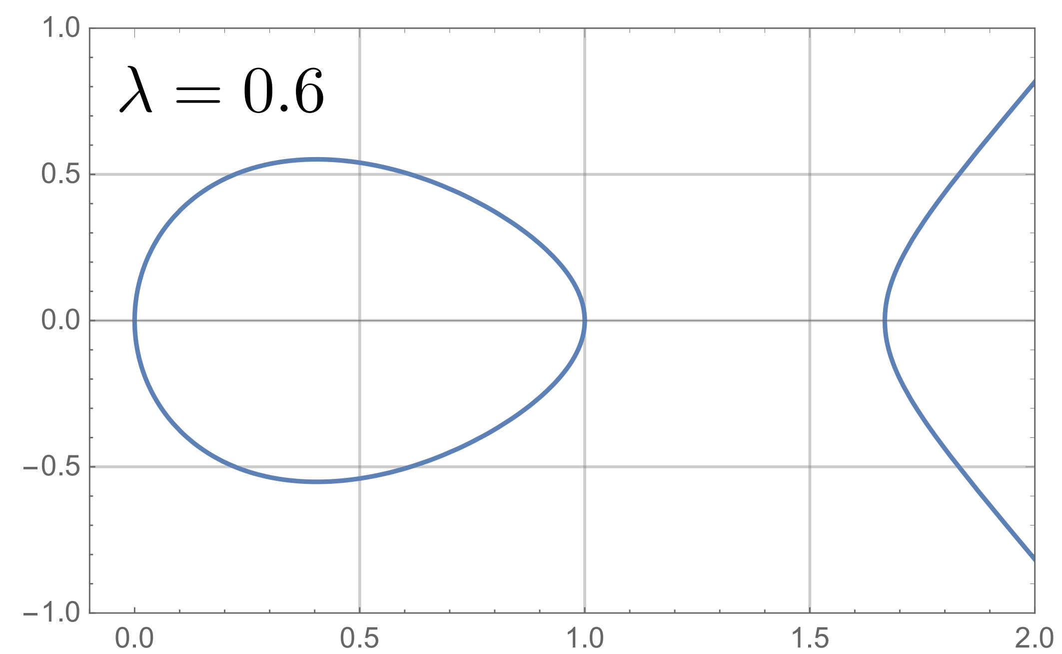
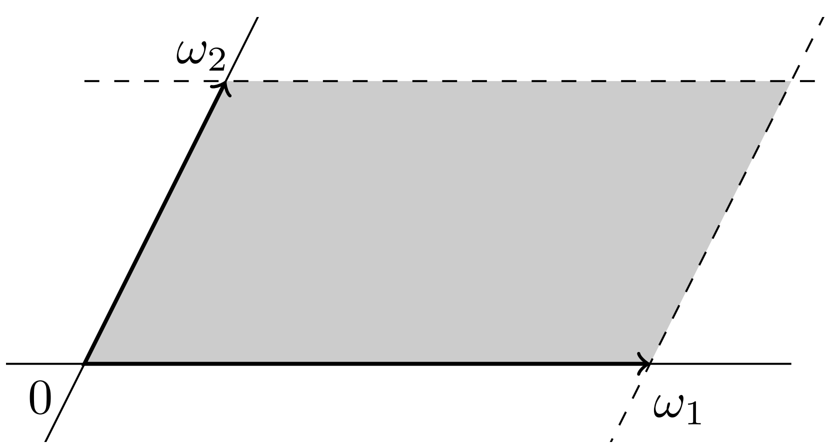


$$\tau = i\frac{\mathrm{K}(1-\lambda)}{\mathrm{K}(\lambda)} = i\,0.911\ldots$$

- A torus is defined by (ω_2, ω_1) .
  - ➡ Rescale to $(\tau, 1) \quad = (\omega_2/\omega_1, 1)$.
  - ➡ Rotation of the basis defines same torus:



$$\begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix} \sim \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_2\\ \omega_1\end{pmatrix} \qquad \tau \sim \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$$

Iterated int. of modular forms

- Modular form ~ holomorphic function with nice transformation properties:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^n\, f(\tau)$$

- Definition: Iterated integral of modular forms: [Manin; Brown]

$$I(f_{i_1},\ldots,f_{i_k};\tau) = \int_{i\infty}^{\tau} \frac{d\tau'}{2\pi i}\, f_{i_1}(\tau')\, I(f_{i_2},\ldots,f_{i_k};\tau')$$

f_{i_a} = modular forms of weight n_{i_a}

Weight = $-k+\sum_{a=1}^{k}|n_{i_a}|$

Length = k

- Looks very different from eMPLs….

➡ What is the connection to eMPLs…?

The differential of eMPLs

- Total differential of eMPLs: $A_i^{[r]} \equiv \left(\begin{smallmatrix} n_i+r \\ z_i \end{smallmatrix}\right)$

[Brödel, CD, Dulat, Penante, Tancredi]

$$d\widetilde{\Gamma}\left(A_1\cdots A_k;z,\tau\right) = \sum_{p=1}^{k-1}(-1)^{n_{p+1}}\,\widetilde{\Gamma}\left(A_1\cdots A_{p-1}\,{}^{0}_{0}\,A_{p+2}\cdots A_k;z,\tau\right)\,\omega_{p,p+1}^{(n_p+n_{p+1})}$$

$$+\sum_{p=1}^{k}\sum_{r=0}^{n_p+1}\Bigg[\binom{n_{p-1}+r-1}{n_{p-1}-1}\widetilde{\Gamma}\left(A_1\cdots A_{p-1}^{[r]}\,\hat{A}_p\,A_{p+1}\cdots A_k;z,\tau\right)\omega_{p,p-1}^{(n_p-r)}$$

$$-\binom{n_{p+1}+r-1}{n_{p+1}-1}\widetilde{\Gamma}\left(A_1\cdots A_{p-1}\,\hat{A}_p\,A_{p+1}^{[r]}\cdots A_k;z,\tau\right)\omega_{p,p+1}^{(n_p-r)}\Bigg],$$

$$\omega_{ij}^{(n)} = (dz_j - dz_i)\,g^{(n)}(z_j-z_i,\tau) + \frac{n\,d\tau}{2\pi i}\,g^{(n+1)}(z_j-z_i,\tau)$$

Integral on moduli space

➡ Differential involves 1-forms on moduli space.

➡ Iterated integrals on moduli space.

The differential of eMPLs

- Assume that z_i are 'rational': $z_i = \frac{r_i}{N} + \frac{s_i}{N}\tau$ $\qquad r_i, s_i, N$ integer
  - ➡ $g^{(n)}\left(\frac{r}{N} + \frac{s}{N}\tau, \tau\right)$ is always a combination of modular forms.

$$g^{(n)}\left(\frac{r}{N} + \frac{s}{N}\tau, \tau\right) = \sum_{k=0}^{n} \frac{(-2\pi i s)^k}{k!}\, h^{(n)}_{N,r,s}(\tau)$$

[Brödel, CD, Dulat, Penante, Tancredi; Zagier]

  - ➡ $h^{(n)}_{N,r,s}$ are Eisenstein series of weight n for $\Gamma(N)$.

$$h^{(n)}_{N,r,s}(\tau) = -\mathbf{a}_{n,N,r,s}(\tau) - i\,\mathbf{b}_{n,N,r,s}(\tau) = -\sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}} \frac{e^{2\pi i(bs-ar)/N}}{(a\tau+b)^n}$$

- Conclusion: If all z_i in $\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ z_1 & \dots & z_k\end{smallmatrix}; z, \tau\right)$ are 'rational' [torsion points], then the eMPL can be written in terms of iterated integrals of modular forms [Eisenstein series].

[Brödel, CD, Dulat, Penante, Tancredi]

The differential of eMPLs

$$U_1(\lambda) = 1 + 2\epsilon \left(\mathcal{E}_3\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}; 1\right) + \mathcal{E}_3\left(\begin{smallmatrix}0 & 1\\ 0 & 1\end{smallmatrix}; 1\right) + \mathcal{E}_3\left(\begin{smallmatrix}0 & 1\\ 0 & 1/\lambda\end{smallmatrix}; 1\right)\right) + \mathcal{O}(\epsilon^2)$$

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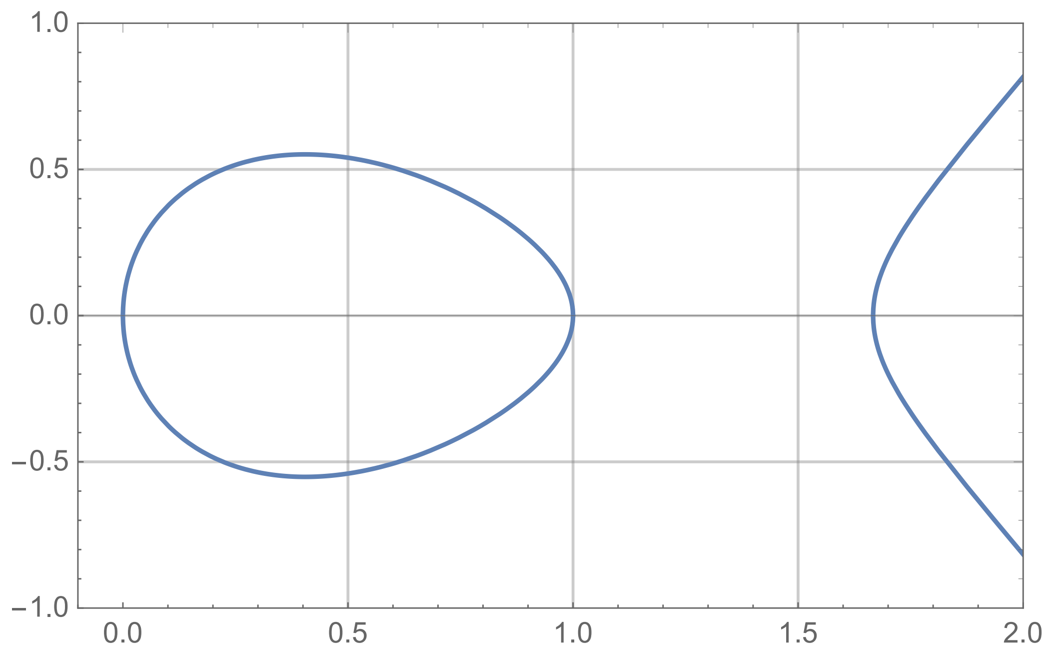

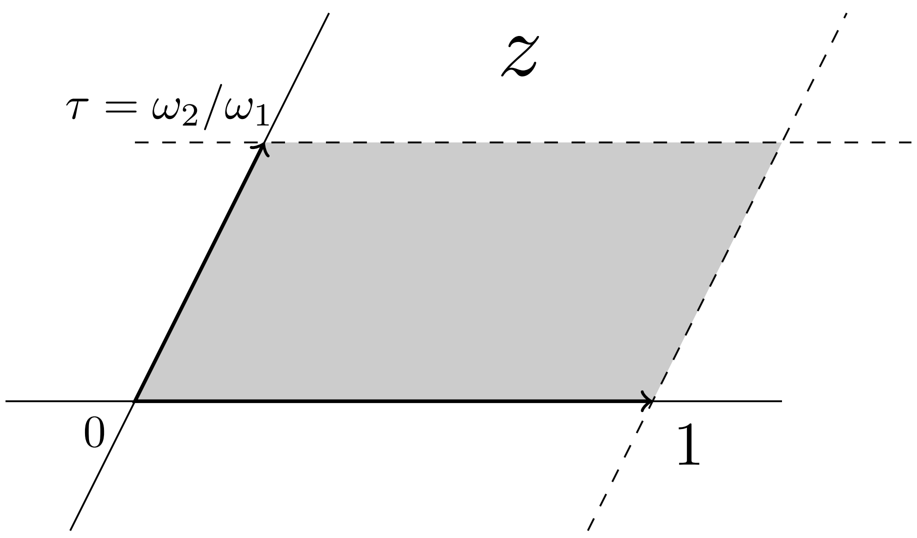

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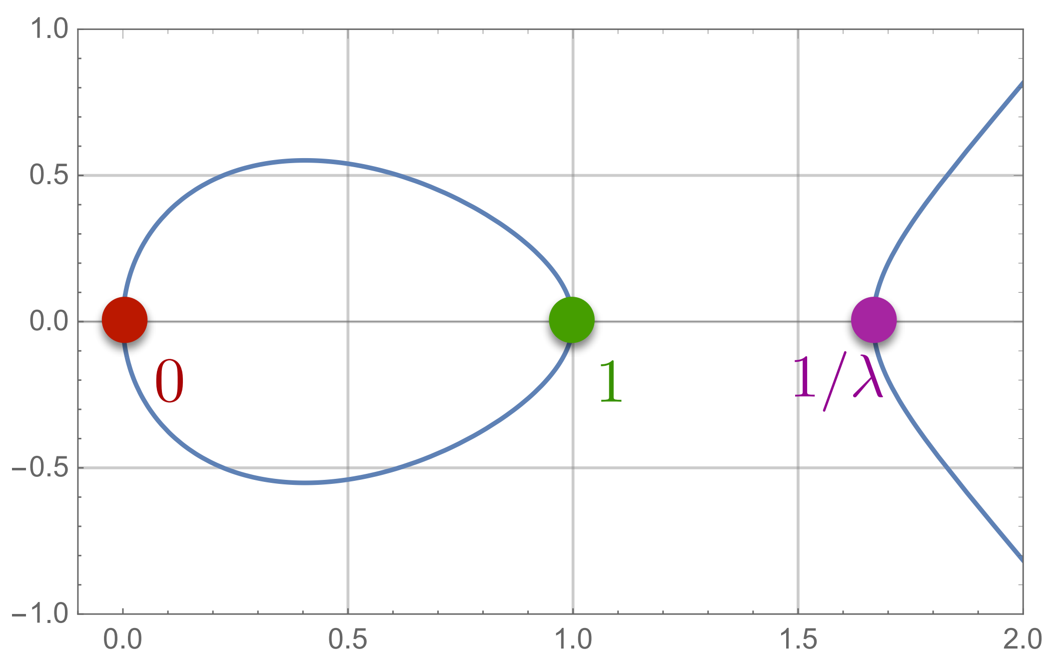

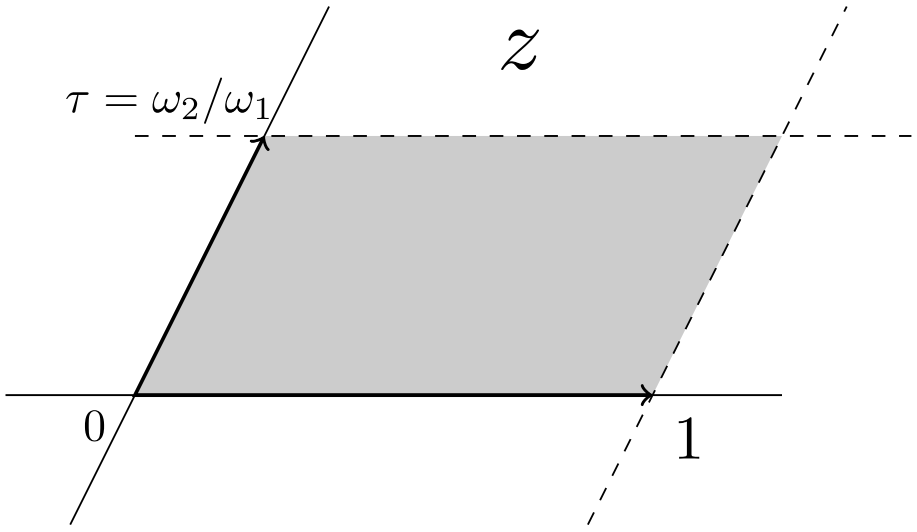

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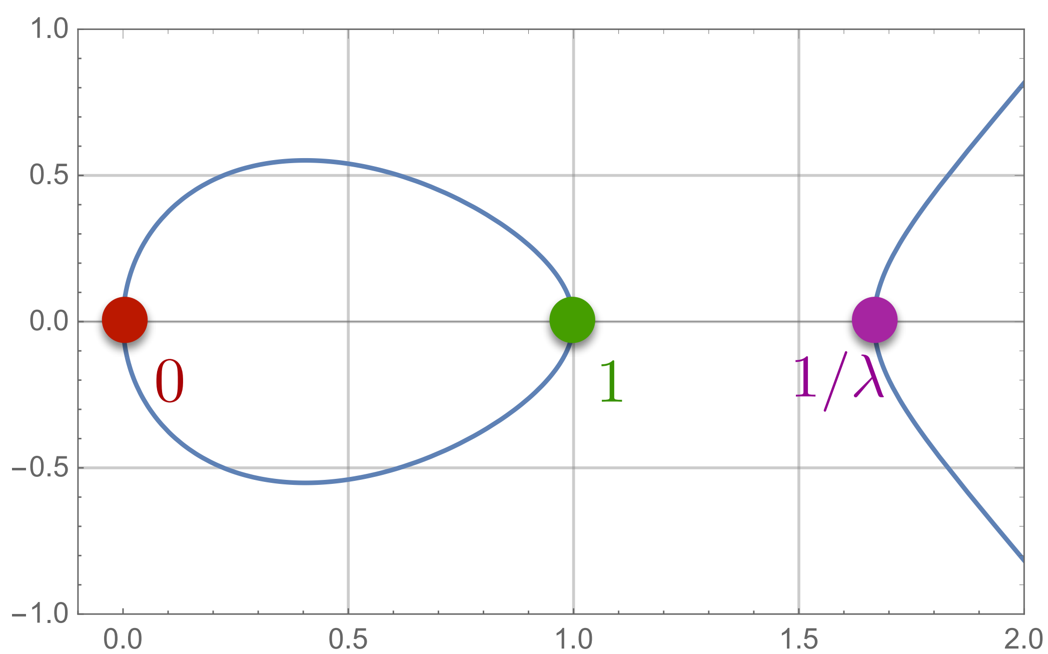

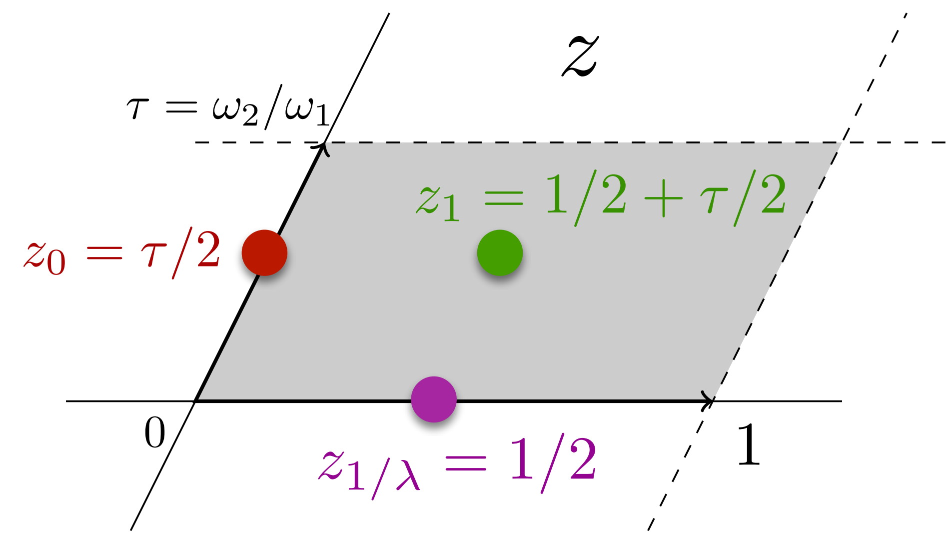

$$\tau = i\frac{\mathrm{K}(1-\lambda)}{\mathrm{K}(\lambda)}$$

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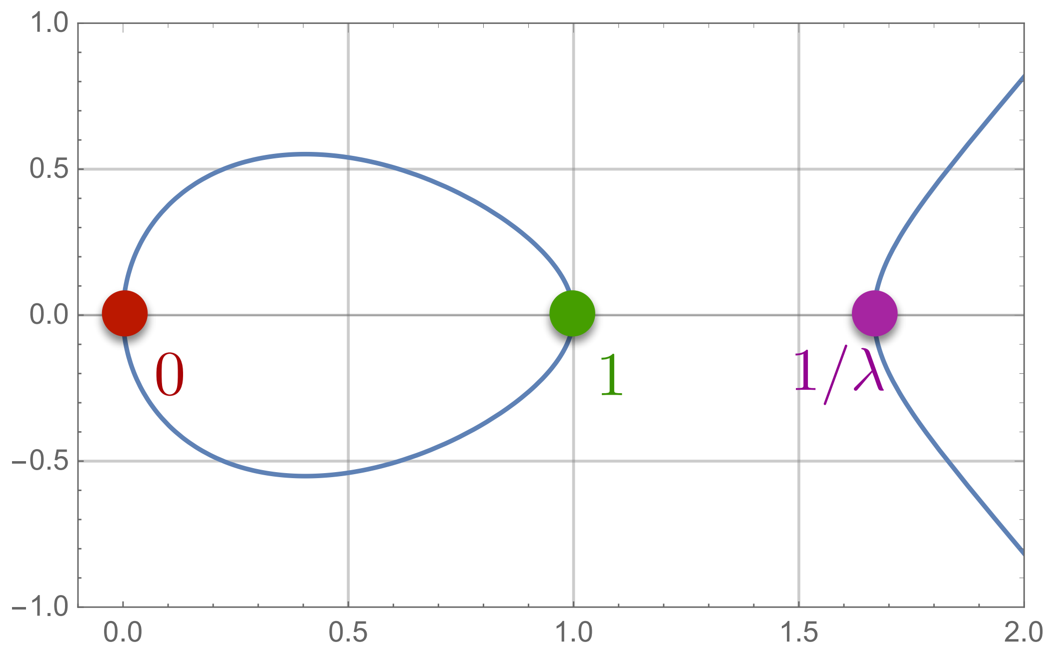

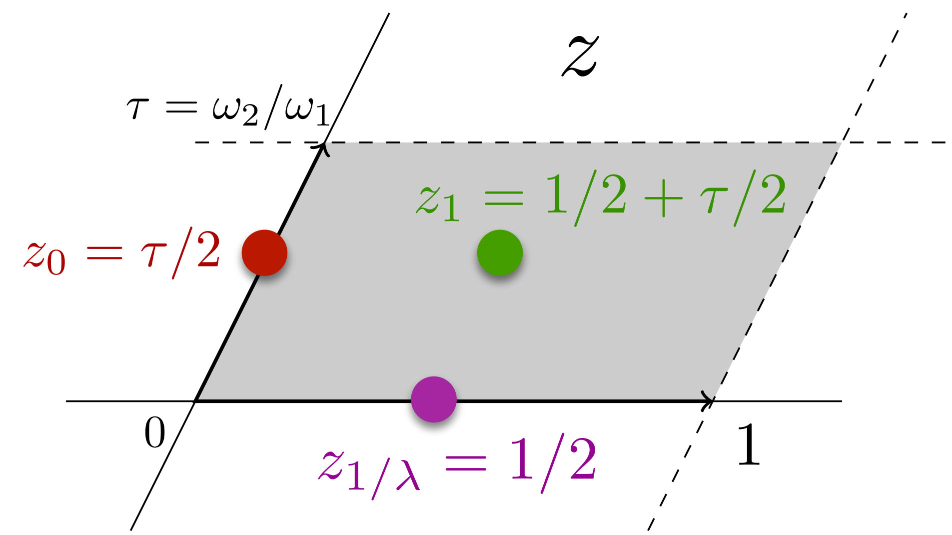

$$\tau = i\,\frac{\mathrm{K}(1-\lambda)}{\mathrm{K}(\lambda)}$$

➡ The $U_i(\lambda)$ can be expressed in terms of iterated integrals of Eisenstein series.

$$U_1(\lambda) = 1 + 2\epsilon\,\left(8\,I(\mathsf{a}_{2,2,1,0};\tau) + 4\,I(\mathsf{a}_{2,2,1,1};\tau) - 2\pi^2 I(1;\tau) - 4\log 2\right) + \mathcal{O}(\epsilon^2)$$

$$U_2(\lambda) = i\pi + \epsilon\,\left(8i\pi I(\mathsf{a}_{2,2,1,0};\tau) + 4i\pi I(\mathsf{a}_{2,2,1,1};\tau) + \frac{90}{i\pi} I(\mathsf{a}_{4,2,0,0};\tau) - 4i\pi\log 2\right) + \mathcal{O}(\epsilon^2)$$

[Brödel, CD, Dulat, Penante, Tancredi; CD, Tancredi]

The differential of eMPLs

- Relation to differential equation?

$$\partial_\lambda \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \epsilon\,\Omega \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \qquad \Omega = \begin{pmatrix} \frac{1}{(\lambda-1)\lambda} & \frac{i\pi}{4(\lambda-1)\lambda \mathrm{K}(\lambda)^2} \\ \frac{4\left(\lambda^2-\lambda+1\right)\mathrm{K}(\lambda)^2}{i\pi(\lambda-1)\lambda} & \frac{1}{(\lambda-1)\lambda} \end{pmatrix}$$

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- Change variables from λ to τ :

$$\tau = i\frac{\mathrm{K}(1-\lambda)}{\mathrm{K}(\lambda)} \qquad \partial_\lambda \tau = \frac{i\pi}{4\,\lambda\,(\lambda-1)\,\mathrm{K}(\lambda)^2}$$

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$$\partial_\tau \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \epsilon\, \tilde{\Omega} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad \tilde{\Omega} = \begin{pmatrix} \frac{4\mathrm{K}(\lambda(\tau))^2}{i\pi} & 1 \\ -\frac{16\left(\lambda(\tau)^2-\lambda(\tau)+1\right)\mathrm{K}(\lambda(\tau))^4}{\pi^2} & \frac{4\mathrm{K}(\lambda(\tau))^2}{i\pi} \end{pmatrix}$$

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- $\mathrm{K}(\lambda(\tau))^{2n}\lambda(\tau)^p,\ 0\le p\le n:$ basis of modular forms of weight n for $\Gamma(2)$.

➡ Example: $\mathrm{K}(\lambda(\tau))^2 = \mathbf{a}_{2,2,1,0}(\tau) + \frac{1}{2}\mathbf{a}_{2,2,1,1}(\tau)$

Elliptic Feynman integrals

	Non-elliptic 2F1	Elliptic 2F1
Direct integration	✓ [MPLs]	✓ [eMPLs & MFs]
Pure basis	✓ [Uniform weight]	✓ [Uniform weight]
Canonical DE	✓ [dlogs]	✓ [modular forms]
Numerical Evaluation	✓	

Numerical evaluation

- Modular forms for $\Gamma(2)$ are invariant under translations by 2.
  - ➡ Fourier expansion in $q_2 = e^{i\pi\tau}$.
  - ➡ Example: $\mathbf{a}_{4,2,1,0}(\tau) = -\frac{7\pi^4}{360} - \frac{2\pi^4}{3}\,q_2 + \frac{2\pi^4}{3}\,q_2^2 + \mathcal{O}(q_2^3)$

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- Carries over to iterated integrals of modular forms:

$$I(\mathbf{a}_{4,2,1,0};\tau) = -\frac{\pi^2}{90}\log q_2 + \frac{4\pi^2}{3}\, q_2^2 + \frac{14\pi^2}{3}\, q_2^4 + \mathcal{O}(q_2^6)$$

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- Convergence: $|q_2| = e^{-\pi \mathrm{Im}\,\tau} < 1$, for $\mathrm{Im}\,\tau > 0$.
  - ➡ Converges fast for large $\mathrm{Im}\,\tau$.
  - ➡ Converges slowly for small $\mathrm{Im}\,\tau$.

Numerical evaluation

- Numerical convergence of $I(\mathbf{a}_{4,2,1,0};\tau)$:

$$\tau = \frac{1}{2} + \frac{i}{10} \qquad \lambda \simeq 0.0000192897 - 0.0062112101i$$

1	$-6.985970464 - 0.172257093i$	
2		
3		
20		
30		
40		
47		
48		
49		
50		

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2	$6.122619677 - 0.172257093i$	
3	$-12.526118272 - 0.172257093i$	
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30	$-8.379345867 - 0.172257093i$	
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30	$-8.379345867 - 0.172257093i$	
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3	$-12.526118272 - 0.172257093i$	
20	$-8.370957843 - 0.172257093i$	
30	$-8.379345867 - 0.172257093i$	
40	$-8.379379366 - 0.172257093i$	
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2	$6.122619677 - 0.172257093i$	
3	$-12.526118272 - 0.172257093i$	
20	$-8.370957843 - 0.172257093i$	
30	$-8.379345867 - 0.172257093i$	
40	$-8.379379366 - 0.172257093i$	
47	$-8.379379472 - 0.172257093i$	
48		
49		
50		

Numerical evaluation

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2	$6.122619677 - 0.172257093i$	
3	$-12.526118272 - 0.172257093i$	
20	$-8.370957843 - 0.172257093i$	
30	$-8.379345867 - 0.172257093i$	
40	$-8.379379366 - 0.172257093i$	
47	$-8.379379472 - 0.172257093i$	
48	$-8.379379470 - 0.172257093i$	
49		
50		

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30	$-8.379345867 - 0.172257093i$	
40	$-8.379379366 - 0.172257093i$	
47	$-8.379379472 - 0.172257093i$	
48	$-8.379379470 - 0.172257093i$	
49	$-8.379379471 - 0.172257093i$	
50		

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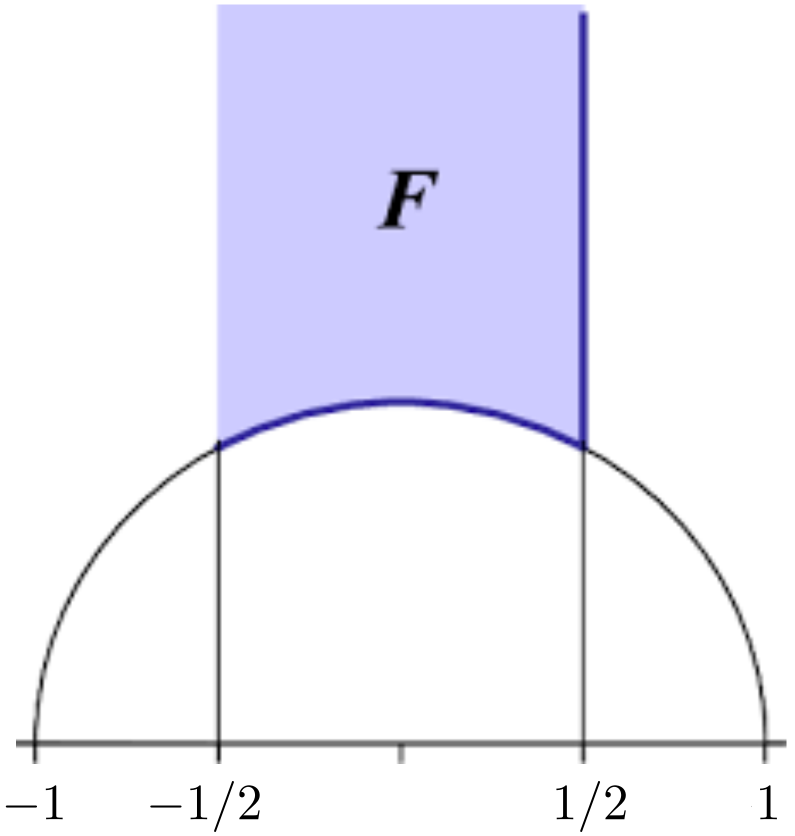
47	$-8.379379472 - 0.172257093i$	
48	$-8.379379470 - 0.172257093i$	
49	$-8.379379471 - 0.172257093i$	
50	$-8.379379471 - 0.172257093i$	

Numerical evaluation

- For every $\tau \in \mathbb{H}$ in each there is $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ such that

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} \in F$$



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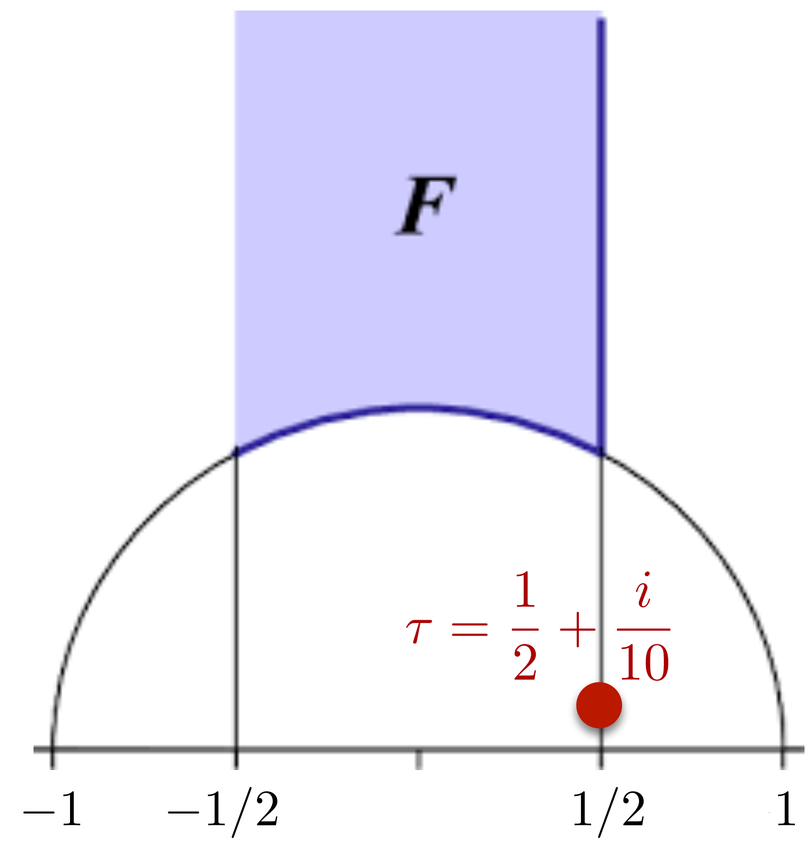
➡ Example:



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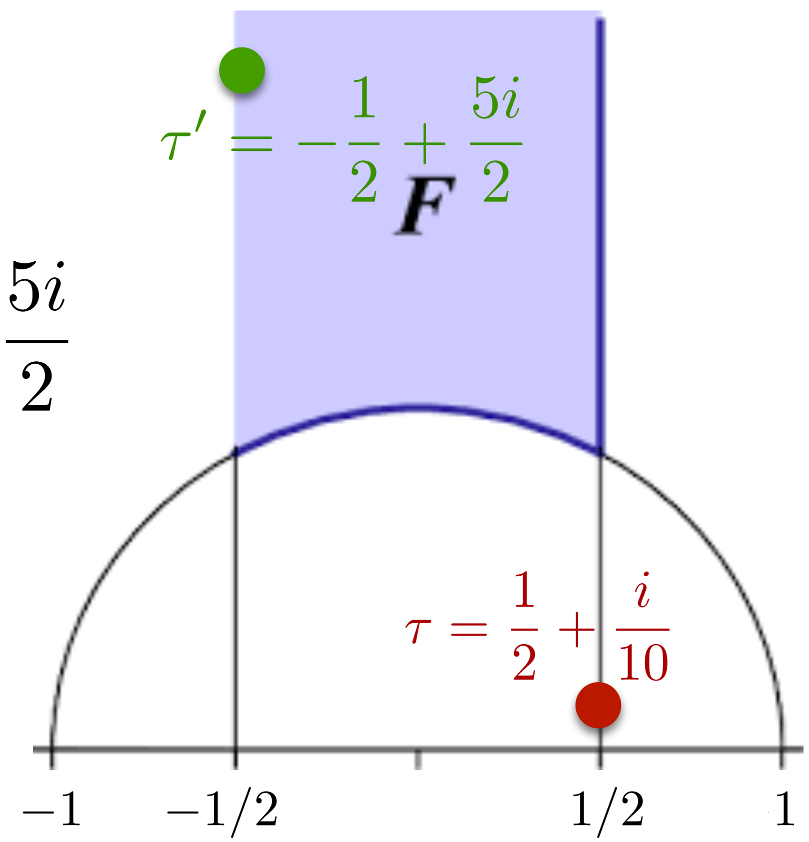
➡ Example: $\gamma = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ $\quad \tau' = \frac{\tau}{1-2\tau} = -\frac{1}{2} + \frac{5i}{2}$



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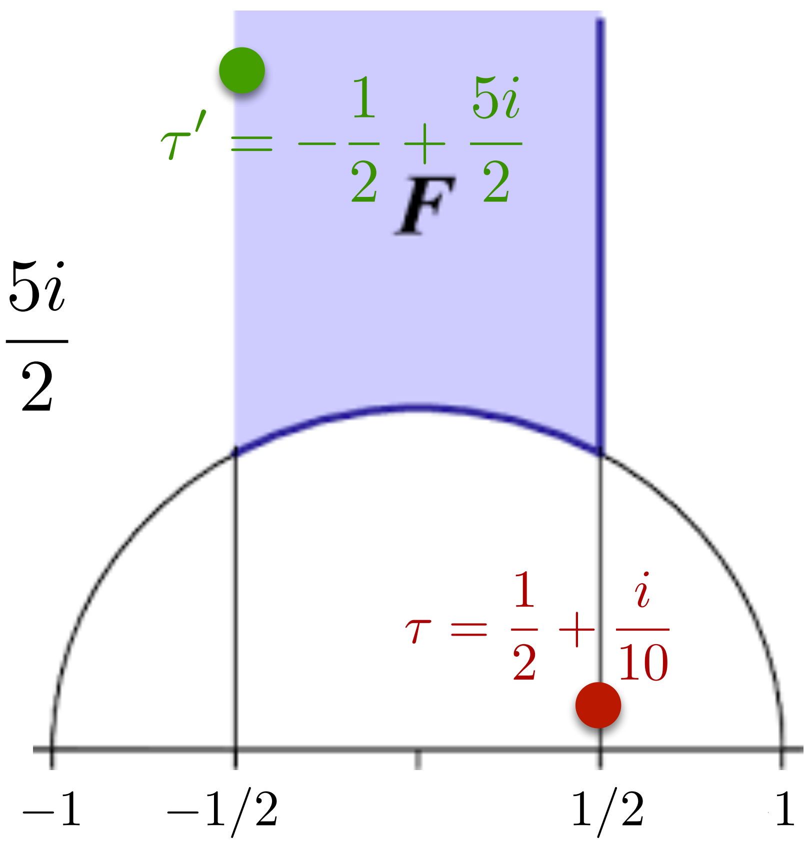

- Eisenstein series transform nicely:

$$h^{(n)}_{N,r,s}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^n\, h^{(n)}_{N,rd+sb,rc+sa}(\tau)$$

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Numerical evaluation

- For every $\tau \in \mathbb{H}$ in each there is $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ such that

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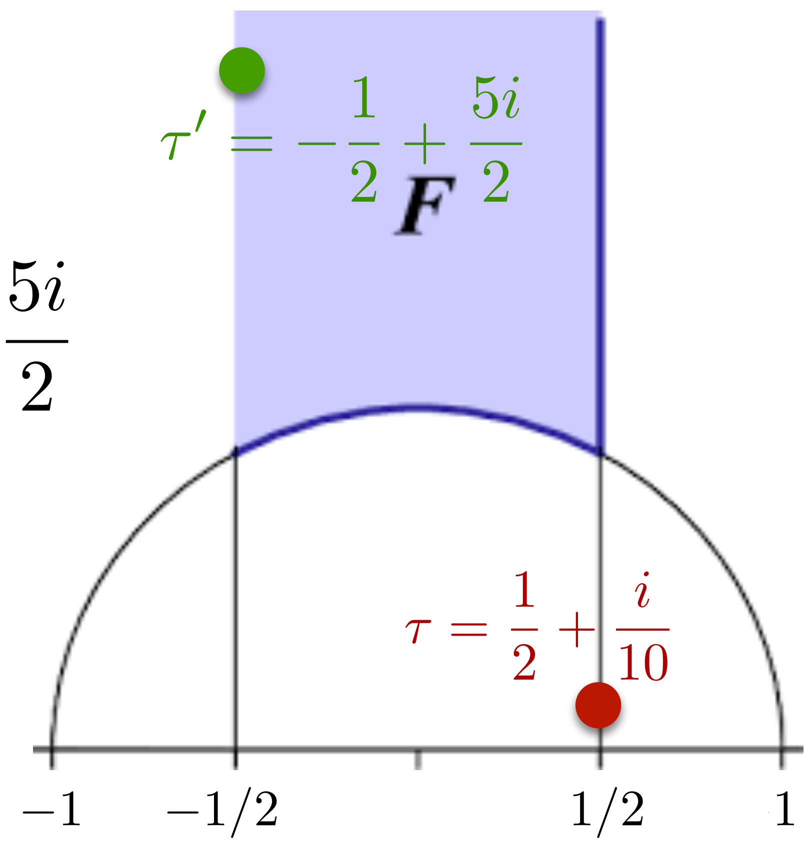

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- Carries over to iterated integrals of Eisenstein series: [CD, Tancredi]

$$I(\mathbf{a}_{4,2,1,0};\tau) = I(\mathbf{a}_{4,2,1,0};\tau') + 2i\pi I(\mathbf{a}_{4,2,1,0},1;\tau') - 32\pi^2 I(\mathbf{a}_{4,2,1,0},1,1;\tau') - \zeta_3 - \frac{i\pi^3}{135}$$

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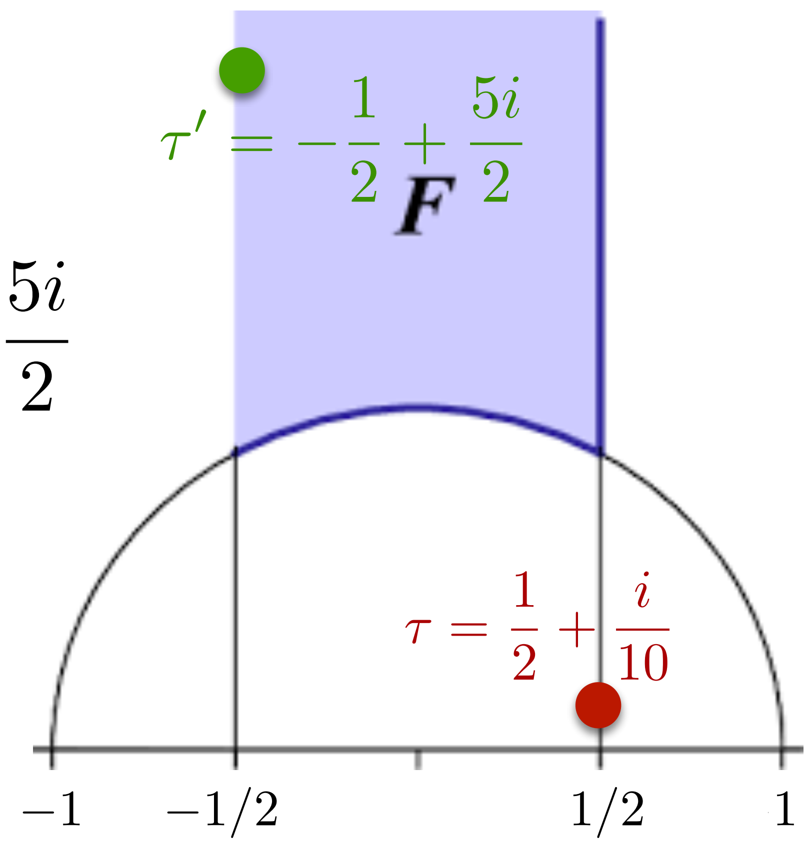

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Multiple Modular Value [Brown]

Numerical evaluation

- Numerical convergence of $I(\mathbf{a}_{4,2,1,0};\tau)$:

$$\tau = \frac{1}{2} + \frac{i}{10} \qquad \lambda \simeq 0.0000192897 - 0.0062112101i$$

1	$-6.985970464 - 0.172257093i$	
2	$6.122619677 - 0.172257093i$	
3	$-12.526118272 - 0.172257093i$	
20	$-8.370957843 - 0.172257093i$	
30	$-8.379345867 - 0.172257093i$	
40	$-8.379379366 - 0.172257093i$	
47	$-8.379379472 - 0.172257093i$	
48	$-8.379379470 - 0.172257093i$	
49	$-8.379379471 - 0.172257093i$	
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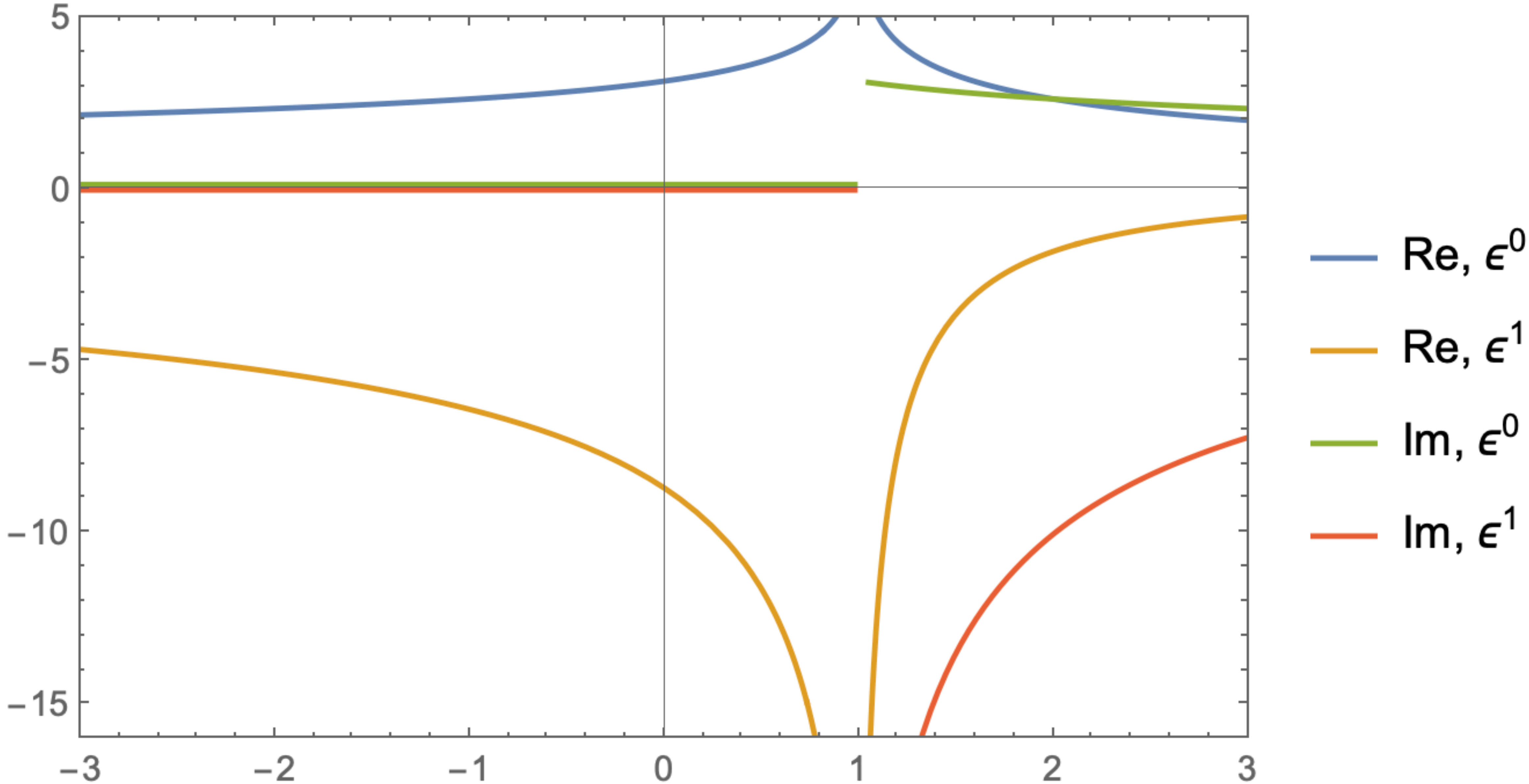
Numerical evaluation

- Numerical results for $T_1(\lambda)$ for all real values of λ :



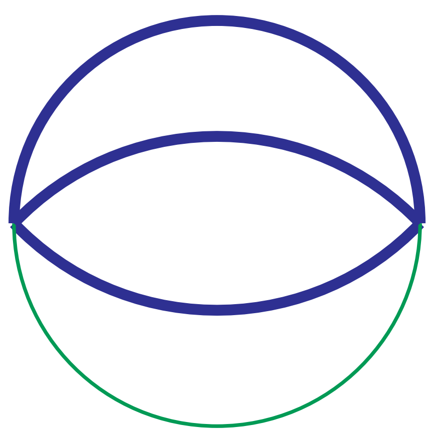
[CD, Tancredi]

Elliptic Feynman integrals

	Non-elliptic 2F1	Elliptic 2F1
Direct integration	✓ [MPLs]	✓ [eMPLs & MFs]
Pure basis	✓ [Uniform weight]	✓ [Uniform weight]
Canonical DE	✓ [dlogs]	✓ [modular forms]
Numerical Evaluation	✓	✓ [Iterated Eisenstein integrals]

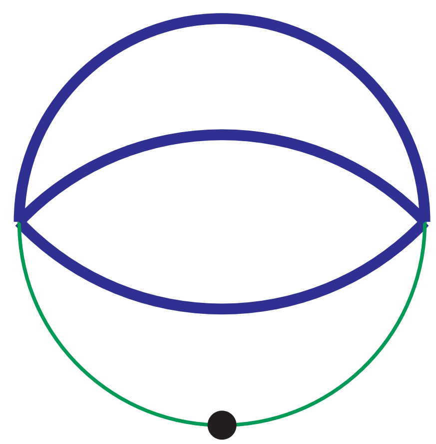
The rho parameter

- All these ideas carry over to Feynman integrals.
- Example: the rho parameter at 3 loops.
  - ➡ Known numerically from [Grigo, Hoff, Marquard, Steinhauser].
  - ➡ Was not known analytically.



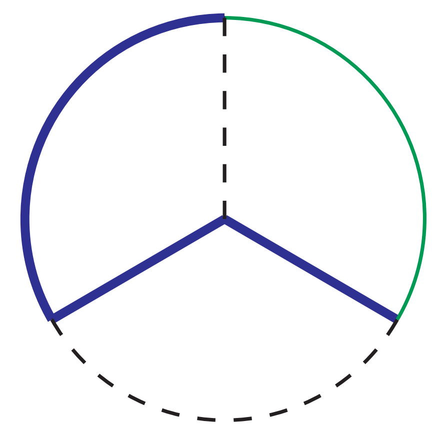
$f_8^{(2)}(t)$

$D = 2 - 2\epsilon$



$f_9^{(2)}(t)$

$D = 2 - 2\epsilon$



$f_{10}(t)$

$D = 4 - 2\epsilon$

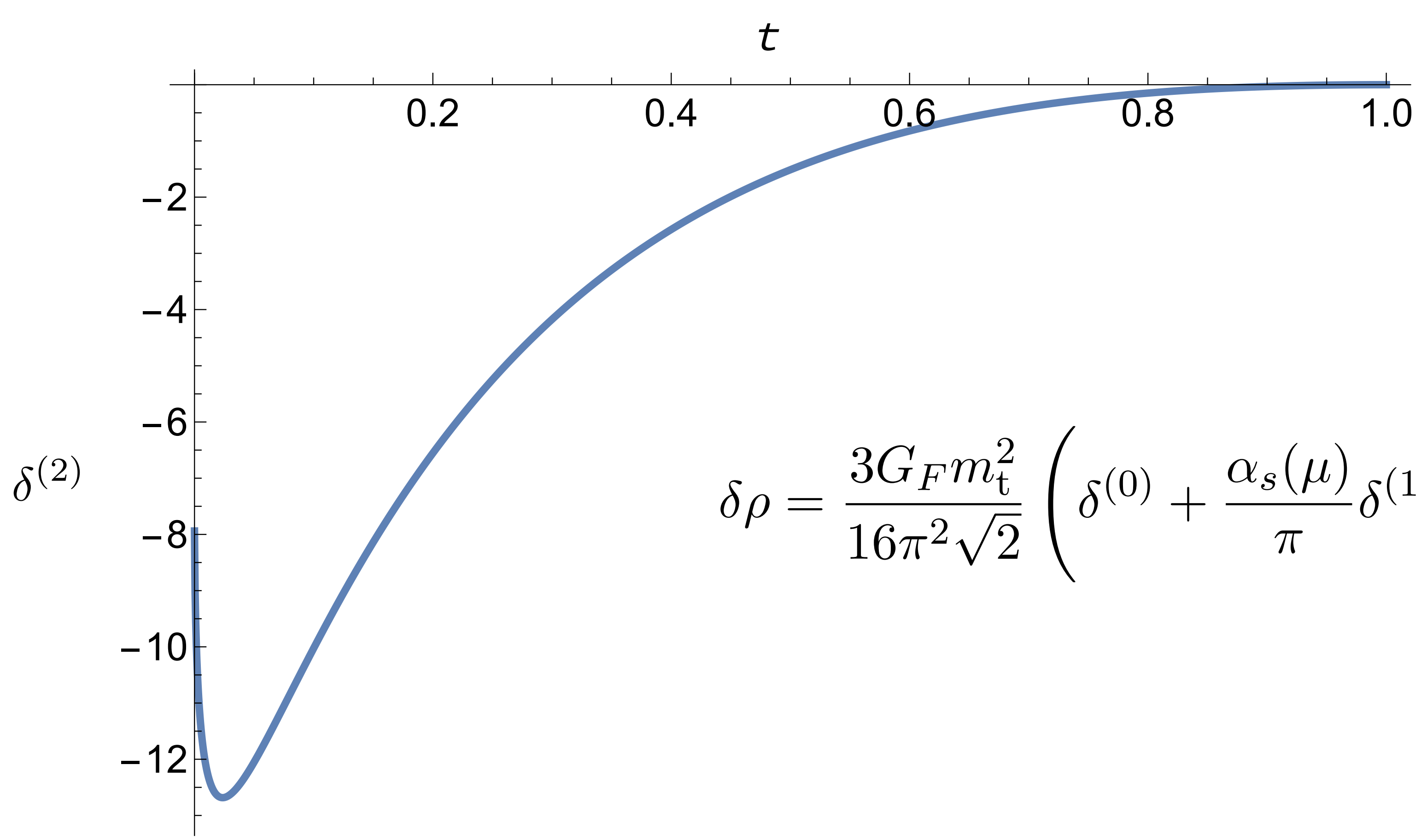
$$t = \frac{m^2}{M^2}$$

The rho parameter

$$\begin{pmatrix} f_8^{(2)}(t) \\ f_9^{(2)}(t) \\ f_{10}(t) \end{pmatrix} = \begin{pmatrix} \Psi_1(t) & 0 & 0 \\ -\Phi_1(t) & \frac{24}{(t-9)(t-1)t\Psi_1(t)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U_8(t) \\ U_9(t) \\ U_{10}(t) \end{pmatrix}$$

Same matrix as for sunrise/kite

Pure functions of uniform weight; expressible via eMPLs or iterated Eisenstein integral



$$\delta\rho = \frac{3G_F m_t^2}{16\pi^2\sqrt{2}}\left(\delta^{(0)} + \frac{\alpha_s(\mu)}{\pi}\delta^{(1)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \delta^{(2)} + \mathcal{O}(\alpha_s(\mu)^3)\right)$$

[Abreu, Becchetti, CD, Marzucca]

Conclusion

- We have learned a lot about Feynman integrals that cannot be expressed in terms of MPLs.
- For Feynman integrals that evaluate to iterated Eisenstein integrals, we have now a solid understanding:
  - ➡ Pure functions & differential equations in canonical form.
  - ➡ Numerical evaluation & analytic continuation.
- Still a lot to do!
  - ➡ More than one variable?
  - ➡ More than one elliptic curve?
  - ➡ Additional singularities (e.g., coming from subtopologies):

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}}\,\frac{1}{x-c} \qquad \longleftrightarrow \qquad \int_0^\lambda \frac{d\lambda'\,\mathrm{K}(\lambda')}{\lambda'-c}$$