0). Periods.

$$I = \int_\gamma \omega$$

X alg. variety $/\mathbb{Q}$

ω closed algebraic n-form

$\gamma \subseteq X(\mathbb{C})$ closed n-chain.

View: $[\omega] \in H^n_{dR}(X;\mathbb{Q})$

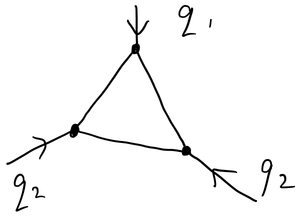
$[\gamma] \in H_n(X(\mathbb{C});\mathbb{Q})$

I has a cohomological interpretation.

Variants:

- ω, γ depend on parameters, $X \longrightarrow S$
- γ has boundary $\partial\gamma \subseteq D$ — X replaced with (X, D) relative cohomology

Example: G any Feynman graph



$$I_G(q_i, m_i) = \cdot \int_0^\infty \cdots \int_0^\infty \frac{\Omega}{(\mathcal{F}+\mathcal{U})^p}$$

pre factors

explicit polynomials depending on parameters m_i, particle masses

q_i particle momenta.

Theorem (Bloch–Esnault–Kreimer '07 no masses, momenta,
B— '15, generic q_i, m_i)

I_G is a period of a canonical cohomology spaces:

$$\left(H^n_{dR}(X, D),\ H^n(X(\mathbb{C}), D(\mathbb{C}))\right)$$

local resolution of singularities of graph hypersurface complement, relative to boundaries.

What do we gain?

- Canonical ODE's (Picard–Fuchs equations)
- Weights
- Action of "motivic" Galois group

⋮

lots more

Coaction in brief.

a "motivic period" is an equivalence class

$$I^m = [H, [\omega], [\gamma]]^m$$

Coaction is a rule to break it up into simpler pieces "atoms"

$$\Delta I^m = \sum_{i=1}^{N} a_i^m \otimes b_i^{\partial R}$$

RHS knows "everything"!

respects

- all algebraic relations
- all ODE's
- monodromy

Today: Integrals of the form

$$I(s_1,..,s_r) = \int_\gamma f_1^{s_1} \cdots f_r^{s_r} \omega$$

viewed as a function of s_i.

where $f_1,..,f_r : X \longrightarrow \mathbb{G}_m$.

joint work with C. Dupont
in progress with C. Dupont, J. Fresán, M. Tapušković

1907.06603, 1910.01107, 1810.07682

Examples:
- Hypergeometric functions
- String perturbation
- Dim reg.

Two different points of view

"GLOBAL" $s_1,..,s_r \in \mathbb{C}$ generic, fixed.

"LOCAL" Laurent expand in s_i at non-generic point $s_1 = .. = s_r = 0$

GLOBAL

$$I(s_1,..,s_r) = \int_\gamma \omega f_1^{s_1} \cdots f_r^{s_r}$$

s_i fixed $\in \mathbb{C}$ generic

$I(s_1,..,s_r)$ is a period of cohomology with coefficients

$H^r_{dR}(X, (V, \nabla_s))$ $\qquad (V, \nabla_s)$ alg. v. bundle on X

$H_r(X(\mathbb{C}), \mathcal{L}_s^\vee)$ $\qquad \mathcal{L}_s^\vee$ local syst. of sols.

FINITE RANK

FINITELY MANY PERIODS

LOCAL

$s_1,..,s_r \in \mathbb{C}$ formal parameters Taylor (Laurent) expansion at non-generic point

$$I(s_1,..,s_r) = \sum_{k_i \geq 0} \frac{s_1^{k_1}}{k_1!} \cdots \frac{s_r^{k_r}}{k_r!} \underbrace{\int_\gamma \log^{k_1} f_1 .. \log^{k_2} f_2 \omega}$$

classical periods of auxilliary varieties $X_{k_1,..,k_r} \subseteq X \times \mathbb{G}_m^{k_1+..+k_r}$

INFINITELY MANY PERIODS

Toy example: Euler Beta function

$$\frac{s_0 s_1}{s_0+s_1}\,\beta(s_0,s_1) = s_1\int_0^1 x^{s_0}(1-x)^{s_1}\,\frac{dx}{1-x} = \frac{\Gamma(s_0)\Gamma(s_1)}{\Gamma(s_0+s_1)}$$

GLOBAL point of view $\quad s_0, s_1 \in \mathbb{C} \qquad s_0, s_1, s_0+s_1 \notin \mathbb{Z}$.

$$X = \mathbb{P}^1 \setminus \{0,1,\infty\}$$

$$V = \mathcal{O}_X, \qquad \nabla_{\underline{s}} = d + s_0\frac{dx}{x} + s_1\frac{dx}{1-x}$$

$\mathcal{L}_s^\vee$ rank 1 loc. sys over $\mathbb{Q}(e^{2\pi i s_j})$, $\quad x^{s_0}(1-x)^{s_1}\,\mathbb{Q}(e^{2\pi i s_j})$

$$\left.\begin{array}{l} H^1_{dR}(X, (V,\nabla_s)) \cong \left[\frac{dx}{1-x}\right]\mathbb{Q}(s_0,s_1) \\ H_1^{lf}(X(\mathbb{C}), \mathcal{L}_s^\vee) \cong (0,1)\otimes x^{s_0}(1-x)^{s_1} \end{array}\right\} \text{rank } 1$$

period matrix: 1x1 matrix $\left(\frac{s_0 s_1}{s_0+s_1}\beta(s_0,s_1)\right)$

LOCAL POINT OF VIEW : EXPAND AROUND BAD POINT

$s_0 = s_1 = 0$

$$s_1 \int_0^1 x^{s_0} (1-x)^{s_1} \frac{dx}{1-x} = \sum \frac{s_0^{k_0}}{k_0!} \frac{s_1^{k_1+1}}{k_1!} \int_{dch} \log(x_0)^{k_0} \log(1-x_1)^{k_1} \frac{dx}{1-x}$$

regularized integral.

How to interpret the expansion coefficients? Write

$$\log x = \int_1^x \frac{dv}{v}$$

Coefficients of $s_0^{k_0}, s_1^{k_1}$ of the form:

$$\int_{\{0<u_1<\ldots<u_{k_1}<x<v_1<\ldots<v_{k_0}<1\}} \frac{du_1}{1-u_1} \cdots \frac{du_{k_1}}{1-u_{k_1}} \frac{dx}{1-x} \frac{dv_1}{v_1} \cdots \frac{dv_{k_0}}{v_{k_0}}$$

Classical periods on $\mathcal{M}_{0,n}$

(iterated integrals on $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$)

Multiple zeta values $\zeta(1,\ldots,1,m)$

Not obvious, but classical:

period (GLOBAL) = period (LOCAL)

$$\frac{s_0 s_1}{s_0+s_1}\,\beta(s_0,s_1) = \exp\left(\sum_{n\geq 2} \frac{(-1)^{n-1}\zeta(n)}{n}\left((s_0+s_1)^n - s_0^n - s_1^n\right)\right)$$

single period on $\mathcal{M}_{0,4}$ with coeffs.

∞ sequence of periods of $\mathcal{M}_{0,n+3}$

Goal is to check that natural "motivic" constructions on local & global sides agree.

New structures

(i) Single-valued periods

(ii) Coactions

1) "single-valued periods"

Complex beta function:

$$\frac{s_0 s_1}{s_0+s_1}\,\beta_{\mathbb{C}}(s_0,s_1) = \frac{-s_1}{2\pi i}\int_{\mathbb{C}} |z|^{2s_0}\,|1-z|^{2s_1}\left(\frac{d\bar{z}}{\bar{z}-1}-\frac{d\bar{z}}{\bar{z}}\right)\wedge\frac{dz}{1-z}$$

Three ways to compute $\beta_{\mathbb{C}}(s_0,s_1)$ from $\beta(s_0,s_1)$

① "Double copy":

$$\frac{s_0 s_1}{s_0+s_1}\,\beta_{\mathbb{C}}(s_0,s_1) = -\frac{\beta(s_0,s_1)}{\beta(-s_0,-s_1)}$$

Applying twisted period relations:

② (KLT formula):

$$\beta_{\mathbb{C}}(s_0,s_1) = -\frac{1}{2\pi i}\left(\frac{2}{i}\,\frac{\sin(\pi s_0)\,\sin(\pi s_1)}{\sin \pi(s_0+s_1)}\right)\beta(s_0,s_1)^2$$

③ (Local - global)

$$\frac{s_0 s_1}{s_0+s_1}\,\beta_{\mathbb{C}}(s_0,s_1) = \exp\left(\sum_{\substack{n\geq 2\\ n\text{ odd}}}\frac{2\,\zeta(n)}{n}\left((s_0+s_1)^n - s_0^n - s_1^n\right)\right)$$

"throw away all even zeta values & double odd zeta values"

2). Coactions

General formalism forces:

G $$\Delta\left(\frac{s_0 s_1}{s_0+s_1}\beta^{m}(s_0,s_1)\right) = \frac{s_0 s_1}{s_0+s_1}\beta^{m}(s_0,s_1) \otimes \frac{s_0 s_1}{s_0+s_1}\beta^{\partial k}(s_0,s_1)$$

Does this mean anything? Compare with local version.

$$\frac{s_0 s_1}{s_0+s} \beta^{m}(s_0,s_1) = \exp\left(\sum_{n\geq 2} (-1)^{n-1} \frac{\zeta^{m}(n)}{n} \left((s_0+s_1)^n - s_0^n - s_1^n\right)\right) \qquad \bullet = m, \partial k$$

L $$\Delta \zeta^{m}(n) = \zeta^{m}(n) \otimes 1 + 1 \otimes \zeta^{\partial k}(n) \qquad (B.\ 2011)$$

non-trivial.

Check: G $\Longleftrightarrow$ L

(Proof: Recall $\Delta \alpha = \alpha \otimes 1 + 1 \otimes \alpha$ $\qquad \Delta\alpha^n = (\alpha\otimes 1 + 1\otimes\alpha)^n$

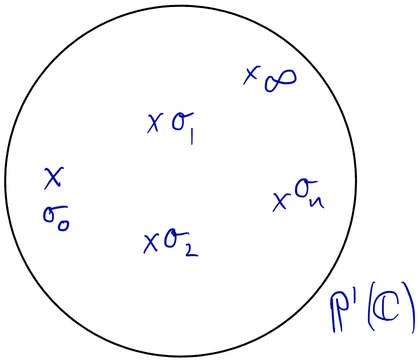
$\Longleftrightarrow \quad \Delta e^{\alpha} = e^{\alpha} \otimes e^{\alpha}$.)

Theorem (B – Dupont '19)

All the above results hold for Lauricella hypergeometric functions, in particular ${}_2F_1$.

$$\Sigma = \{\sigma_0, \ldots, \sigma_n\}$$

$$\sigma_0 = 0$$



$$(L_\Sigma)_{ij} = -s_j \int_0^{\sigma_i} x^{s_0} \prod_{k=1}^{n} (1 - x\sigma_k^{-1})^{s_k} \frac{dx}{x - \sigma_j}$$

$$1 \leq i, j \leq n$$

Matrix of Lauricella functions.

Multivalued functions of $\sigma_1, \ldots, \sigma_n$.

__Example__:

- $\Sigma = \{0, 1\}$, $\quad L_{\{0,1\}} = \dfrac{s_0 s_1}{s_0 + s_1} \beta(s_0, s_1)$

- $\Sigma = \{0, 1, 1/y\}$

$$\beta(b, c-b)\, {}_2F_1(a, b, c; y) = \int_0^1 x^{b-1} (1-x)^{c-b-1} (1-yx)^{-a} dx$$

(i) Single-valued Lauricella fns

$$\left(L^{\mathfrak{s}}_{\Sigma}\right)_{ij} = \frac{s_j}{2\pi i} \int_{\mathbb{C}} |z|^{2s_0} \prod_{k=1}^{n} \left|1 - z\sigma_k^{-1}\right|^{2s_k} \left(\frac{d\bar{z}}{\bar{z}-\bar{\sigma}_i} - \frac{d\bar{z}}{\bar{z}}\right) \wedge \frac{dz}{z-\sigma_j}$$

$s_i \in \mathbb{R}$, $s_i > 0$ $2(s_0 + \dots + s_n) < 1$

Double copy !

$$L^{\mathfrak{s}}_{\Sigma} = \left(L_{\overline{\Sigma}}(-s_0, \dots, -s_n)\right)^{-1} L_{\Sigma}(s_0, \dots, s_n)$$

complex conjugate $\overline{\Sigma} = \{\bar{\sigma}_i\}$

(ii) Coactions

$$\Delta L^{\mathfrak{m}}_{\Sigma} = L^{\mathfrak{m}}_{\Sigma} \otimes L^{\partial R}_{\Sigma}$$

(cf formula conjectured by Abreu, Britto, Duhr, Gardi)

Idea of proof:
- Tangential base point regularization & "renormalisation" of poles in twisted cohomology
- Prove that L_{Σ} is meta-abelian quotient of generalised Drinfeld associators.