Four-loop master integrals and hypergeometric functions

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Summary

- Four-loop QED g-2 and slope
- Analytical and conjectured coefficients
- Polylogarithmic and non-polylogarithmic (“elliptic”) objects
- Hypergeometric ${}_4F_3$

In Quantum Electrodynamics the *anomalous magnetic momentr*, expressed in Bohr magnetons can be written as a power series in the small quantity$\left(\frac{\alpha}{\pi}\right)$ ($\alpha \approx 1/137$).

$$F_2(0) = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$

Another similar quantity, the so-called *slope* of the Dirac form factor, important for bound states calculation, can be expanded in power series

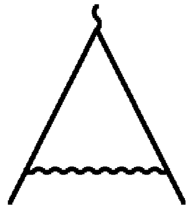
$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$

the coefficients C_i and A_i are pure numbers and can be extracted from the Feynman diagrams of the theory as linear combination of (a large number of) Feynman integrals.

These combinations are to be reduced to a linear combination of (irreducible) master integrals by solving of large systems of IBP identities, or, hopefully in near future, avoiding this step through intersection theory.

For a given n, both C_n and A_n are espressible in term of the same master integrals, and as a consequence have analytical expressions with similar structure.

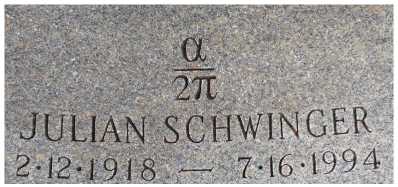
$$(g-2)/2 = C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + \dots$$
$$F_1'(0) = A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + \dots$$



1 diagram $\to$ 1 master integral

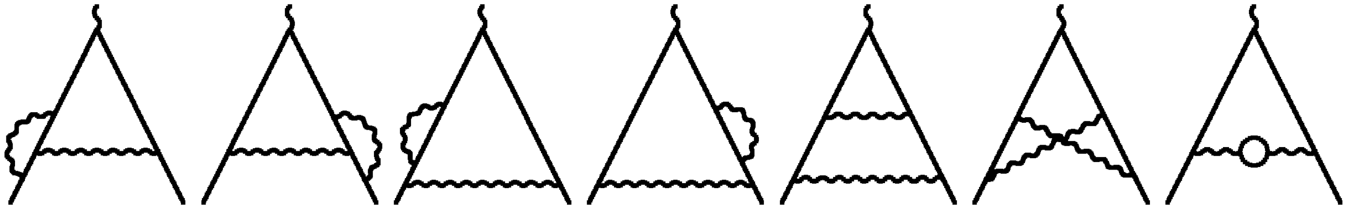
$$C_1 = \frac{1}{2}$$

Obtained by Julian Schwinger in 1948



$$A_1 = -\frac{1}{8} - \frac{1}{3}\ln\frac{\Delta E}{m} + \frac{5}{18} \qquad \text{(Bethe 1947)}$$

$$
\begin{aligned}
(g-2)/2 =& C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + \dots \\
F_1'(0) =& A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + \dots
\end{aligned}
$$



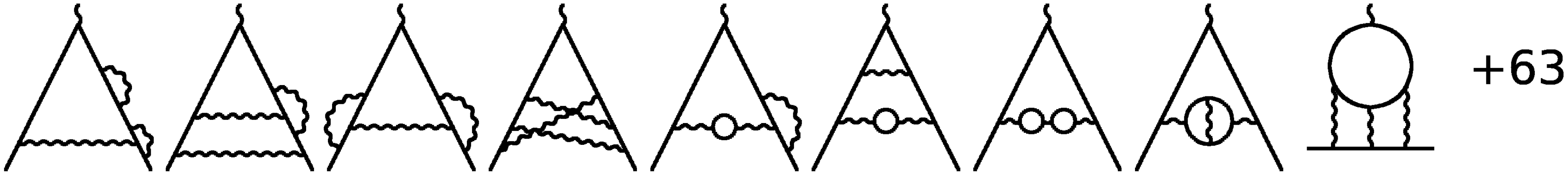
7 diagrams → 3 master integrals

$$
\begin{aligned}
C_2 =& \quad \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) = -0.328\,478\,965\dots \\
A_2 =& -\frac{4819}{5184} - \frac{49}{432}\pi^2 + \frac{1}{2}\pi^2 \ln 2 - \frac{3}{4}\zeta(3) = \ 0.469\,941\,487\dots
\end{aligned}
$$

- C_2 was computed by Petermann and Sommerfeld in 1957.
- A_2 was computed by R. Barbieri, J. A. Mignaco and E. Remiddi in 1972.

Contributions at three loops

$$
\begin{aligned}
(g-2)/2 &= C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + \dots \\
F_1'(0) &= A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + \dots
\end{aligned}
$$



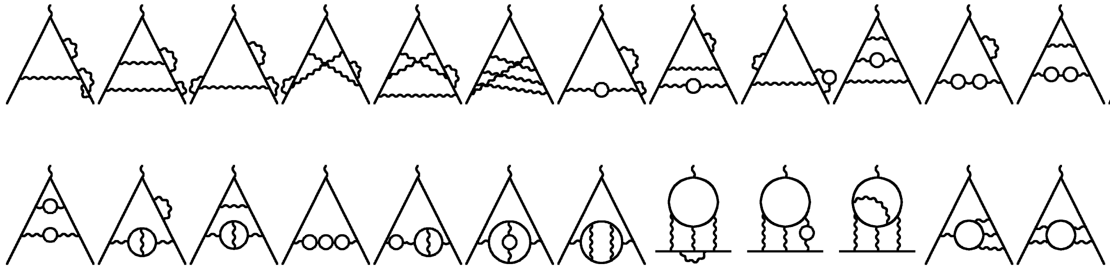
72 diagrams → 17 master integrals

$$
C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2 \ln^2 2}{24}\right] - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2 \ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184} = 1.181\ 24
$$

$$
A_3 = -\frac{17}{24}\pi^2\zeta(3) + \frac{25}{8}\zeta(5) - \frac{217}{9}\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{103}{1080}\pi^2 \ln^2 2 + \frac{3899}{25920}\pi^4 - \frac{2929}{288}\zeta(3) + \frac{41671}{2160}\pi^2 \ln 2 - \frac{454979}{38880}\pi^2 - \frac{77513}{186624} = 0.171\ 720\ 018\dots
$$

- C_3 was obtained by **S.L.** and Ettore Remiddi in 1996.
- A_3 was obtained by Melnikov and Ritbergen in 1999.

$$(g-2)/2 = C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + \dots$$
$$F_1'(0) = A_1(\alpha/\pi) + A_2(\alpha/\pi)^2 + A_3(\alpha/\pi)^3 + A_4(\alpha/\pi)^4 + \dots$$



891 diagrams → 334 master integrals

$C_4 =$ `-1.9122457649264455741526471674398300540608733906587253451713298480060384439806517061427...`

$A_4 =$ `+0.8865456739464431458368217306103153593904240326600647453680559093208403164656289274548...`

- (**S.L** 2017, **S.L** 2019)
- Numerical values calculated with 1100 digits of precision
- Semi-analytical expressions fitted with the PSLQ algorithm with very high reliability. Components of analytical expressions known with at least 4800 digits.
- Jump in complexity: analytical fits contain ∼120 terms

$$
\begin{aligned}
C_4 =& \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) + \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
& - \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) - \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 \\
& + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 - \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) + \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) \\
& + \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 + \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 + \frac{407771}{432}\zeta^2(3)\ln 2 \\
& - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 - \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2 \\
& + \sqrt{3}\Big[-\frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) \\
& + 19\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
& - \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi - \frac{223}{243}\zeta(4)\pi\ln 2 \\
& + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 - \frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2\Big] + \frac{13487}{60}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) \\
& + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& - \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi + \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
& + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
& + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 + 104\left(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\right) \\
& + \sqrt{3}\Big[\pi\left(-\frac{28458503}{691200}B_3 + \frac{250077961}{18662400}C_3\right) + \frac{483913}{77760}\pi f_2(0,0,1) + \pi\left(\frac{4715}{1944}\ln 2\, f_2(0,0,1) + \frac{270433}{10935}f_2(0,2,0) - \frac{188147}{4860}f_2(0,1,1) + \frac{188147}{12960}f_2(0,0,2)\right) \\
& + \pi\Big(\frac{826595}{248832}\zeta(2)f_2(0,0,1) - \frac{5525}{432}\ln 2\, f_2(0,0,2) + \frac{5525}{162}\ln 2\, f_2(0,1,1) - \frac{5525}{243}\ln 2\, f_2(0,2,0) + \frac{526015}{248832}f_2(0,0,3) - \frac{4675}{768}f_2(0,1,2) + \frac{1805965}{248832}f_2(0,2,1) \\
& - \frac{3710675}{1119744}f_2(0,3,0) - \frac{75145}{124416}f_2(1,0,2) - \frac{213635}{124416}f_2(1,1,1) + \frac{168455}{62208}f_2(1,2,0) + \frac{69245}{124416}f_2(2,1,0)\Big)\Big] - \frac{4715}{1458}\zeta(2)f_1(0,0,1) + \zeta(2)\Big(\frac{2541575}{82944}f_1(0,0,2) \\
& - \frac{556445}{6912}f_1(0,1,1) + \frac{54515}{972}f_1(0,2,0) - \frac{75145}{20736}f_1(1,0,1)\Big) - \frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c}.
\end{aligned}
$$

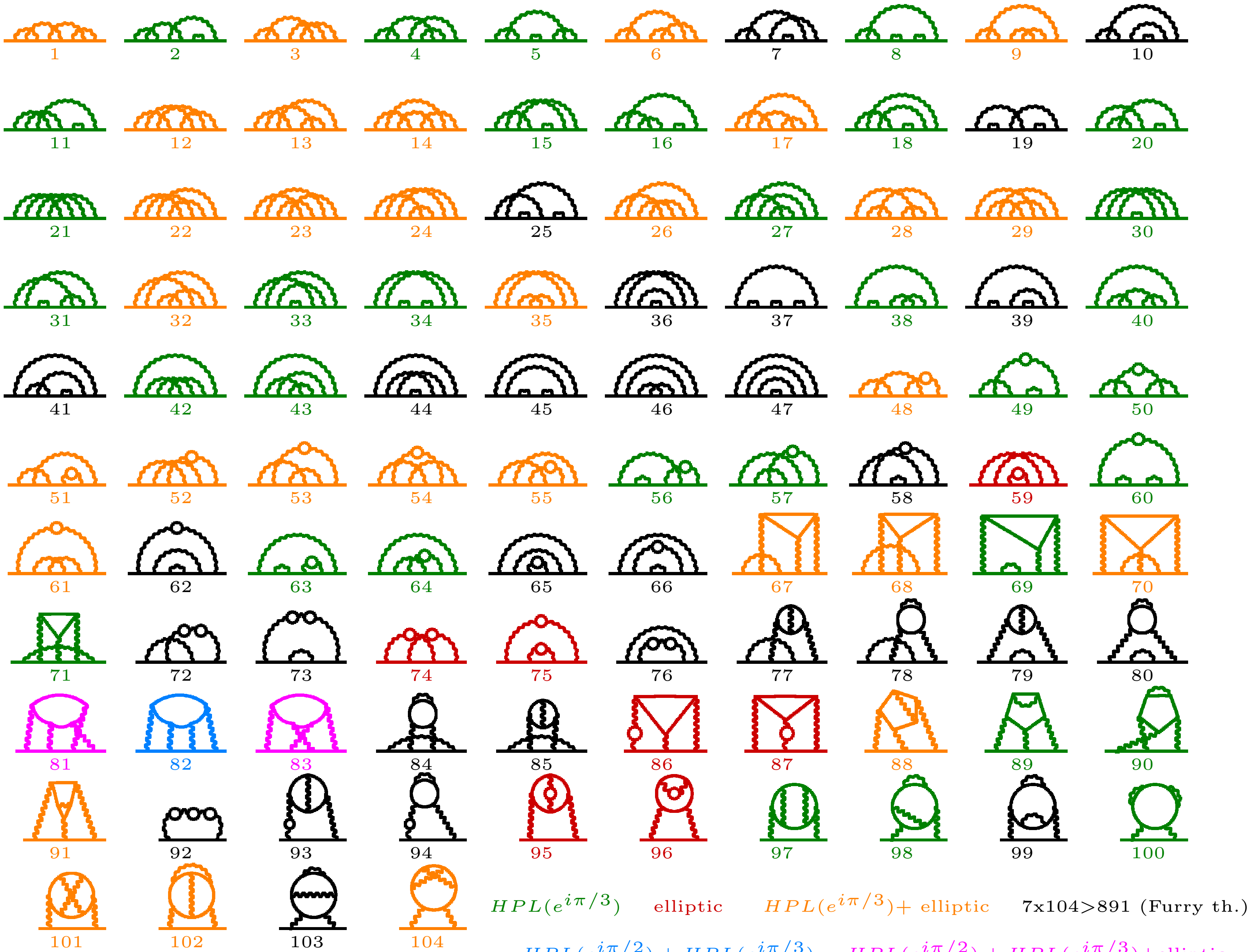
(polylogarithms) (harmonic polylogarithms) (elliptic) (unknown elliptic)

analytical fit of A_4

$$
\begin{aligned}
A_4 =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) + \frac{4572662443}{12247200}\ln^2 2\zeta(2) - \frac{1449791143}{3061800}\left(a_4 + \frac{1}{24}\ln^4 2\right) + \frac{90355973}{134400}\zeta(5) \\
& + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 - \frac{68168}{135}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{26062}{27}a_6 - \frac{18215}{27}b_6 + \frac{18215}{27}a_5\ln 2 \\
& - \frac{18215}{27}\zeta(5)\ln 2 + \frac{402152509}{189000}a_4\zeta(2) + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 - \frac{18215}{162}\zeta(3)\ln^3 2 + \frac{188648503}{1512000}\zeta(2)\ln^4 2 - \frac{21671}{6480}\ln^6 2 - \frac{7224951103}{1741824}\zeta(7) \\
& - \frac{1267114025}{387072}\zeta(4)\zeta(3) - \frac{427145}{504}a_4\zeta(3) - \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{1420289}{180}a_5\zeta(2) + \frac{116987}{21}a_7 - \frac{116987}{63}b_7 + \frac{256321}{756}d_7 + \frac{971827}{128}\zeta(6)\ln 2 + \frac{607282}{189}a_6\ln 2 \\
& - \frac{256321}{378}b_6\ln 2 - \frac{1794247}{3456}\zeta^2(3)\ln 2 + \frac{104041}{20}a_4\zeta(2)\ln 2 - \frac{1888991}{24192}\zeta(5)\ln^2 2 + \frac{75222353}{60480}\zeta(3)\zeta(2)\ln^2 2 + \frac{256321}{378}a_5\ln^2 2 - \frac{9699379}{6048}\zeta(4)\ln^3 2 - \frac{2574883}{36288}\zeta(3)\ln^4 2 \\
& + \frac{37144753}{226800}\zeta(2)\ln^5 2 - \frac{218465}{127008}\ln^7 2 + \sqrt{3}\,[-\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{458299}{36855}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10540877}{442260}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{178619489}{3980340}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{1833196}{45927}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{12563350487}{2579260320}\zeta(5)\pi \\
& + \frac{533401067}{459270}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{844343}{18900}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{844343}{28350}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{458299}{21870}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{458299}{14580}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& - \frac{263673944}{295245}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{39924629}{6889050}\zeta(3)\zeta(2)\pi + \frac{844343}{1224720}\zeta(4)\pi\ln 2 - \frac{844343}{11340}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 - \frac{844343}{7560}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 + \frac{458299}{275562}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2 \\
& + \frac{19130869}{367416}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2] + \frac{212671}{2400}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) - \frac{1031987}{14400}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{507}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
& - 507\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13689}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{68445}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{13689}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{507}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
& - \frac{1521}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{24505}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi - \frac{295}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{295}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{2655}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
& - \frac{2655}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{295}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{885}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) + \frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) \\
& - 118\left(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\right) + \sqrt{3}\,[\pi\left(+\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right) \\
& - \frac{11495611}{3265920}\pi f_2(0,0,1) + \pi\left(\frac{751}{972}\ln 2\, f_2(0,0,1) - \frac{365478661}{24494400}f_2(0,2,0) + \frac{119022487}{5443200}f_2(0,1,1) - \frac{119022487}{14515200}f_2(0,0,2)\right) - \frac{751}{729}\zeta(2)f_1(0,0,1) \\
& + \pi\left(-\frac{1735283}{497664}\zeta(2)f_2(0,0,1) + \frac{1105}{108}\ln 2\, f_2(0,0,2) - \frac{2210}{81}\ln 2\, f_2(0,1,1) + \frac{4420}{243}\ln 2\, f_2(0,2,0) - \frac{1104271}{497664}f_2(0,0,3) + \frac{272833}{41472}f_2(0,1,2) - \frac{4011005}{497664}f_2(0,2,1)\right. \\
& \left. + \frac{8417635}{2239488}f_2(0,3,0) + \frac{157753}{248832}f_2(1,0,2) + \frac{354323}{248832}f_2(1,1,1) - \frac{298711}{124416}f_2(1,2,0) - \frac{157753}{497664}f_2(2,0,1) - \frac{98285}{248832}f_2(2,1,0)\right)] + \zeta(2)\left(-\frac{4629335}{165888}f_1(0,0,2)\right. \\
& \left. + \frac{112357}{1536}f_1(0,1,1) - \frac{99731}{1944}f_1(0,2,0) + \frac{157753}{41472}f_1(1,0,1)\right) + \frac{174623}{288000}C_{81a} + \frac{29479}{7200}C_{81b} - \frac{43}{6}C_{81c} + \frac{10871}{14400}C_{83a} - \frac{157}{1620}C_{83b} - \frac{95}{24}C_{83c}
\end{aligned}
$$

(polylogarithms) (harmonic polylogarithms) (elliptic) (unknown elliptic)

A coloured view of the 104 self-mass diagrams



$HPL(e^{i\pi/3})$ elliptic $HPL(e^{i\pi/3})$+ elliptic 7x104>891 (Furry th.)

$HPL(e^{i\pi/2}) + HPL(e^{i\pi/3})$ $HPL(e^{i\pi/2}) + HPL(e^{i\pi/3})$+elliptic

The slope at 4 loops: analytical fit part 1

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
T =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) \\
&+ \frac{4572662443}{12247200}\zeta(2)\ln^2 2 - \frac{1449791143}{3061800}\left(a_4 + \frac{1}{24}\ln^4 2\right) + \frac{90355973}{134400}\zeta(5) + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 \\
&- \frac{68168}{135}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{26062}{27}a_6 - \frac{18215}{27}b_6 \\
&+ \frac{402152509}{189000}a_4\zeta(2) + \frac{18215}{27}a_5\ln 2 - \frac{18215}{27}\zeta(5)\ln 2 + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 \\
&- \frac{18215}{162}\zeta(3)\ln^3 2 + \frac{188648503}{1512000}\zeta(2)\ln^4 2 - \frac{21671}{6480}\ln^6 2 - \frac{7224951103}{1741824}\zeta(7) - \frac{1267114025}{387072}\zeta(4)\zeta(3) - \frac{427145}{504}a_4\zeta(3) \\
&- \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{1420289}{180}a_5\zeta(2) + \frac{116987}{21}a_7 - \frac{116987}{63}b_7 + \frac{256321}{756}d_7 + \frac{971827}{128}\zeta(6)\ln 2 + \frac{607282}{189}a_6\ln 2 \\
&- \frac{256321}{378}b_6\ln 2 - \frac{1794247}{3456}\zeta^2(3)\ln 2 + \frac{104041}{20}a_4\zeta(2)\ln 2 - \frac{1888991}{24192}\zeta(5)\ln^2 2 + \frac{75222353}{60480}\zeta(3)\zeta(2)\ln^2 2 \\
&+ \frac{256321}{378}a_5\ln^2 2 - \frac{9699379}{6048}\zeta(4)\ln^3 2 - \frac{2574883}{36288}\zeta(3)\ln^4 2 + \frac{37144753}{226800}\zeta(2)\ln^5 2 - \frac{218465}{127008}\ln^7 2
\end{aligned}$$

$$\zeta(n) = \sum_{i=1}^{\infty} i^{-n} \quad a_n = \mathrm{Li}_n(1/2) \;\; b_6 = H_{0,0,0,0,1,1}(1/2) \;\; b_7 = H_{0,0,0,0,0,1,1}(1/2) \;\; d_7 = H_{0,0,0,0,1,-1,-1}(1)$$

$H_{i_1,i_2,\ldots}(x)$ harmonic polylogarithms $\quad H_{a_1,a_2,\ldots a_n}(x) = \int^{x} \frac{dy_1}{y_1 - a_1} \cdots \int^{y_{n-2}} \frac{dy_{n-1}}{y_{n-1} - a_{n-1}} \int^{y_{n-1}} \frac{dy_n}{y_n - a_n} \qquad a_i \in \{-1, 0, 1\}$

The slope at 4 loops: analytical fit part 2

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_a =& -\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+\frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{458299}{36855}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10540877}{442260}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
&+\frac{178619489}{3980340}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{1833196}{45927}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{12563350487}{2579260320}\zeta(5)\pi + \frac{533401067}{459270}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
&+\frac{844343}{18900}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{844343}{28350}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{458299}{21870}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
&+\frac{458299}{14580}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{263673944}{295245}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{39924629}{6889050}\zeta(3)\zeta(2)\pi + \frac{844343}{1224720}\zeta(4)\pi\ln 2 \\
&-\frac{844343}{11340}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{19130869}{367416}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2 - \frac{844343}{7560}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 \\
&+\frac{458299}{275562}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2
\end{aligned}$$

$\mathrm{Cl}_n(\theta) = \mathrm{Im}\ \mathrm{Li}_n(e^{i\theta})$ $\quad H_{i_1,i_2,\ldots}(x)$ harmonic polylogarithms

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
V_b =& \frac{212671}{2400}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) - \frac{1031987}{14400}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{507}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
& -507\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13689}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{68445}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& + \frac{13689}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{507}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{1521}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& - \frac{24505}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi - \frac{295}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{295}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
& - \frac{2655}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{2655}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{295}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
& - \frac{885}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
W =& -\frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) + \frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) - 118\Bigg(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) \\
& -2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\Bigg),
\end{aligned}$$

The slope at 4 loops: analytical fit part 5

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
E_a =& \pi\left(\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right) - \frac{11495611}{3265920}\pi f_2(0,0,1) + \pi\left(\frac{751}{972}\ln 2\ f_2(0,0,1) - \frac{365478661}{24494400}f_2(0,2,0)\right.\\
&\left.+\frac{119022487}{5443200}f_2(0,1,1) - \frac{119022487}{14515200}f_2(0,0,2)\right) - \frac{751}{729}\zeta(2)f_1(0,0,1) + \pi\left(-\frac{1735283}{497664}\zeta(2)f_2(0,0,1)\right.\\
&+\frac{1105}{108}\ln 2\ f_2(0,0,2) - \frac{2210}{81}\ln 2\ f_2(0,1,1) + \frac{4420}{243}\ln 2\ f_2(0,2,0) - \frac{1104271}{497664}f_2(0,0,3) + \frac{272833}{41472}f_2(0,1,2)\\
&-\frac{4011005}{497664}f_2(0,2,1) + \frac{8417635}{2239488}f_2(0,3,0) + \frac{157753}{248832}f_2(1,0,2) + \frac{354323}{248832}f_2(1,1,1) - \frac{298711}{124416}f_2(1,2,0)\\
&\left.-\frac{157753}{497664}f_2(2,0,1) - \frac{98285}{248832}f_2(2,1,0)\right)
\end{aligned}$$

$$E_b = \zeta(2)\left(-\frac{4629335}{165888}f_1(0,0,2) + \frac{112357}{1536}f_1(0,1,1) - \frac{99731}{1944}f_1(0,2,0) + \frac{157753}{41472}f_1(1,0,1)\right)$$

$$f_1(i,j,k) = \int_1^9 ds\ D_1^2\left[s - \frac{9}{5}\right]\ln^i(9-s)\ln^j(s-1)\ln^k(s) \qquad f_2(i,j,k) = \int_1^9 ds\ D_1(s)\sqrt{3}\mathrm{Re}D_m(s)\left[s-\frac{9}{5}\right]\ln^i(9-s)\ln^j(s-1)\ln^k(s)$$

$$D_1(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(\frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right) \qquad D_2(s) = \frac{2}{\sqrt{(\sqrt{s}+3)(\sqrt{s}-1)^3}}K\left(1 - \frac{(\sqrt{s}-3)(\sqrt{s}+1)^3}{(\sqrt{s}+3)(\sqrt{s}-1)^3}\right)$$

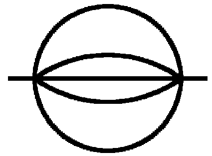
$K(x)$ complete elliptic integral of the first kind

The slope at 4 loops: analytical fit part 6

$$A_4 = T + \sqrt{3}V_a + V_b + W + \sqrt{3}E_a + E_b + U$$

$$U = \frac{174623}{288000}C_{81a} + \frac{29479}{7200}C_{81b} - \frac{43}{6}C_{81c} + \frac{10871}{14400}C_{83a} - \frac{157}{1620}C_{83b} - \frac{95}{24}C_{83c}$$

C_{xy} known only numerically; they contain two elliptic kernels at weight six.

The basic elliptic constants come from the sunrise diagram $S_4 =$  with one line differentiated and/or cutted. We can define 4 constants with a relatively simple integral representation: A_3, B_3, C_3, D_3.

$$A_3 = \int_0^1 dx \frac{K_c(x)K_c(1\text{-}x)}{\sqrt{1-x}} \qquad B_3 = \int_0^1 dx \frac{K_c^2(x)}{\sqrt{1-x}} \qquad K_c(x) = {}_2F_1\left(\begin{matrix}\frac{1}{3}\;\frac{2}{3}\\1\end{matrix};x\right)$$

$$C_3 = \int_0^1 dx \frac{E_c^2(x)}{\sqrt{1-x}} \qquad D_3 = \int_0^1 dx \frac{E_c(x)E_c(1-x)}{\sqrt{1-x}} \qquad E_c(x) = {}_2F_1\left(\begin{matrix}\frac{1}{3}\;-\frac{1}{3}\\1\end{matrix};x\right)$$

$$S_4(D{=}4\text{-}2\epsilon) = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon} + \frac{\pi\sqrt{3}}{240}(297B_3 - 1477C_3) - \frac{2320981}{20736} + O(\epsilon)$$

$$S_4(D{=}2) = \sqrt{3}\pi B_3$$

A_3 disappears in the final results. D_3 appears only cutting a line.

This constants have a hypergeometric expression (**S.L.** 2017, Y.Zhou 2018)

$$A_3 = \frac{\pi}{54}\sqrt{3}\left[{}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2}\\ \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}\end{matrix};1\right) + {}_4\tilde{F}_3\left(\begin{matrix}\frac{5}{6}\ \frac{2}{3}\ \frac{2}{3}\ \frac{1}{2}\\ \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}\end{matrix};1\right)\right] \quad \text{sum}$$

$$B_3 = \frac{\pi}{27}\sqrt{3}\left[{}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{2}\\ \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}\end{matrix};1\right) - {}_4\tilde{F}_3\left(\begin{matrix}\frac{5}{6}\ \frac{2}{3}\ \frac{2}{3}\ \frac{1}{2}\\ \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}\end{matrix};1\right)\right] \quad \text{difference}$$

$$C_3 = \frac{\pi}{27}\sqrt{3}\left[{}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{4}{3}\ -\frac{1}{2}\\ -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}\end{matrix};1\right) - {}_4\tilde{F}_3\left(\begin{matrix}-\frac{7}{6}\ -\frac{1}{3}\ \frac{2}{3}\ -\frac{1}{2}\\ -\frac{5}{6}\ \frac{1}{6}\ \frac{1}{3}\end{matrix};1\right)\right] \quad \text{difference}$$

$$D_3 = \frac{-\pi}{180}\sqrt{3}\left[{}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{6}\ \frac{1}{3}\ \frac{4}{3}\ -\frac{1}{2}\\ -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}\end{matrix};1\right) + {}_4\tilde{F}_3\left(\begin{matrix}-\frac{7}{6}\ -\frac{1}{3}\ \frac{2}{3}\ -\frac{1}{2}\\ -\frac{5}{6}\ \frac{1}{6}\ \frac{1}{3}\end{matrix};1\right)\right] + \frac{3}{10}A_3 + \frac{2}{9}\pi\sqrt{3} \quad \text{sum}$$

The hypergeometric regularized ${}_4\tilde{F}_3$ is

$${}_4\tilde{F}_3\left(\begin{matrix}a_1\ a_2\ a_3\ a_4\\ b_1\ b_2\ b_3\end{matrix};x\right) = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}\,{}_4F_3\left(\begin{matrix}a_1\ a_2\ a_3\ a_4\\ b_1\ b_2\ b_3\end{matrix};x\right)$$

A_3 and D_3 do not appear in 4-loop QED g-2 integrals.

(Bailey, Borwein, Broadhurst, Glasser 2008)

$$M(a,b,c) = \int_0^\infty I_0^a(x) K_0^b(x) x^c \, dx$$

$I_0(x)$, $K_0(x)$: modified Bessel functions.

4-loop sunrise in $D = 2$ dimensions

$$S_4(1) = 2^4 M(1,5,1) = 2^4 \int_0^\infty I_0(x) K_0^5(x) \, dx$$

(Broadhurst, Mellit 2016) (Zhou 2018)

$$\mathrm{Det} \begin{bmatrix} M(1,5,1) & M(1,5,3) \\ M(2,4,1) & M(2,4,3) \end{bmatrix} = \frac{\pi^4}{24^2}$$

$$7\ {}_4F_3\left({\tfrac{1}{2}\ \tfrac{2}{3}\ \tfrac{2}{3}\ \tfrac{5}{6} \atop \tfrac{7}{6}\ \tfrac{7}{6}\ \tfrac{4}{3}};1\right){}_4F_3\left({-\tfrac{1}{2}\ \tfrac{1}{3}\ \tfrac{4}{3}\ \tfrac{1}{6} \atop -\tfrac{1}{6}\ \tfrac{5}{6}\ \tfrac{5}{3}};1\right)+10\ {}_4F_3\left({\tfrac{1}{2}\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{6} \atop \tfrac{5}{6}\ \tfrac{5}{6}\ \tfrac{2}{3}};1\right){}_4F_3\left({-\tfrac{1}{2}\ -\tfrac{1}{3}\ \tfrac{5}{6}\ -\tfrac{4}{3} \atop -\tfrac{1}{6}\ \tfrac{5}{6}\ \tfrac{5}{3}};1\right)=40$$

introducing the regularized ${}_4\tilde{F}_3$

$${}_4\tilde{F}_3\left({\tfrac{1}{2}\ \tfrac{2}{3}\ \tfrac{2}{3}\ \tfrac{5}{6} \atop \tfrac{7}{6}\ \tfrac{7}{6}\ \tfrac{4}{3}};1\right){}_4\tilde{F}_3\left({-\tfrac{1}{2}\ \tfrac{1}{3}\ \tfrac{4}{3}\ \tfrac{1}{6} \atop -\tfrac{1}{6}\ \tfrac{5}{6}\ \tfrac{5}{3}};1\right)-{}_4\tilde{F}_3\left({\tfrac{1}{2}\ \tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{6} \atop \tfrac{5}{6}\ \tfrac{5}{6}\ \tfrac{2}{3}};1\right){}_4\tilde{F}_3\left({-\tfrac{1}{2}\ -\tfrac{1}{3}\ \tfrac{5}{6}\ -\tfrac{4}{3} \atop -\tfrac{1}{6}\ \tfrac{5}{6}\ \tfrac{5}{3}};1\right)=\frac{240\sqrt{3}\pi}{7}$$

Conjecture: fractions appearing in the indices are actually $\frac{1}{2}\pm k\delta$ or $1\pm k\delta$ with small integer k and $\delta=\frac{1}{6}$, that is, a deformation of the “simplest” hypergeometric

$${}_4F_3\left({\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2} \atop 1\ 1\ 1};1\right)=\frac{4}{\pi^4}M(0,4,0) \tag{1}$$

i.e.

$$\begin{aligned}\frac{1}{3}&\to\frac{1}{2}-\delta\\ \frac{2}{3}&\to\frac{1}{2}+\delta\\ \frac{5}{6}&\to\frac{1}{2}+2\delta\\ -\frac{4}{3}&\to-\frac{3}{2}+\delta \text{ or } -1-2\delta\ldots\end{aligned}$$

Experimenting a bit we found

$$
{}_4\tilde{F}_3\left({\frac{1}{2}\ \frac{2}{3}\ \frac{2}{3}\ \frac{5}{6} \atop \frac{7}{6}\ \frac{7}{6}\ \frac{4}{3}};1\right){}_4\tilde{F}_3\left({-\frac{1}{2}\ \frac{1}{3}\ \frac{4}{3}\ \frac{1}{6} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right) - {}_4\tilde{F}_3\left({\frac{1}{2}\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{6} \atop \frac{5}{6}\ \frac{5}{6}\ \frac{2}{3}};1\right){}_4\tilde{F}_3\left({-\frac{1}{2}\ -\frac{1}{3}\ \frac{5}{6}\ -\frac{4}{3} \atop -\frac{1}{6}\ \frac{5}{6}\ \frac{5}{3}};1\right) = \frac{240\sqrt{3}\pi}{7}
$$

corresponds to the $\delta = 1/6$ case of the general conjecture

$$
{}_4\tilde{F}_3\left({\frac{1}{2}\ \ \frac{1}{2}{+}\delta\ \ \frac{1}{2}{+}\delta\ \ \frac{1}{2}{+}2\delta \atop 1{+}\delta\ \ 1{+}\delta\ \ 1{+}2\delta};1\right){}_4\tilde{F}_3\left({-\frac{1}{2}\ \ \frac{1}{2}{-}\delta\ \ \frac{3}{2}{-}\delta\ \ \frac{1}{2}{-}2\delta \atop -\delta\ \ 1{-}\delta\ \ 2{-}2\delta};1\right) - {}_4\tilde{F}_3\left({\frac{1}{2}\ \ \frac{1}{2}{-}\delta\ \ \frac{1}{2}{-}\delta\ \ \frac{1}{2}{-}2\delta \atop 1{-}\delta\ \ 1{-}\delta\ \ 1{-}2\delta};1\right){}_4\tilde{F}_3\left({-\frac{1}{2}\ \ -\frac{1}{2}{+}\delta\ \ \frac{1}{2}{+}\delta\ \ -\frac{3}{2}{+}2\delta \atop -\delta\ \ 1{-}\delta\ \ 2{-}2\delta};1\right) =
$$

$$
= \frac{4\Gamma(1/2-2\delta)\Gamma(1/2-\delta)\Gamma(-(1/2)+\delta)\Gamma(-(3/2)+2\delta))}{\Gamma(1-\delta)\Gamma(-1+\delta)}
$$

$$
= 128\pi \frac{\delta-1}{(4\delta-1)(2\delta-3)(2\delta-1)} \frac{\sin(\pi\ \delta)^2}{\sin(4\pi\ \delta)^2}
$$

numerically checked

$$
{}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}{+}d\ \frac{1}{2}{+}d\ \frac{1}{2}{+}2d\\ 1{+}d\ 1{+}d\ 1{+}2d\end{matrix};1\right){}_4\tilde{F}_3\left(\begin{matrix}-\frac{1}{2}\ \frac{1}{2}{-}d\ \frac{3}{2}{-}d\ \frac{1}{2}{-}2d\\ -d\ 1{-}d\ 2{-}2d\end{matrix};1\right)
$$

$$
-\,{}_4\tilde{F}_3\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}{-}d\ \frac{1}{2}{-}d\ \frac{1}{2}{-}2d\\ 1{-}d\ 1{-}d\ 1{-}2d\end{matrix};1\right){}_4\tilde{F}_3\left(\begin{matrix}-\frac{1}{2}\ -\frac{1}{2}{+}d\ \frac{1}{2}{+}d\ -\frac{3}{2}{+}2d\\ -d\ 1{-}d\ 2{-}2d\end{matrix};1\right)=
$$

$$
=\frac{4\Gamma(1/2-2d)\Gamma(1/2-d)\Gamma(-(1/2)+d)\Gamma(-(3/2)+2d))}{\Gamma(1-d)\Gamma(-1+d)}
$$

$$
=128\pi\frac{d-1}{(4d-1)(2d-3)(2d-1)}\frac{\sin(\pi\ d)^2}{\sin(4\pi\ d)^2}
$$

This conjecture can be further generalized introducing at least another free parameter. It can be rigorously proved likely by using intersections theory along the same lines of similar identities for the simpler hypergeometric functions ${}_2F_1$ and ${}_3F_2$.

$$+{}_2F_1\left({\frac{1}{2}+\lambda \;\; -\frac{1}{2}-\nu \atop 1+\lambda+\mu};r\right){}_2F_1\left({\frac{1}{2}-\lambda \;\; \frac{1}{2}+\nu \atop 1+\nu+\mu};1-r\right)+{}_2F_1\left({\frac{1}{2}+\lambda \;\; \frac{1}{2}-\nu \atop 1+\lambda+\mu};r\right){}_2F_1\left({-\frac{1}{2}-\lambda \;\; \frac{1}{2}+\nu \atop 1+\nu+\mu};1-r\right)$$

$$-{}_2F_1\left({\frac{1}{2}+\lambda \;\; \frac{1}{2}-\nu \atop 1+\lambda+\mu};r\right){}_2F_1\left({\frac{1}{2}-\lambda \;\; \frac{1}{2}+\nu \atop 1+\nu+\mu};1-r\right)=\frac{\Gamma\left(1+\lambda+\mu\right)\Gamma\left(1+\nu+\mu\right)}{\Gamma\left(\frac{3}{2}+\lambda+\mu+\nu\right)\Gamma\left(\frac{1}{2}+\mu\right)}$$

for $\lambda=\mu=\nu=0 \to$ Legendre's identity (easily obtainable with intersections)

Conclusions

- We have found/conjectured a generalization of a quadratic relation between the ${}_4F_3$ hypergeometrics which appear in some 4-loop QED quantities.
- Proof it is likely to be found through intersections.

The End

Backup: A simple example of analytical fit by using the PSLQ algorithm

$$
\begin{aligned}
G_7 = &-2342.2075141060230754235225405907927098853287320565594708 07\\
&3594814835713846916806455916973185992614831948904197343569 86\\
&6405364828391809277375993763069797378291106083117076717679 35\\
&9831391259607669183299238838719305848684965160728687292431 83\\
&3178005196947599399147517611412834358100307911368387937080 71\\
&1573460997870203023575268524120954362873328464489262424305 03\\
&2364495474744073075812911236379210785864186765175498779728 67\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
= &\frac{1671597}{512} - \frac{4381}{96}\pi^2 - \frac{22193}{24}\zeta(3) - 144\,\pi^2\ln 2 - \frac{3617}{240}\pi^4 - \frac{71}{2}\zeta(5)\\
&- \frac{393}{2}\pi^2\zeta(3) - \frac{869}{162}\pi^6 - 24\pi^4\ln^2 2 + 576\pi^2 a_4 + 24\pi^2\ln^4 2 - \frac{803}{2}\zeta(3)^2\\
&+ 504\pi^2\zeta(3)\ln 2 - \frac{1735}{4}\zeta(7) + \frac{799}{6}\pi^2\zeta(5) - \frac{661}{180}\pi^4\zeta(3)
\end{aligned}
$$

black: ansatz (the input)

brown: coefficients found by PSLQ (the output)

This particular fit can be found using input data with a minimum precision of 415 digits.

Backup: The slope at 4 loops: analytical fit part 1 (rewritten)

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$\begin{aligned}
T =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) \\
&+ \frac{4572662443}{12247200}\zeta(2)\ln^2 2 - \frac{1449791143}{3061800}t_4 + \frac{90355973}{134400}\zeta(5) + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 \\
&- \frac{68168}{135}t_5 - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 \\
&+ \frac{402152509}{189000}t_4\zeta(2) - \frac{18215}{27}t_{61} + \frac{26062}{27}t_{62} - \frac{7224951103}{1741824}\zeta(7) - \frac{1267114025}{387072}\zeta(4)\zeta(3) \\
&- \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{971827}{128}\zeta(6)\ln 2 - \frac{6242389}{6048}\zeta(3)\zeta(2)\ln^2 2 - \frac{427145}{504}t_4\zeta(3) + \frac{1420289}{180}t_5\zeta(2) \\
&+ \frac{256321}{756}t_{71} - \frac{116987}{63}t_{72} + \frac{104041}{20}t_{73}
\end{aligned}$$

$a_n = \mathrm{Li_n}(1/2)$, $b_6 = H_{0,0,0,0,1,1}(1/2)$, $b_7 = H_{0,0,0,0,0,1,1}(1/2)$, $d_7 = H_{0,0,0,0,1,-1,-1}(1)$

Decomposition in constants valid for each diagram contribution to $F_1'(0)$ & $F_2(0)$

$t_4 = a_4 + \frac{1}{24}\ln^4 2 \qquad t_5 = a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2 \qquad t_{62} = a_6 - \frac{1}{48}\zeta(2)\ln^4 2 + \frac{1}{720}\ln^6 2$

$$t_{61} = b_6 - a_5\ln 2 + \zeta(5)\ln 2 + \frac{1}{6}\zeta(3)\ln^3 2 - \frac{1}{12}\zeta(2)\ln^4 2 + \frac{1}{144}\ln^6 2$$

$$t_{71} = d_7 - 2b_6\ln 2 + 4a_6\ln 2 + 2a_5\ln^2 2 - \frac{49}{32}\zeta^2(3)\ln 2 - \frac{95}{32}\zeta(5)\ln^2 2 + \frac{1}{8}\zeta(4)\ln^3 2 - \frac{1}{3}\zeta(3)\ln^4 2 + \frac{1}{12}\zeta(2)\ln^5 2 - \frac{\ln^7 2}{120}$$

$$t_{72} = b_7 - 3a_7 - a_6\ln 2 - \frac{1}{2}\zeta(5)\ln^2 2 + \frac{1}{48}\zeta(4)\ln^3 2 - \frac{1}{24}\zeta(3)\ln^4 2 + \frac{1}{120}\zeta(2)\ln^5 2 - \frac{\ln^7 2}{1680}$$

$$t_{73} = \left(a_4 - \frac{1}{4}\zeta(2)\ln^2 2 + \frac{7}{16}\zeta(3)\ln 2 + \frac{1}{24}\ln^4 2\right)\zeta(2)\ln 2$$

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$V_a = -\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{916598}{76545}v_{61} + \frac{844343}{28350}v_{62} + \frac{178619489}{3980340}v_{63} - \frac{263673944}{295245}v_{64}$$

$$\mathrm{Cl}_n(\theta) = \mathrm{Im}\mathrm{Li}_n(e^{i\theta})$$

$$v_{61} = \mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{27}{26}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{207}{104}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10}{3}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{7}{4}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{21}{8}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{5}{72}\zeta(3)\zeta(2)\pi - \frac{5}{6}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{5}{36}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2$$

$$v_{62} = \zeta(2)\left[\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{1}{6}\zeta(3)\pi + \frac{1}{108}\zeta(2)\pi\ln 2 - \frac{5}{2}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\ln 2 - \frac{15}{4}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\ln 2 + \frac{25}{12}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2 - \frac{661}{1188}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\right]$$

$$v_{63} = \mathrm{Cl}_6\left(\frac{\pi}{3}\right) - \frac{3}{4}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$$

$$v_{64} = \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{91}{66}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)$$

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$V_b = \frac{212671}{2400} v_{65} - \frac{1031987}{14400}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right) - \frac{507}{4} v_{71} - \frac{295}{4} v_{72}$$

$$v_{65} = \mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_2\left(\frac{\pi}{3}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right),$$

$$v_{71} = \mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 4\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{27}{8}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{135}{16}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$- \frac{27}{2}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_4\left(\frac{\pi}{3}\right) + \frac{145}{132}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi$$

$$v_{72} = \zeta(2)\Bigg[\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 2\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{9}{4}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)$$
$$+ \frac{9}{2}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{3}{2}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\Bigg]$$

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$W_a = -\frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) \qquad \text{New}$$

$$W_b = +\frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) - 472 v_{73}$$

$$v_{73} = \zeta(2)\left(\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right) + \mathrm{Cl}_2\left(\frac{\pi}{2}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right) - \frac{1}{2}\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\pi + \frac{1}{4}\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\ln 2\right)$$

- $\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)$ appears in $F_1'(0)$, cancels out in $F_2(0)$

$\mathrm{Cl}_2\left(\frac{\pi}{2}\right)$ Catalan's constant $\beta_2 = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$

Backup: The slope at 4 loops: analytical fit part 5 (rewritten)

$$A_4 = T + \sqrt{3}V_a + V_b + W_a + W_b + \sqrt{3}E_a + E_b + U$$

$$E_a = \pi\left(\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right) - \frac{11495611}{3265920}\pi f_2(0,0,1) - \frac{365478661}{24494400}e_{61} + \frac{119022487}{5443200}e_{62} - \frac{98285}{248832}e_{71} - \frac{157753}{497664}e_{72}$$

$$E_b = -\frac{751}{729}\zeta(2)f_1(0,0,1) + \frac{157753}{41472}e_{73} - \frac{99731}{1944}e_{74}$$

$$e_{71} = \pi\Big(f_2(2,1,0) + \frac{7}{3}f_2(1,2,0) - 2f_2(1,1,1) + \frac{40}{27}f_2(0,3,0) - \frac{7}{3}f_2(0,2,1) + f_2(0,1,2) - 30\ln 2 f_2(0,2,0) + 45\ln 2 f_2(0,1,1) - \frac{135}{8}\ln 2 f_2(0,0,2)\Big)$$

$$e_{72} = \pi\Big(f_2(2,0,1) + \frac{14}{3}f_2(1,2,0) - 2f_2(1,1,1) - 2f_2(1,0,2) - \frac{370}{27}f_2(0,3,0) + \frac{85}{3}f_2(0,2,1) - 22f_2(0,1,2) + 7f_2(0,0,3) + 11\zeta(2)f_2(0,0,1) - 20\ln 2 f_2(0,2,0) + 30\ln 2 f_2(0,1,1) - \frac{45}{4}\ln 2 f_2(0,0,2)\Big)$$

$$e_{73} = \zeta(2)\left(f_1(1,0,1) - f_1(0,1,1) + \frac{1}{4}f_1(0,0,2)\right)$$

$$e_{74} = \zeta(2)\left(f_1(0,2,0) - \frac{3}{2}f_1(0,1,1) + \frac{9}{16}f_1(0,0,2)\right)$$

other combinations: $e_{73} - \dfrac{e_{72}}{4\sqrt{3}}$