From DIAGRAMMAR to DIAGRAMMALGEBRA

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MathemAmplitudes 2019

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Università degli Studi di Padova

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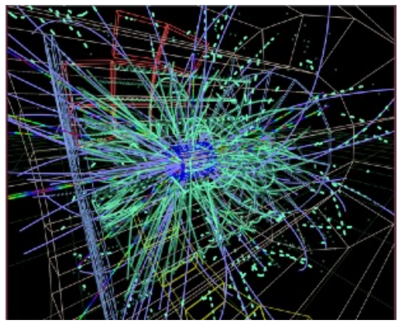
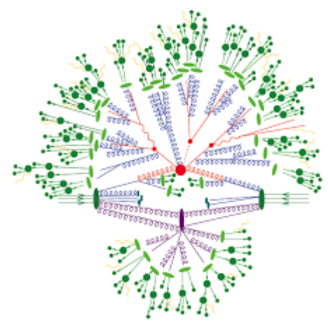
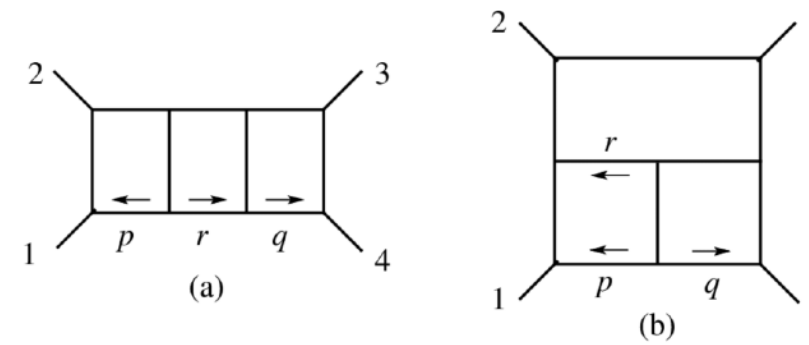
Physics and Astronomy Department
Galileo Galilei
University of Padova - Italy

The Ubiquitous Feynman Integrals

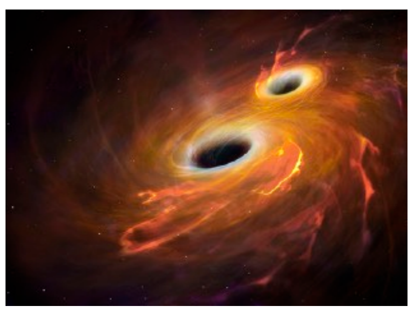


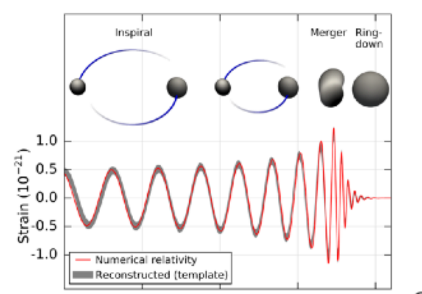




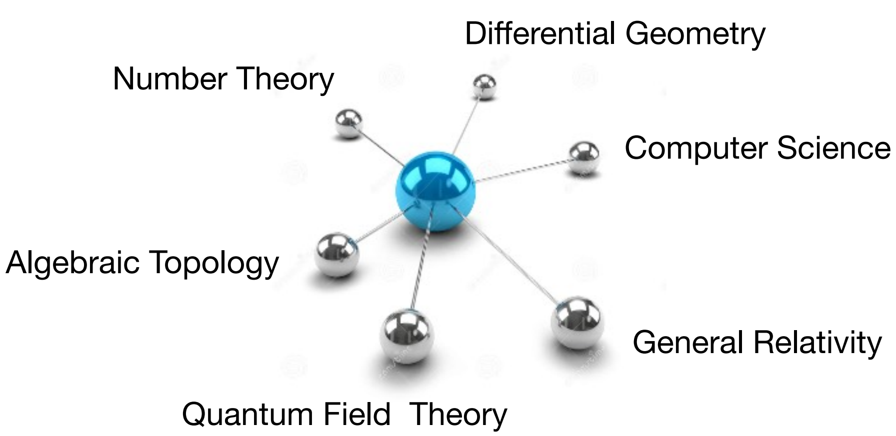
» **SCATTERING AMPLITUDES for *precision physics***
60 orders of magnitudes in Energy scales:
from quarks and gluons to black-hole binary systems

» **one tool: Feynman diagrams**
crucial for Elementary Particles
and Gravitational Waves Phenomenology:
form hard scattering cross-sections to astrophysical coalescing systems

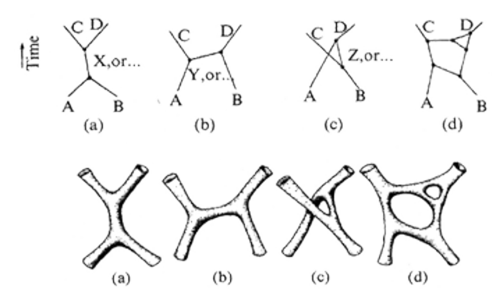




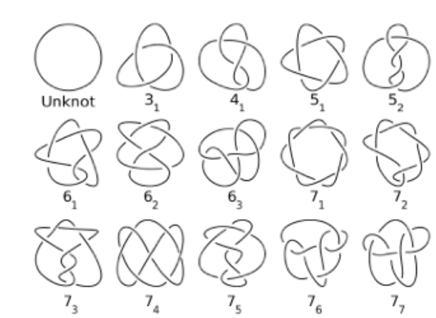
» **Interdisciplinary competences** required

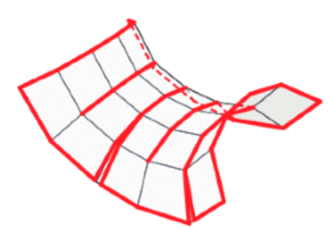


Particles and Strings



Knots theory

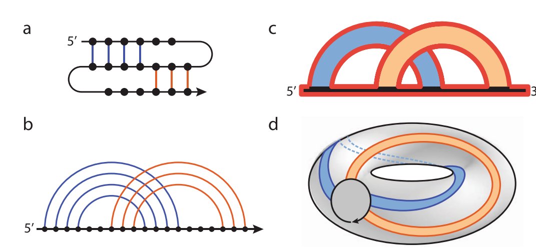


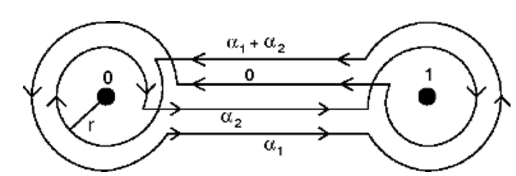


Surface folding

» **Impact**
Physics and **Mathematics,** but also **Biology, Chemistry, Statistics and Economy**

RNA folding





Differential Geometry

Dimensionally Regulated Integrals

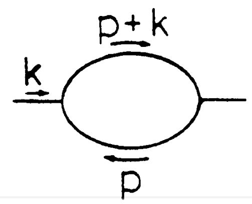
DIAGRAMMAR

G. 't Hooft and M. Veltman

GENEVA

1973

Consider a self-energy diagram with two scalar intermediate particles in n dimensions:



$$I_n = \int d_n p \frac{1}{(p^2 + m^2 - i\varepsilon)[(p+k)^2 + M^2 - i\varepsilon]} . \qquad (13.1)$$

k rest-frame

(k = 0, 0, 0, iμ).

$$I_n = \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} \omega^{n-2}\, d\omega \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \left\{ \frac{1}{(-p_0^2 + \omega^2 + m^2)[-(p_0 + \mu)^2 + \omega^2 + M^2]} \right\} . \qquad (13.4)$$

that I_1 from Eq. (13.4) is an undetermined form ∞/∞. Let us first fix n to be in the region where the above expression exists, for instance $1.5 < n < 1.75$. Next we perform a partial integration with respect to ω^2

$$d\omega\, \omega^{n-2} = \frac{1}{2} d\omega^2 (\omega^2)^{(n-3)/2} = \frac{1}{2} d\omega^2 \frac{2}{n-1} \frac{d}{d\omega^2} (\omega^2)^{(n-1)/2} .$$

For n in the given domain the surface terms are zero. Using $z\Gamma(z) = \Gamma(z+1)$ and repeating this operation λ times we obtain

$$I_n = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2} + \lambda\right)} \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} d\omega\, \omega^{n-2+2\lambda} \left(-\frac{\partial}{\partial\omega^2}\right)^{\lambda} \{\cdots\} . \qquad (13.5)$$

We have derived this equation for $1 < n < 4$. However it is meaningful also for $1 - 2\lambda < n < 4$. The ultraviolet behaviour is unchanged, but the divergence near $\omega = 0$ is

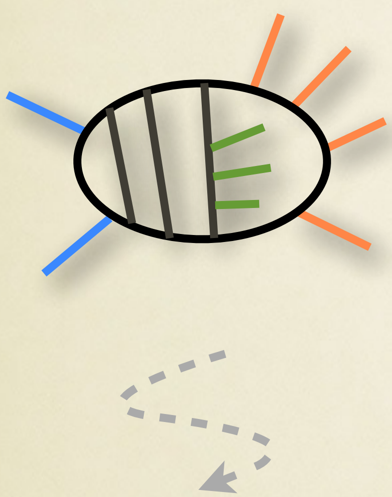
Shifting integration variables ($p' = p + kx$), making the Wick rotation, and introducing n-dimensional polar coordinates, one computes

$$I_n = \frac{i\pi^{n/2}\,\Gamma\left(2 - \frac{n}{2}\right)}{\Gamma(2)} \int_0^1 dx \frac{1}{[M^2 x + m^2(1-x) + k^2 x(1-x)]^{2-n/2}} .$$

- Analytic continuation
- Integration-by-parts Identites
- Graph-polynomial

Feynman Integrals

- **Momentum-space Representation**



N-denominator
generic Integral

$$= I_{a_1,\ldots,a_N} = \int \prod_{i=1}^{L} d^d k_i \left(\prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right)$$

L loops, $E+1$ external momenta,

$N = LE + \frac{1}{2}L(L+1)$ (generalised) denominators

total number of *reducible* and *irreducible* scalar products

't Hooft & Veltman

- **Integration-by-parts Identites** Tkachov; Chetyrkin & Tkachov

Laporta, Remiddi, Kuehn, Baikov, Smirnov, Melnikov, Gehrmann, Weinzierl, Anastasiou,…

$$\int \prod_{i=1}^{L} d^d k_i \, \frac{\partial}{\partial k_j^{\mu}} \left(v_\mu \prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right) = 0$$

$v_\mu = v_\mu(p_i, k_j)$ arbitrary

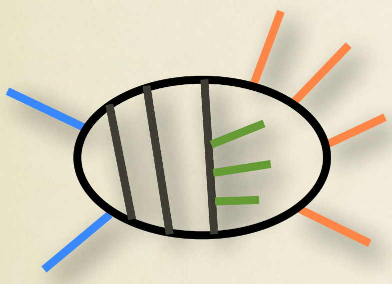
- **IBP identities** $\sum_i b_i \, I_{a_1,\ldots,a_i \pm 1,\ldots,a_N} = 0$

Parametric Representation(s)

- **Upon a change of integration variables**

$$\varphi_N(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^N\mathbf{z} \qquad \textbf{differential } m\textbf{-form}$$

$$d^N\mathbf{z} = dz_1 \wedge \ldots \wedge dz_N$$



$$= I_{a_1,\ldots,a_N} = \int_{\mathcal{C}} u(\mathbf{z}) \;\; \varphi_N(\mathbf{z})$$

$$\hat{\varphi}_N(\mathbf{z}) = f(\mathbf{z}) \prod_i z_i^{-a_i}$$

$$u(\mathbf{z}) = \mathcal{P}(\mathbf{z})^{\gamma}$$

$$\mathcal{P}(\mathbf{z}) = \textbf{graph-Polynomial}$$

$$\gamma(d) = \textbf{generic exponent}$$

- **Integration-by-parts:** two situations may occur

$$\int_{\mathcal{C}} d\Big(u(\mathbf{z}) \;\; \varphi_N(\mathbf{z})\Big) \quad \begin{cases} \neq 0 \,, & \\ = 0 \,, & u(\partial\mathcal{C}) = 0. \end{cases}$$

- Baikov representation

- IBP identities

$$\sum_i b_i \, I_{a_1,\ldots,a_i \pm 1,\ldots,a_N} = 0$$

Vector Space Decomposition

$\nu =$ dimension of the vector space

- **Vector decomposition** $$I = \sum_{i=1}^{\nu} c_i \, J_i$$

- **Projections** $$c_i = \begin{cases} I \cdot J_i \ , & J_i \cdot J_j = \delta_{ij} \\ \sum_{j=1}^{\nu} I \cdot J_j (C^{-1})_{ji} \ , & J_i \cdot J_j = C_{ij} \neq \delta_{ij} \end{cases}$$

- **Completeness** $$\sum_{i,j} J_j \, (C^{-1})_{ji} \, J_i = \mathbb{I}_{\nu \times \nu}$$

The two questions:

1) what is the vector space dimension ν ?

2) what is the *scalar product* "$\cdot$" between integrals ?

Basics of Intersection Theory

Aomoto, Cho, Goto, Kita, Matsubara-Heo,
Mazumoto, Mimachi, Mizera, Yoshida,...

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \underbrace{\int_{\mathcal{C}}}_{\text{twisted cycle}} u(\mathbf{z}) \underbrace{\varphi_m(\mathbf{z})}_{\text{twisted cocycle}}$$

$\varphi_m(\mathbf{z})$ is a differential m-form

$u(\mathbf{z})$ is a multivalued function (regulating *all* poles of φ_m)

$u(\partial\mathcal{C}) = 0$

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$u(\partial\mathcal{C}) = 0$

There could exist many forms φ_m that upon integration give the same result I

- **Covariant Derivatives** $\omega \equiv d \log u \qquad \nabla_\omega \equiv d + \omega\wedge \qquad \nabla_{-\omega} \equiv d - \omega\wedge$
- **Integrals** $I = \int_{\mathcal{C}} u\, \varphi_m = \int_{\mathcal{C}} u \left(\varphi_m + \nabla_\omega \varphi_{m-1}\right)$
- **Dual Integrals** $\tilde{I} = \int_{\mathcal{C}} u^{-1}\, \phi_m = \int_{\mathcal{C}} u^{-1} \left(\phi_m + \nabla_{-\omega} \phi_{m-1}\right)$

Pairings of Cycles and Co-cycles

- **Co-Homology Groups**

$$H^m_\omega \equiv \{m\text{−forms } \varphi_{\rm m} \,|\, \nabla_\omega \varphi_{\rm m} = 0\}/\{\nabla_\omega \varphi_{\rm m-1}\}$$

$$H^m_{-\omega} \equiv \{m\text{−forms } \phi_{\rm m} \,|\, \nabla_{-\omega} \phi_{\rm m} = 0\}/\{\nabla_{-\omega} \phi_{\rm m-1}\}$$

- **Basic building blocks**

$$\langle \varphi_L | \equiv \varphi_L(\mathbf{z}) \in H^m_\omega \qquad |\varphi_R\rangle \equiv \varphi_R(\mathbf{z}) \in H^m_{-\omega} \qquad |\mathcal{C}_R] \equiv \int_{\mathcal{C}_R} u(\mathbf{z}) \qquad [\mathcal{C}_L| \equiv \int_{\mathcal{C}_L} u(\mathbf{z})^{-1}$$

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- **Integrals ::** pairings of cycles and co-cycles

$$\langle\, \varphi_L \,|\, \mathcal{C}_R \,] \;\equiv\; \int_{\mathcal{C}_R} u(\mathbf{z})\; \varphi_L(\mathbf{z}) \;=\; I$$

- **Dual Integrals ::** pairings of cycles and co-cycles

$$[\, \mathcal{C}_L \,|\, \varphi_R \,\rangle \;\equiv\; \int_{\mathcal{C}_L} u(\mathbf{z})^{-1}\; \varphi_R(\mathbf{z}) \;=\; \tilde{I}$$

- **Intersection numbers for cycles ::** pairings of cycles

$$[\, \mathcal{C}_{\rm L} \,|\, \mathcal{C}_{\rm R} \,] \equiv \text{intersection number}$$

- **Intersection numbers for co-cycles ::** pairings of co-cycles

$$\langle\, \varphi_{\rm L} \,|\, \varphi_{\rm R} \,\rangle \equiv \int_{\mathcal{C}} \iota(\varphi_{\rm L}) \wedge \varphi_{\rm R}$$

Linear Relations

Feynman Integrals and Intersection Theory

Mizera **& P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

$$I = \langle \varphi | \mathcal{C}]$$

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}} u(\mathbf{z})\, e_i(\mathbf{z}) = \langle e_i | \mathcal{C}] \,, \qquad i = 1, \ldots, \nu \,,$$

$$I = \sum_{i=1}^{\nu} c_i\, J_i$$

$$\langle \varphi | = \sum_{i=1}^{\nu} c_i \, \langle e_i | \;=\; \sum_{i,j=1}^{\nu} \underbrace{\langle \varphi | e_j \rangle \, (\mathbf{C}^{-1})_{ij} \, \langle e_i |}_{=1}$$

Quadratic Relations

Riemann Twisted Periods Relations (RTPR)

- **Completeness for forms**

$$\sum_{i,j=1}^{\nu} |e_j\rangle \, (\mathbf{C}^{-1})_{ij} \, \langle e_i| = \mathbb{I}_c \qquad \mathbf{C}_{ij} \equiv \langle e_i | e_j \rangle$$

- **Completeness for contours**

$$\sum_{i,j=1}^{\nu} |\mathcal{C}_j] \, (\mathbf{H}^{-1})_{ij} \, [\mathcal{C}_i| = \mathbb{I}_h \qquad \mathbf{H}_{ij} \equiv [\mathcal{C}_i | \mathcal{C}_j]$$

Riemann Twisted Periods Relations (RTPR)

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- **Riemann Twisted Period Relations**

Cho, Matsumoto (1995)

$$\langle \, \varphi_{\rm L} \mid \varphi_{\rm R} \, \rangle = \sum_{i,j} \langle \, \varphi_{\rm L} \mid \mathcal{C}_{{\rm R},j} \,] \; [\, \mathcal{C}_{{\rm L},i} \mid \mathcal{C}_{{\rm R},j} \,]^{-1} \; [\, \mathcal{C}_{{\rm L},i} \mid \varphi_{\rm R} \, \rangle$$

$$[\, \mathcal{C}_{\rm L} \mid \mathcal{C}_{\rm R} \,] = \sum_{i,j} [\, \mathcal{C}_{{\rm L},i} \mid \varphi_{{\rm R},j} \, \rangle \; \langle \, \varphi_{{\rm L},i} \mid \varphi_{{\rm R},j} \, \rangle^{-1} \; \langle \, \varphi_{\rm L} \mid \mathcal{C}_{\rm R} \,]$$

Elliot's Identity from Intersections

Matsumoto **& P.M.** (w.i.p.)

The complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$ of the first and second kind

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - r^2 \sin^2\phi}} \qquad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2\phi}\, d\phi \qquad r \in (0, 1)$$

- **Legendre Identity**

$$\mathcal{E}\mathcal{K}' + \mathcal{E}'\mathcal{K} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}$$

$$\mathcal{K}'(r) = \mathcal{K}(r') \text{ and } \mathcal{E}'(r) = \mathcal{E}(r')$$
$$r^2 + r'^2 = 1$$

- **Elliot's Identity and Hypergeometric Functions**

Balasubramanian, Naik, Ponnusamy, Vuorinen (2001)

$$\begin{aligned}
&F(\tfrac{1}{2}+\lambda, -\tfrac{1}{2}-\nu, 1+\lambda+\mu; r)F(\tfrac{1}{2}-\lambda, \tfrac{1}{2}+\nu, 1+\mu+\nu; 1-r)\\
&+F(\tfrac{1}{2}+\lambda, \tfrac{1}{2}-\nu, 1+\lambda+\mu; r)F(-\tfrac{1}{2}-\lambda, \tfrac{1}{2}+\nu, 1+\mu+\nu; 1-r)\\
&-F(\tfrac{1}{2}+\lambda, \tfrac{1}{2}-\nu, 1+\lambda+\mu; r)F(\tfrac{1}{2}-\lambda, \tfrac{1}{2}+\nu, 1+\mu+\nu; 1-r)\\
&=\frac{\Gamma(1+\lambda+\mu)\Gamma(1+\mu+\nu)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}.
\end{aligned}$$

the choice $\lambda = \mu = \nu = 0$ gives the Legendre relation.

Elliot's Identity from Intersections

Matsumoto **& P.M.** (w.i.p.)

- *Hypothesys*: **too close to RTPR to be accidental** **P.M.**

Elliot's Identity from Intersections

Matsumoto & **P.M.** (w.i.p.)

- *Hypothesys*: **too close to RTPR to be accidental** **P.M.**
- *Proof* Matsumoto

$$u(t) = t^{1/2+\lambda}(1-t)^{-1/2+\mu}(1-rt)^{1/2+\nu},$$

$$\varphi_1 = \frac{dt}{t}, \quad \varphi_2 = \frac{dt}{t(1-rt)} = \Big(\frac{1}{t} - \frac{1}{t-1/r}\Big)dt,$$

$$\psi_1 = \frac{dt}{1-t} = \frac{-dt}{t-1}, \quad \psi_2 = \frac{dt}{t(1-t)} = \Big(\frac{1}{t} - \frac{1}{t-1}\Big)dt.$$

$$\gamma = (0,1)\otimes u(t) \text{ and } \delta = (-\infty,0)\otimes 1/u(t)$$

- *Riemann Twisted Period Relation*

$${}^t\Pi_\omega\, {}^tH_c^{-1}\Pi_{-\omega} = H_h.$$

$$\Big(\int_0^1 u(t)\varphi_1, \int_0^1 u(t)\varphi_2\Big)\, {}^tH_c^{-1}\begin{pmatrix}\int_{-\infty}^0 \frac{1}{u(t)}\psi_1\\ \int_{-\infty}^0 \frac{1}{u(t)}\psi_2\end{pmatrix} = \frac{-1}{e^{2\pi\sqrt{-1}\lambda}+1}.$$

$$\Big(F(\frac{1}{2}+\lambda, -\frac{1}{2}-\nu, 1+\lambda+\mu; r), F(\frac{1}{2}+\lambda, \frac{1}{2}-\nu, 1+\lambda+\mu; r)\Big)\cdot\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\cdot\begin{pmatrix}F(\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r)\\ F(-\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r)\end{pmatrix}$$

$$= \frac{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}$$

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$$\Big(F(\frac{1}{2}+\lambda, -\frac{1}{2}-\nu, 1+\lambda+\mu; r), F(\frac{1}{2}+\lambda, \frac{1}{2}-\nu, 1+\lambda+\mu; r)\Big)\cdot\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} F(\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r) \\ F(-\frac{1}{2}-\lambda, \frac{1}{2}+\nu, 1+\mu+\nu; 1-r) \end{pmatrix}$$

$$= \frac{\Gamma(\lambda+\mu+1)\Gamma(\mu+\nu+1)}{\Gamma(\lambda+\mu+\nu+\frac{3}{2})\Gamma(\mu+\frac{1}{2})}$$

- Paradigmatic case for studying **quadratic relations for Feynman Integrals**

Broadhurst, Roberts (2018)
Lee, Pomeranski (2019)

Completeness Relations: factoring "1"

- the richness of factorization

$$i\,(\text{-}i) = 1$$

$$\sum_n |\,\psi_\mathrm{n}\,\rangle\,\langle\,\psi_\mathrm{n}| \;=\; \mathbb{1}$$

$$(p^2 - m^2) = (\not{p} - m)(\not{p} + m)$$

$$\varepsilon^{\mu\nu} = \varepsilon^{\mu}\varepsilon^{\nu}$$

$$\sum_{i,j=1}^{\nu} |e_j\rangle\,(\mathbf{C}^{-1})_{ij}\,\langle e_i| = \mathbb{I}_c$$

$$\sum_{i,j=1}^{\nu} |\mathcal{C}_j]\,(\mathbf{H}^{-1})_{ij}\,[\mathcal{C}_i| = \mathbb{I}_h$$

Summary

- **Twisted De Rahm (co)-Homology Theory and Aomoto-Gel'fand Hypergeometric F's**
  - Proper mathematical framework for Feynman integrals

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- **Twisted De Rahm (co)-Homology Theory and Aomoto-Gel'fand Hypergeometric F's**
  - Proper mathematical framework for Feynman integrals
- **A Fundamental Property Discovered**
  - The **algebra** of Feynman Integrals is controlled by **Intersection Numbers**
  - Feynman Integrals admit **a scalar product**

Summary

- **Twisted De Rahm (co)-Homology Theory and Aomoto-Gel'fand Hypergeometric F's**
  - Proper mathematical framework for Feynman integrals
- **A Fundamental Property Discovered**
  - The **algebra** of Feynman Integrals is controlled by **Intersection Numbers**
  - Feynman Integrals admit **a scalar product**
- **"The number of master integrals" from being the *question* to being the *answer***

$\nu =$ number of independent *master* integrals — Chetyrkin, Tkachov (1981); Remiddi, Laporta (1996); Laporta (2000)

$=$ is finite — Smirnov, Petuckhov (2010)

$=$ number of critical points of graph polynomials — Lee, Pomeranski (2013)

$=$ is related to Euler characteritics χ_E — Aluffi, Marcolli (2008) — Bitoun, Bogner, Klausen, Panzer (2018)

$=$ number of independent integration contours — Bosma, Sogaard, Zhang (2017) — Primo, Tancredi (2017)

$=$ number of independent forms — Mizera **& P.M.** (2018)

$= \dim H^m_{\pm\omega}$ — Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019); Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

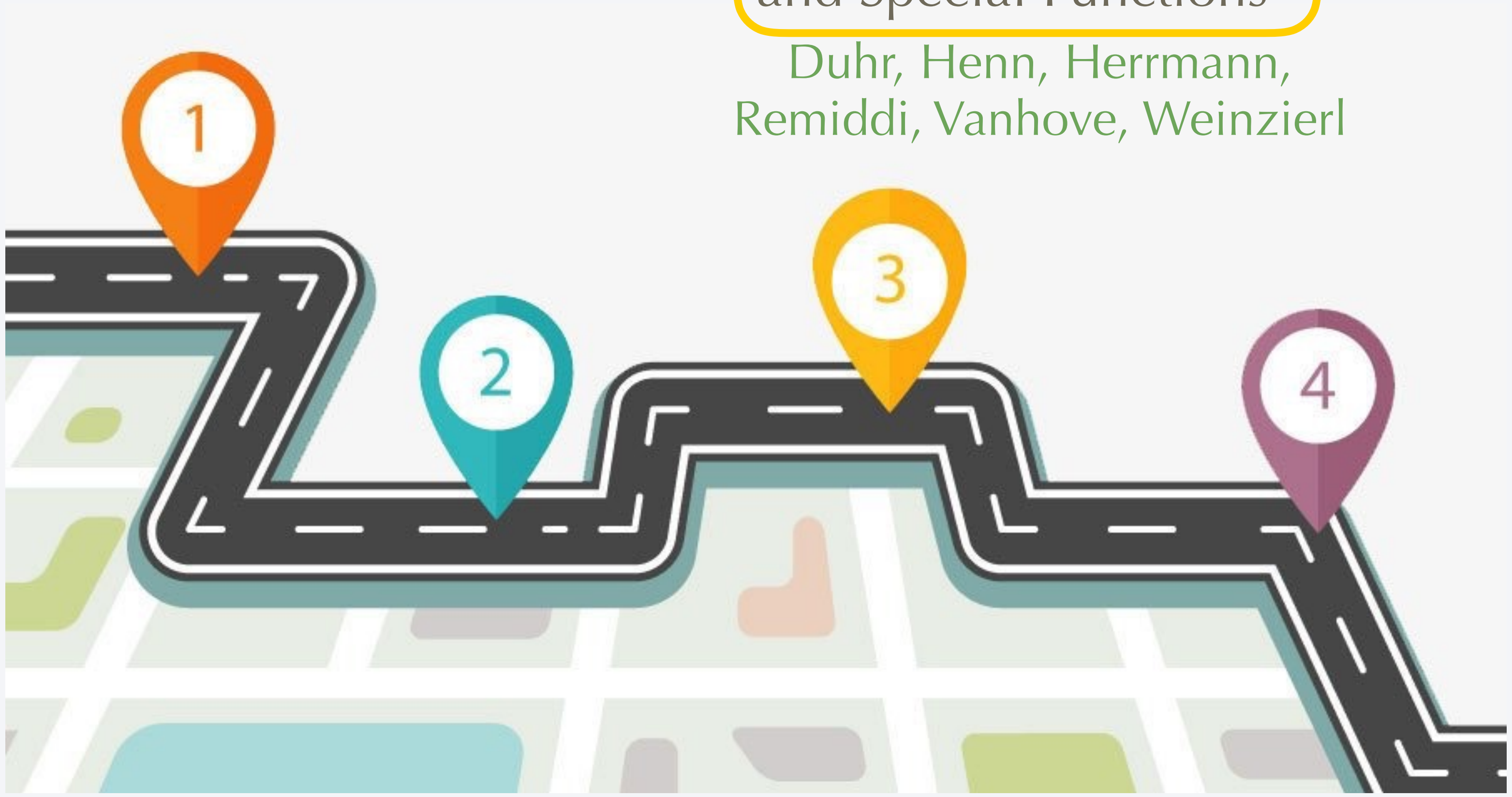
MatheAmplitudes 2019 :: the Roadmap

Mathematical Developments
and Geometric Aspects
of Intersection Theory

Aluffi, Aomoto, Brown,
Matsubara-Heo, Mimachi, Yoshida

Scattering Amplitudes,
Master Integrals,
Differential Equations
and Special Functions

Duhr, Henn, Herrmann,
Remiddi, Vanhove, Weinzierl



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Computational Algebraic Geometry
and Computer Algebra

Bohem, Zhang

Intersection Theory
and Feynman Calculus

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