A simple algorithm for finding canonical differential equations for Feynman integrals

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Feynman integrals are crucial in perturbative quantum field theory (QFT)

- Required to compute observables/predictions beyond the leading order in perturbation theory

  - Collider physics experiments: underlying scattering processes computed from integrals in momentum space

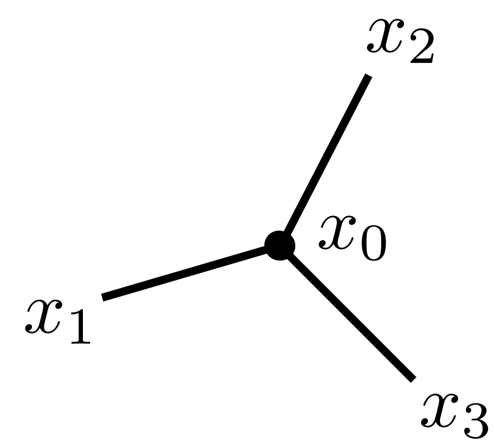
  - Position-space correlation functions (e.g. in conformal field theory), scaling dimensions of fields, renormalisation group coefficients

- Interesting connections to mathematics: periods, special functions, differential equations; algebraic geometry

Innovative field: many new methods valid in any QFT

Example: one-loop star/triangle integral

$$f(x_{12}^2, x_{23}^2, x_{13}^2) = \int \frac{d^4x_0}{\pi^2} \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2}$$



Definitions & notation: $x_i \in \mathbb{R}^4$ $\quad x_{ij}^2 = (x_i - x_j)^2 = |x_i - x_j|^2$

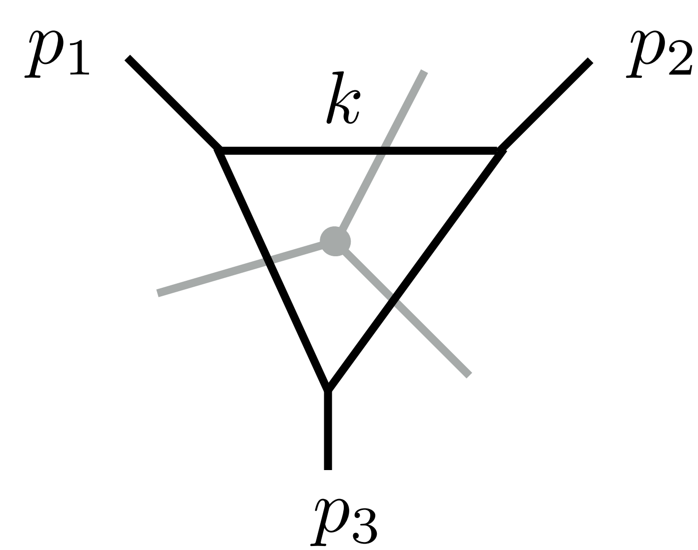
$$f(x_{12}^2, x_{23}^2, x_{13}^2) = \frac{1}{\sqrt{\Delta}} \times \text{ (Bloch-Wigner dilogarithm)}$$

Dual representation:

$$p_1 = x_2 - x_1 \qquad p_2 = x_3 - x_2$$

$$p_3 = x_1 - x_3 \qquad k = x_2 - x_0$$

$$p_1 + p_2 + p_3 = 0$$



$$f(p_1^2, p_2^2, p_3^2) = \int \frac{d^4k}{\pi^2} \frac{1}{k^2 (k - p_1)^2 (k + p_2)^2}$$

Questions about Feynman integrals

We are interested in multi-loop integrals defined from Feynman graphs

$$f(p_i \cdot p_j) = \int d^4k_1 d^4k_2 \dots d^4k_L \, \mathcal{I}(p_i; k_j)$$

Either Euclidean or Minkowski space

Sometimes necessary (and interesting!) to consider generalisation to non-integer dimensions $d^4k \longrightarrow d^{4-2\epsilon}k$

Typical questions:

- What special functions do Feynman integrals evaluate to?
- What singularities do they have?
- How can we determine the functions efficiently?

Special functions appearing in Feynman integrals

One loop: logarithms and dilogarithm sufficient

$$\log z = \int_1^z \frac{dt}{t} \qquad \mathrm{Li}_2(z) = -\int_0^z \frac{dt}{t} \log(1-t)$$

Natural generalization: 'Hyperlogarithms' cover large classes of multi-loop Feynman integrals

$$G_{a_1,\ldots a_n}(z) = \int_0^z \frac{dt}{t-a_1} G_{a_2,\ldots,a_n}(t)$$

Weight: number of indices = integrations

Starting from two loops, also new functions related to elliptic integrals

$$K(z) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-zt^2)}}$$

Multiple elliptic polylogarithms

Canonical differential equation method

Feynman integrals satisfy n-th order partial differential equations (DE)

Equivalently, system of 1st order DE

Typically complicated

Idea: (rational) loop integrand contains key information on special functions appearing after integration

Special functions defined from ‘canonical’ DE

Very simple

Canonical differential equations (for Feynman integrals evaluating to multiple polylogarithms)

$$\frac{d}{dx} f(x,\epsilon) = \epsilon \left[\sum_k m_k \frac{1}{x - x_k} \right] f(x,\epsilon)$$

Basis of uniform weight Feynman integrals

Constant matrices

Singular points

Iterative solution in terms of multiple polylogarithms

Uniform transcendental weight (UT)

We use dlog forms and leading singularities to choose the uniform weight basis

Example d-log integrand: [Arkani-Hamed, Bourjaily, Cachzo, Goncharov, Postnikov, Trnka, 2012] [Caron-Huot, talk at Trento, 2012] [Lipstein, Mason, 2013]

$$\frac{d^4\ell \;\; (p_1+p_2)^2(p_1+p_3)^2}{\ell^2(\ell+p_1)^2(\ell+p_1+p_2)^2(\ell-p_4)^2}$$
$$= d\log\left(\frac{\ell^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell+p_1+p_2)^2}{(\ell-\ell^*)^2}\right) d\log\left(\frac{(\ell-p_4)^2}{(\ell-\ell^*)^2}\right)$$

Closely related to constant leading singularities

- Conjecturally, integrate to uniform weight functions
  - ☑ Explored and checked in many cases
  - ☑ Guides basis choice for differential equations [JMH, 2013]
  - ☑ Direct link to differential equations: [Herrmann, Parra-Martinez, 2019]
- Other method based on Moser algorithm and improvements: [JMH, '14; Lee '14; Prausa '17; Meyer '17; Gituliar, Magerya '17]

Our research question:

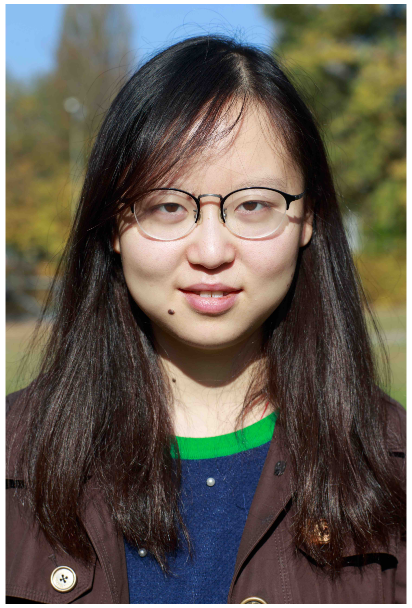
Some integral (family) is given. The differential equations in a random basis are complicated.

We have some idea about the canonical form of the differential equations

The leading singularity and Moser methods help find UT functions, but are not always easily applicable. Can we find the canonical DE with less information?

The collaboration at the Max Planck Institute for Physics

Christoph Dlapa
(PhD student)



Kai Yan
(Postdoc)

Based on work to appear soon on arXiv:...

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One integral to rule them all…

or

From the Picard-Fuchs equation of a uniform weight (UT) integral to a canonical system of differential equations (DE)

We develop further an idea by Höschele et al.

[Höschele, Hoff, Ueda '14]

- Idea: first-order DE are only canonical if all integrals are UT, but Picard-Fuchs eq. single integral is unique, and contains valuable information.
- They applied this knowledge to find the remaining UT integrals, for cases with 2 or 3 master integrals, and outlined a general procedure.

Here:

☑ We formulate the method in matrix form, and show how to solve the equations systematically

☑ We use `dlog` integrands to find first UT integral

☑ We apply the method to cutting edge examples (eg. sectors with 17 master integrals), and full systems of differential equations

From the Picard-Fuchs equation of a UT integral to a canonical system of DE

Assumption: know one UT integral f_1

Automated steps (computer algebra):

- Complete (in any way) to basis f_1
- Computer differential equations: $\frac{d}{dx}\vec{f} = A(x,\epsilon)\vec{f}\,.$
- Independent derivatives of f_1 :

$$(f_1', f_1'', \cdots f_1^{(n)})^T = \Psi(x,\epsilon)\,\vec{f}$$

Ψ^{-1} determines the Picard-Fuchs equation of f_1

Goal: Switch to new basis $\vec{g}$ with $g_1 = f_1$,
with all integrals UT.

From the Picard-Fuchs equation of a UT integral to a canonical system of DE

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- Independent derivatives of f_1 : $(f_1', f_1'', \cdots f_1^{(n)})^T = \Psi(x,\epsilon)\,\vec{f}$
- Ψ given by derivatives: $A^{[1]} \equiv A,\ A^{[n]} \equiv \frac{d}{dx}A^{[n-1]} + A^{[n-1]}A.$

$$\Psi = \left(A^{[1]}, A^{[2]}, \ldots, A^{[n]}\right)^T$$

Ψ^{-1} determines the Picard-Fuchs equation of f_1

Idea: use the infinite amount of information provided by f_1 being UT.

From the Picard-Fuchs equation of a UT integral to a canonical system of DE

Goal: Switch to new basis $\vec{g}$ with $g_1 = f_1$, with all integrals UT.

$$\frac{d}{dx}\vec{g} = \epsilon\,\tilde{A}(x)\,\vec{g}\,.$$

$$\vec{f} = T\,\vec{g}\,. \quad \text{where } T \equiv \Psi^{-1}\Phi\,.$$

$$(g_1', g_1'', \cdots g_1^{(n)})^T = \Phi(x,\epsilon)\,\vec{g}.$$

Deduce from assumption first line to T is $\vec{v}_0 = (1, 0, \cdots, 0)$

$\vec{v}_0 \Psi^{-1}\Phi = \vec{v}_0$ yields system of equations at each order in ϵ

Constraint on Φ from expected canonical form:

$$\tilde{A}(x) = \sum_i \frac{d\ln a_i(x)}{dx}\, m_i$$

$\frac{d\ln a_i(x)}{dx}$ → Follow from singularities of A(x)

m_i → constant matrices: to be found

Systematic solution of the equations

$$\vec{v}_0 \Psi^{-1} \Phi = \vec{v}_0$$

- $\vec{v}_0$: Unit vector
- Ψ: Known matrix x, ϵ
- Φ: Parametrized by set of unknown constant matrices m_i; Known polynomial ϵ dependence

Equations valid for any x : can use finite field methods.

Solve at each order in ϵ .

Eg., at first step, linear equations for $\vec{v}_{1,i} := \vec{v}_0 m_i$

at next step, linear equations for $\vec{v}_{2,i,j} := \vec{v}_{1,i} m_j$ and so on.

(Freedom corresponds to similarity transformations. Fix such that T invertible.)

Higher orders in ϵ provide consistency check.

Corollary of our method: test of UT property of a given integral

- Leading singularities / dog integrands provide a useful tool to find uniform weight integrals a priori (before calculation).

- However, there are some limitations to this:

  - Sometimes, a more careful D-dimensional analysis may be needed. [Chicherin, Gehrmann, JMH, Wasser Zhang, Zoia'18]

  - Practical issues in residue calculations

  - the UT conjecture is not (yet) proven

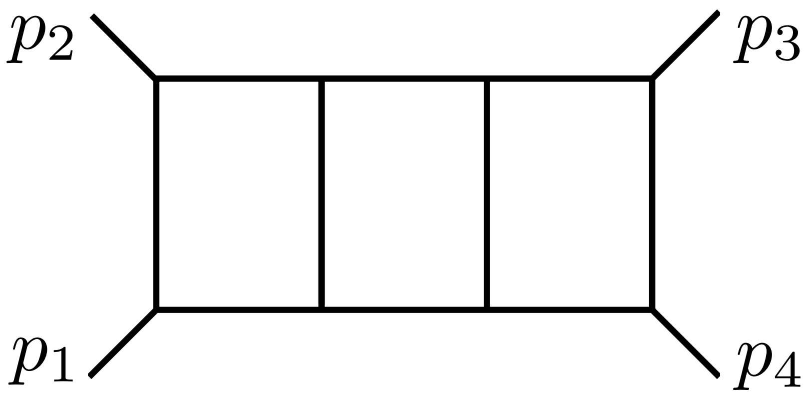
☑ Our method provides a test for the UT property, and can suggest modifications if an integral is 'almost' UT

State-of-the art applications

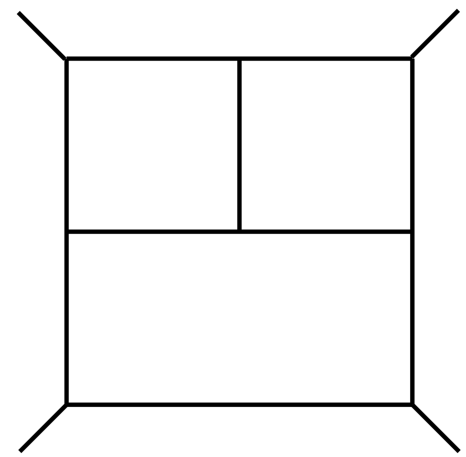
Application 1: Planar three-loop on-shell integrals

Four-particle scattering

[Henn, Smirnov^2, '14]



(a)



(b)

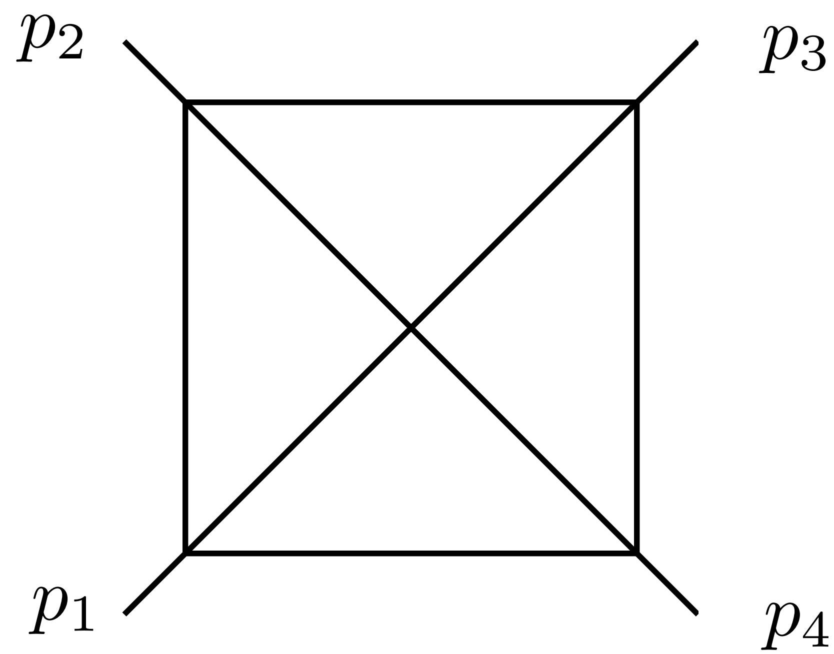
Kinematics: $\sum_{i=1}^{4} p_i = 0\,, \quad p_i^2 = 0\,. \qquad s = (p_1+p_2)^2\,, \quad t = (p_2+p_3)^2\,, \quad x = t/s\,.$

- UT integral in top sector easily found using d-log integrand analysis [Arkani-Hamed et al '11, JMH '13, Wasser, '16]
- Obtained full system of differential equations
  Matrix size 26x26 for case (a), 41x41 for case (b)

$$\tilde{A}(x) = m_0 \frac{d}{dx} \ln x + m_1 \frac{d}{dx} \ln(1+x)$$

Matrix block structure: at most 3 master integrals per sector.

Application 2: Four-loop integrals for four-particle scattering



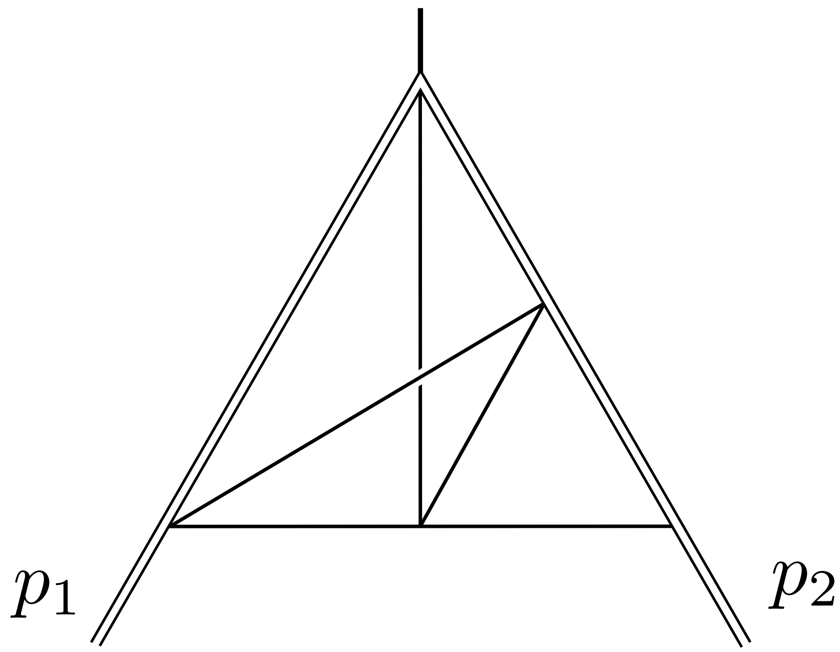
- Solved the system on the cut (8 MI)
- Found further UT integrals (off the cut) by testing a large list of candidates
- Full canonical system (19 MI) of DE obtained

Application 3: many coupled master integrals

- Four-loop non-planar heavy quark effective theory (HQET) integrals

Kinematics $\cos\phi = \frac{p_1 \cdot p_2}{\sqrt{p_1^2 p_2^2}} = x + \frac{1}{x}$

Correspond to Wilson line integrals in position space.



- State of the art: 3 loops [JMH, Korchemsky Marquard '15]
- 17 coupled master integrals
- Form of singularities:

$$\tilde{A}(x) = m_0 \frac{d}{dx} \ln x + m_1 \frac{d}{dx} \ln(1+x) + m_{-1} \frac{d}{dx} \ln(1-x)$$

Solved easily (~10 min) using our algorithm.

Discussion

- ☑ We developed further a method of Höschele et al. to find a system of canonical differential equations, given just one uniform weight integral.

‘One integral to rule them all’

- ☑ Being in matrix form, our equations are solved easily.

- ☑ As a byproduct, one obtains a test of the UT property of a given integral.

- ☑ Our method is efficient. We presented state-of-the art applications where other methods fail or require more assumptions.

Our work provides an automated tool for the calculation of canonical differential equations. It removes an important bottleneck in the calculation of Feynman integrals.

Outlook: more complicated integrals

- Unlike other methods, the inclusion of multiple scales does not pose a significant problem. Hence we expect applications to a wide class of Feynman integrals

$$d\, f(\mathbf{x}, \epsilon) = \epsilon \left[\sum_k m_k \, d \log(\alpha_k(\mathbf{x})) \right] f(\mathbf{x}, \epsilon)$$

- The idea of canonical form of differential equations has also been explicitly applied for elliptic polylogarithms. We find it conceivable that our new ideas can be applied here as well.

$$d\, g(\mathbf{x}, \epsilon) = [dA_0(\mathbf{x}) + \epsilon \, dA_1(\mathbf{x})] \, g(\mathbf{x}, \epsilon)$$

[Henn '14; Mizera, Pokraka '19]

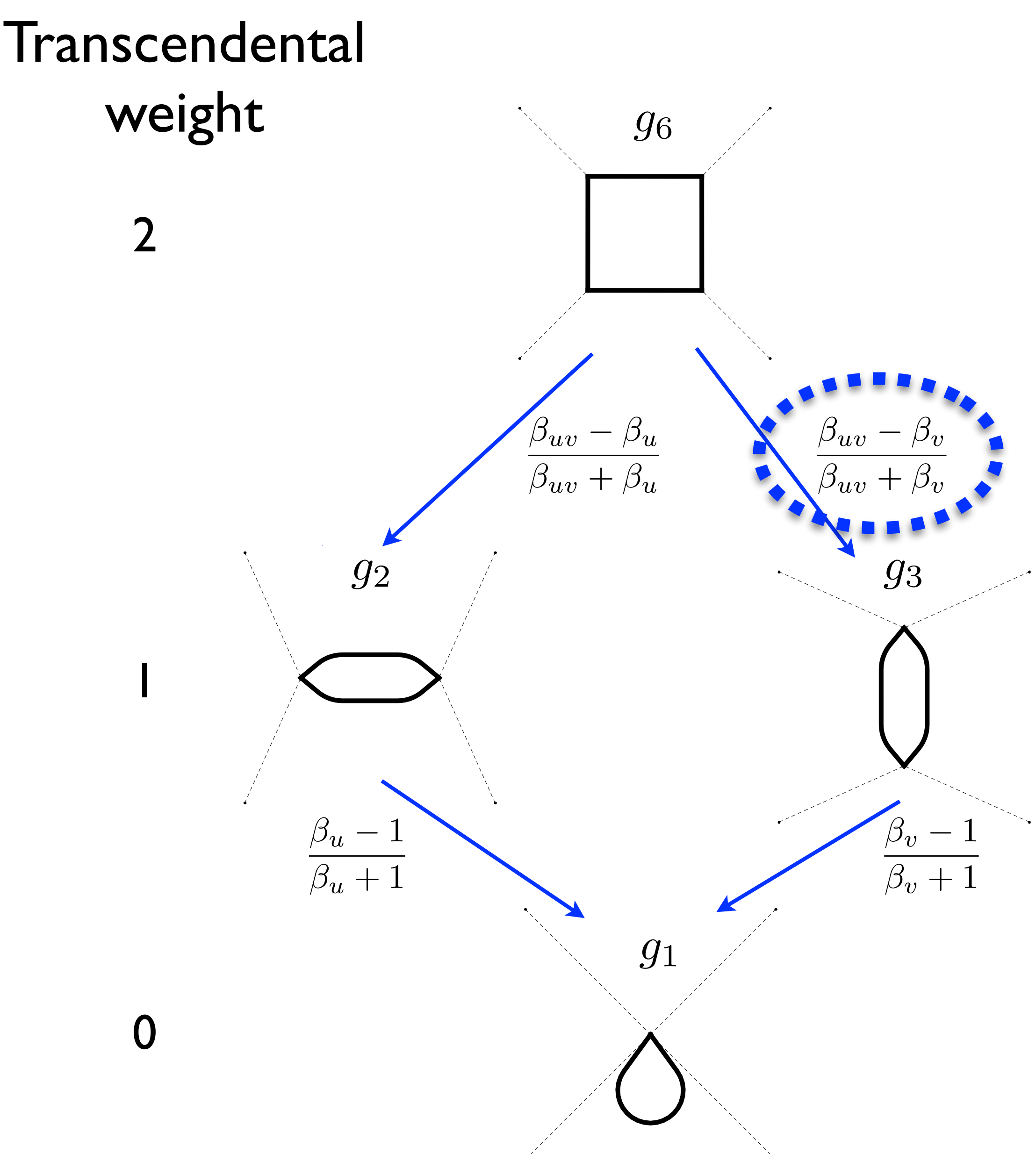
Integrating out A_0 introduces elliptic functions:

$$d\, f(\mathbf{x}, \epsilon) = \epsilon \, dA(\mathbf{x}) f(\mathbf{x}, \epsilon)$$

Beyond logarithmic kernels;[Duhr et al.; Adams and Weinzierl '18]

Outlook: finite integrals (e.g. D=4)

- In the case of finite integrals, the matrices become <u>nilpotent</u>. This leads to further simplifications, as shown in [Caron-Huot, JMH `14]. It would be interesting to apply our ideas to this case.



Arrows represent non-zero matrix entries

$$d\begin{pmatrix} g_6 \\ g_3 \\ g_2 \\ g_1 \end{pmatrix} = d\begin{pmatrix} 0 & \log\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v} & \log\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u} & 0 \\ 0 & 0 & 0 & \log\frac{\beta_u-1}{\beta_u+1} \\ 0 & 0 & 0 & \log\frac{\beta_v-1}{\beta_v+1} \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_6 \\ g_3 \\ g_2 \\ g_1 \end{pmatrix}$$

Outlook: recurrence relations

- Differential equations are closely related to recurrence relations, for example in the dimension. We find it likely that the UT information provides important input for that method

[Tarasov et al.; Lee et al; Schneider et al.]

Thank you for your attention!