Product of Hessians and Discriminant of Critical Points of Level Function for Hypergeometric Integrals

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1 Introductory explanation (Divergent integral and twisted cycle)

The function x_+^λ on $\mathbf{R}$ for $\mathfrak{Re}\lambda > -1$ is an ordinary function but for $\lambda \in \mathbf{C} - \mathbf{Z}, \lambda \leq -1$ is a generalized function defined as follows :

Suppose $f(x)$ is an arbitrary holomorphic function near the origin. Fix a point $a > 0$ near the origin. Consider the integral

$$\begin{aligned} \langle x_+^\lambda, f\rangle &= \int_0^a x^\lambda f(x)dx \\ &= \lim_{\varepsilon\downarrow 0}\int_\varepsilon^a x^\lambda f(x)dx. \end{aligned} \tag{1}$$

Case (i) Suppose first $-n-1 < \mathfrak{Re}\lambda < -n\,(n = 1, 2, 3, \ldots)$. Then (1) is divergent. $f(x)$ has a Taylor expansion at the origin

$$f(x) = \sum_{m=0}^{n-1}\frac{f^m(0)}{m!}x^m + x^n g(x)$$

where $g(x)$ is holomorphic on $[0, a]$. The finite part of (1) in the sense of J.Hadamard is given as follows :

$$\begin{aligned} J(\lambda) &= \text{f.p.} \int_0^a x^\lambda f(x)dx \\ &= \sum_{m=0}^{n-1} \frac{f^{(m)}(0)}{m!} \frac{a^{\lambda+m+1}}{\lambda+m+1} + \int_0^a x^{\lambda+n} g(x)dx. \end{aligned} \tag{2}$$

This is the generalized function x_+^λ which has been defined by I.M.Gelfand and G.E.Shilov in the mid 20th century (see [5]), i.e.,

$$\langle x_+^\lambda, f \rangle = \text{f.p.} \int_0^a x^\lambda f(x)dx$$

In a neighborhood of the origin we take a path σ_0 starting from and ending in a going around the origin counter-clockwise ("loop based on the point a going around the origin ")

$$\frac{1}{e^{2\pi i\lambda}-1}\sigma_0 = [\varepsilon, a] + \frac{1}{e^{2\pi i\lambda}-1}\delta_\varepsilon \quad (\varepsilon > 0)$$

where δ_ε is a scalar multiple of a loop with base point ε in a neighborhood of 0.

Then the integral

$$\frac{1}{e^{2\pi i\lambda}-1} \int_{\sigma_0} x^\lambda f(x)dx$$

equals (2). This is called "detoured cycle at the origin"). (This idea already can be found in the work of J.Leray in the middle of 20th century).

Case (ii) When $\lambda = -n \, (n = 1, 2, 3, \ldots)$ the finite part is defined as

$$\text{f.p.} \int_0^a x^{-n} f(x)dx = \sum_{m=0}^{n-2} \frac{f^{(m)}(0)}{m!} \frac{a^{-n+m+1}}{-n+m+1} + \frac{f^{(n-1)}(0)}{n!} \log a + \int_0^a g(x)dx. \tag{3}$$

The generalized function x_+^{-n} is then defined by the finite part

$$\langle x_+^{\lambda}, f\rangle = \text{f.p.} \int_0^a x^{-n} f(x)dx.$$

$J(\lambda)$ has Laurent expansion at $\lambda = -n$

$$J(\lambda) = \frac{c_{-1}}{\lambda + n} + c_0 + c_1(\lambda + n) + \cdots$$

Then the finite part coincides with c_0 , i.e.,

$$\text{f.p.} \int_0^a x^{-n} f(x)dx = c_0 = \lim_{\lambda \to -n} \frac{d}{d\lambda}(\lambda + n)J(\lambda)$$
$$= \frac{1}{2\pi i} \int_{\sigma_0} x^{-n}(\log x - \pi i) f(x)dx.$$

Example 1

(i)f.p. $\displaystyle\int_a^{\infty} (x - a)^{\lambda} dx = 0 \quad (\text{for all}\, \lambda \in \mathbf{R}).$

(ii) f.p. $\displaystyle\int_a^b \frac{f(x)}{x} dx = \text{ p.v.} \int_a^b \frac{f(x)}{x} dx = \int_a^b \frac{f(x) - f(0)}{x} dx + f(0) \log \frac{b}{-a}\ (a < 0 < b).$

(p.v. denotes the principal value)

(iii) f.p. $\displaystyle\int_0^{\infty} \frac{e^{-x}}{x} dx = \int_0^{\infty} (\frac{e^{-x}}{x} - \frac{x}{e^x - 1})dx = \Gamma'(1) = -C,$

C denotes Euler Constant.

Example 2 Beta function

For $\alpha, \beta \notin \mathbf{Z}$

$$J(\alpha, \beta) = \text{f.p.} \int_0^1 x^{\alpha}(1 - x)^{\beta} dx \tag{4}$$

which is equal to Beta function $B(\alpha, \beta)$. Take σ_0, σ_1 the loops with the base point $x = \frac{1}{2}$ going around $0, 1$ in a positive direction respectively. Then

$$J(\alpha, \beta) = \frac{1}{e^{2\pi i\alpha} - 1} \int_{\sigma_0} x^{\alpha}(1 - x)^{\beta} dx - \frac{1}{e^{2\pi i\beta} - 1} \int_{\sigma_1} x^{\alpha}(1 - x)^{\beta} dx.$$

The monodromy $\mathcal{M}$ associated with the function $\Phi(x) = x^{\alpha}(1-x)^{\beta}$

$$\sigma_0 \longrightarrow M(\sigma_0) = e^{2\pi i\alpha} \in \mathbf{C}^*, \sigma_1 \longrightarrow M(\sigma_1) = e^{2\pi i\beta} \in \mathbf{C}^*$$

defines the local system $\mathcal{L}$ and its dual $\mathcal{L}^*$ on the space $X = \mathbf{C} - \{0, 1\}$. The boundary operator ∂ acts on the linear space of chains $\mathbf{c} = c_0\sigma_0 + c_1\sigma_1$ ($c_0, c_1 \in \mathbf{C}$) with values in $\mathcal{L}^*$ as follows :

$$\partial(c_0\sigma_0 + c_1\sigma_1) = \big(c_0(e^{2\pi i\alpha} - 1) + c_1(e^{2\pi i\alpha} - 1)\big)\{\frac{1}{2}\}.$$

It is closed (twisted cycle) if and only if

$$c_0(e^{2\pi i\alpha} - 1) + c_1(e^{2\pi i\alpha} - 1) = 0$$

Hence the one dimensional homology $H_1(X, \mathcal{L}^*)$ is just one dimenisional with the basis $\mathbf{c} = \frac{1}{e^{2\pi i\alpha}-1}\sigma_0 - \frac{1}{e^{2\pi i\beta}-1}\sigma_1$.

We have

$$J(\alpha, \beta) = \langle \mathbf{c}, dx \rangle. \tag{5}$$

On the other hand if $\alpha = -n-1$ ($n = 0, 1, 2, 3, \ldots$) then

$$J(-n-1, \beta) = \text{f.p.} \int_0^1 x^{-n-1}(1-x)^{\beta} dx \quad (\beta > -1)$$
$$= \frac{1}{2\pi i} \int_{\sigma_0} (1-x)^{-n-1}(1-x)^{\beta}(\log x - \pi i)dx - \frac{1}{(e^{2\pi i\beta} - 1)} \int_{\sigma_1} x^{-n-1}(1-x)^{\beta} dx. \tag{6}$$

The vector function of two components ${}^T\big((1-x)^{\beta}, (1-x)^{\beta}\log x\big)$ (T denotes the transposition) defines the monodromy and the associated local system $\mathcal{L}$ of rank two and its dual $\mathcal{L}^*$. The fundamental 2×2 matrix function Φ is defined by the lower triangular matrix

$$\Phi(x) = \begin{pmatrix} (1-x)^{\beta} & \\ (1-x)^{\beta} \log x & (1-x)^{\beta} \end{pmatrix}$$

$$\mathcal{M} \longrightarrow M(\sigma_0) = \begin{pmatrix} 1 & \\ 2\pi i & 1 \end{pmatrix},\ M(\sigma_1) = \begin{pmatrix} e^{2\pi i\alpha} & \\ & e^{2\pi i\beta} \end{pmatrix}$$

The space of chains with coefficients in $\mathcal{L}^*$ is the linear space consisting of two components

$$\mathfrak{c} = (c_{11}, c_{12})\,\sigma_0 + (c_{21}, c_{22})\sigma_1\,(c_{jk} \in \mathbf{C}).$$

The pairing of integral between the chain $\mathfrak{c}$ and two component vector function ${}^T(\varphi_1(x), \varphi_2(x))$ is given by

$$\langle \mathfrak{c}, {}^T(\varphi_1, \varphi_2)\rangle = \int_{\sigma_0} (c_{11}, c_{12})\Phi(x)\,{}^T(\varphi_1, \varphi_2)\,dx + \int_{\sigma_1} (c_{21}, c_{22})\Phi(x)\,{}^T(\varphi_1, \varphi_2)\,dx.$$

The boundary operator is given by

$$\partial(\mathfrak{c}) = \big\{(c_{11}, c_{12})(M(\sigma_0) - I) + (c_{21}, c_{22})(M(\sigma_1) - I)\big\}\big\{\frac{1}{2}\big\}$$

$\mathfrak{c}$ is closed if and only if

$$2\pi i c_{12} + (e^{2\pi i\beta} - 1)c_{21} = 0,\ (e^{2\pi i\beta} - 1)c_{22} = 0,$$

i.e.,

$$c_{22} = 0, c_{21} = -\frac{2\pi i}{e^{2\pi i\beta} - 1}c_{12}.$$

Hence we have two linearly independent twisted cycles

$$\mathfrak{c}_1 = (1, 0)\sigma_0, \quad \mathfrak{c}_2 = (0, \frac{1}{2\pi i})\sigma_0 + (-\frac{1}{e^{2\pi i\beta} - 1}, 0)\sigma_1.$$

The integral (5) is nothing else than the pairing $\langle \mathfrak{c}_2, {}^T(x^{-n-1}dx, -\pi i x^{-n-1}dx)\rangle$, namely

$$J(-n-1, \beta) = \langle \mathfrak{c}_2, {}^T(x^{-n-1}dx, -\pi i x^{-n-1}dx)\rangle. \tag{7}$$

Let $\mathcal{L}_{lf}$ be the same local system on X which is locally finite at the singularity $0, 1, \infty$ and $\mathcal{L}_{lf}^*$ be its dual. There is a canonical morphism "reg" often called "regularization" or "renormalization"

$$\begin{array}{rcccc} \mathrm{reg} & : & H_1(X, \mathcal{L}_{lf}^*) & \rightarrow & H_1(X, \mathcal{L}^*) \\ & & & & \updownarrow \\ & & & & H_1(X, \mathcal{L}_{lf}) \end{array}$$

such that $\mathrm{reg}[0,1] = \mathfrak{c}$ in (5) and $\mathrm{reg}[0,1] = \mathfrak{c}_2$ in (6).

To evaluate this morphism in an explicitly way the intersection theory between twisted cycles play an important role (refer to [11] and also K.Mimachi's talk .)

2 asymptotics for large exponents

Let us begin from a simplest example.

Example 3 For different $a_j \in \mathbf{C}\,(1 \le j \le m)$ and $\lambda = \sum_{j=1}^m \lambda_j \varepsilon_j \in \mathbf{R}^m$ ($\{\varepsilon_j\}_{1 \le j \le m}$ means the standard basis of $\mathbf{R}^m$) we take

$$\Phi(w) = \prod_{j=1}^m (w - a_j)^{\lambda_j}$$

and the integral over a twisted cycle $\mathfrak{z}$ in the space $X = \mathbf{C} - \bigcup_{j=1}^m \{a_j\}$

$$J_\lambda(\varphi) = \int_{\mathfrak{z}} \Phi(w) \varphi(w) dw.$$

where $\varphi(w)dw$ is a rational differential one-form which is holomorphic on X. Denote by $H^1_\nabla(X, \Omega^\cdot)$ the one dimensional twisted de cohomology with respect to the covariant derivation

$$\nabla : \psi \longrightarrow \nabla\psi = d\psi + \sum_{j=1}^m \lambda_j d \log(w - a_j) \wedge \psi \tag{8}$$

for $\psi \in \Omega^0$ (scalar valued)(see [1]).

Denote the logarithmic one forms $\varphi_j(w)dw = d\log(w-a_j)\,(1 \le j \le m)$. One can take $\varphi_j(w)dw\,(1 \le j \le m-1)$ as the representative of the basis of $H^1_\nabla(X, \Omega^{\cdot})$ (Orlik-Solomon basis)[6].

The shift operator T_{ε_j} associated with the shift : $\lambda \to \lambda + \varepsilon_j$ acts on $H^1_\nabla(X, \Omega^{\cdot})$:

$$T_{\varepsilon_j}(\varphi_k\, dw) \sim \sum_{l=1}^{m-1} \varphi_l dw a_{j;lk}(\lambda), \text{ (homologically).}$$

The $(m-1)\times(m-1)$matrices $A_j(\lambda) = (a_{j;lk}(\lambda))$ are rational functions of λ which have the asymptotic expansions

$$A_j(\lambda) = A_j^0 + O(\frac{1}{N}) \quad (\lambda = N\boldsymbol{\nu} + \lambda')$$

where $A_j^{(0)}$ commute with each other under the genericity condition $\mathcal{C}$:

$$(\mathcal{C}) : a_j \ne a_k \;\; (j \ne k).$$

Put $\lambda = N\boldsymbol{\nu} + \lambda'$ with $\boldsymbol{\nu} = \sum_{j=1}^m \nu_j \varepsilon_j \in \mathbf{Z}^m - \{0\}$, where $\lambda' = \sum_{j=1}^m \lambda'_j \varepsilon_j$ is fixed.

We are interested in the asymptotic behavior of $J_\lambda(\varphi)$ when $N \in \mathbf{Z}_{>0}$ tends to the infinity in the direction $\boldsymbol{\nu}$.

Take

$$F = \sum_{j=1}^m \nu_j \,\log(w-a_j)$$

For the real valued level function $\mathfrak{Re}(F)$ the associated critical points $\zeta_j \in \mathbf{C}\,(1 \le j \le m-1)$ satisfy the equality

$$\frac{dF}{dw} = \sum_{j=1}^m \frac{\nu_j}{w-a_j} = 0. \tag{9}$$

Generally there are $m-1$ different critical points ζ_j. To each point ζ_j there exists the one dimensional stable cycle $\mathfrak{z}_j$ which is Lagrangian. This is locally described at ζ_j by

$$\mathfrak{Im}F(w) = \mathfrak{Im}(\zeta_j).$$

There also exists the one dimensional unstable cycle $\mathfrak{z}_j^-$ at ζ_j. Each of the systems $\mathfrak{z}_j\,(1 \le j \le m-1)$ and $\mathfrak{z}_j^-\,(1 \le j \le m-1)$ makes a basis of $H_1(X, \mathcal{L}^*)$. They give the asymtotics of integral in the direction $\boldsymbol{\nu}$ and and $-\boldsymbol{\nu}$ respectively.

Now for simplicity we consider the case $m = 3$ where $\boldsymbol{\nu} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ i.e., $\nu_1 = \nu_2 = \nu_3 = 1$.

$$A_1(\lambda) = \begin{pmatrix} \frac{\lambda_1}{1+\lambda_\infty}(a_3 - a_1) & \frac{\lambda_1}{1+\lambda_\infty}(a_3 - a_1) \\ \frac{\lambda_2}{1+\lambda_\infty}(a_3 - a_2) & \frac{\lambda_2}{1+\lambda_\infty}(a_3 - a_2) + (a_2 - a_1) \end{pmatrix}$$

$$A_1^{(0)} = \begin{pmatrix} \frac{a_3 - a_1}{3} & \frac{a_3 - a_1}{3} \\ \frac{a_3 - a_2}{3} & \frac{a_3 + 2a_2 - 3a_1}{3} \end{pmatrix}$$

where $\lambda_\infty = \lambda_1 + \lambda_2 + \lambda_3$.

The multiplication by the variable $w : T_w = A_1 + a_1 I$ corresponds to the matrix

$$\begin{aligned} A_w^{(0)} &= A_1^{(0)} + a_1 I \\ &= \begin{pmatrix} \frac{a_3 + 2a_1}{3} & \frac{a_3 - a_1}{3} \\ \frac{a_3 - a_2}{3} & \frac{a_3 + 2a_2}{3} \end{pmatrix} \end{aligned}$$

This has the eigenvalues ζ_1, ζ_2.

One can easily show that ζ_1, ζ_2 both lie in the inside of the triangle with vertices a_1, a_2, a_3.

The discriminant of (7) is given by the determinant of Hankel matrix $\mathcal{H}_1$ of $A_w^{(0)}$:

$$\mathcal{H}_1 = \begin{pmatrix} Tr(I) & Tr(A_w^{(0)}) \\ Tr(A_w^{(0)}) & Tr(\{A_w^{(0)}\}^2) \end{pmatrix}$$

and

$$\begin{aligned} \det\mathcal{H}_1 &= (\zeta_1 - \zeta_2)^2 \\ &= a_1^2 + a_2^2 + a_3^2 - a_1 a_2 - a_1 a_3 - a_2 a_3, \end{aligned}$$

Under the condition $(\mathcal{C})$ one can obtain the product formula

$$\Big[\frac{d^2F}{dw^2}\Big]_{w=\zeta_1}\cdot\Big[\frac{d^2F}{dw^2}\Big]_{w=\zeta_2}=\frac{1}{3}\frac{(\zeta_1-\zeta_2)^2}{(a_1-a_2)^2(a_1-a_3)^2(a_2-a_3)^2}. \tag{10}$$

The two critical points meet each other if and only if $\prod_{j=1}^{2}\big[\frac{d^2F}{dw^2}\big]_{w=\zeta_j}$ *vanishes.* This occurs if and only if a_1, a_2, a_3 are the vertices of a regular triangle and $\zeta_1=\zeta_2$ is the center of gravity.

3 Method and Main results

For large exponents the behavior of critical points of a level function gives an influence for asymptotics of corresponding hypergeometric integral. In this talk I want to show in an explicit way how the product of Hessians of the level function at all critical points is involved in the behavior of its critical points.

Let $f_j=f_j(x)\,(1\le j\le m)$ be real polynomials in $x=(x_1,\dots,x_n)$ in the affine space $\mathbf{C}^n$. Let X be the affine manifold which is the complement of the union of the hypersurfaces $S_j: f_j=0$

$$X=\mathbf{C}^n-\bigcup_{j=1}^{m}S_j.$$

The hypergeometric integral with respect to the multiplicative function

$$\Phi(x)=\prod_{j=1}^{m}f_j^{\lambda_j}$$

with exponents $\lambda=\sum_{j=1}^{m}\lambda_j\varepsilon_j\in\mathbf{R}^m(\varepsilon_j$ denotes the standard basis of $\mathbf{R}^m)$ is defined by

$$J(\varphi)=\int\Phi(x)\varphi(x)dx_1\wedge\cdots\wedge dx_n \quad (\varphi\in\Omega^{\cdot}).$$

$H^n_{\nabla}(X,\Omega^{\cdot})$ denotes the n dimensional twisted cohomology on X with respect to the covariant differentiation :

$$\nabla\varphi = d\varphi + \sum_{j=1}^{m} \lambda_j d\log f_j \wedge \varphi.$$

Its dual is isomorphic to the n dimensional twisted homology $H_n(X, \mathcal{L}^*)$ where $\mathcal{L}^*$ denotes the dual local system associated with the function Φ. The perfect pairing between them can be described by the above integral.

Let $\lambda' \in \mathbf{R}^m$ and $\boldsymbol{\nu} = \sum_{j=1}^{m} \nu_j e_j \in \mathbf{Z}^m - \{0\}$ be fixed. Put $\lambda = N\boldsymbol{\nu} + \lambda'$ for a positive integer N. Denote $|\boldsymbol{\nu}| = \sum_{j=1}^{m} |\nu_j|$. We consider the asymptotic behavior of the integral $J(\varphi)$ for a large N. One can define the real valued level function $\mathfrak{Re}F$ from the logarithm

$$F(x) = \sum_{j=1}^{m} \nu_j \log f_j.$$

The singularity of the gradient flow of $\mathbf{v} = \mathrm{grad}\mathfrak{Re}F$ in X coincides with its critical points $\mathbf{c}_k$ of F satisfying the equation :

$$0 = dF = \sum_{j=1}^{m} \nu_j d\log f_j. \tag{11}$$

A system of linearly independent representatives of $H_n(X, \mathcal{L}^*)$ is obtained by stable cycles $\mathfrak{z}_k$ $(1 \leq k \leq \kappa)$ which are Lagrangian.

Suppose the critical point $\mathbf{c}_k$ is non-degenerate. Then there exists a system of local coordinates $\xi = (\xi_1, \ldots, \xi_n)$ such that the origin corresponds to $\mathbf{c}_k$ and ξ is real on the stable cycle $\mathfrak{z}_k$ (see [1] Theorem 4.6).

The Hessian of F at $\mathbf{c}_k$ is defined by

$$[Hess(F)]_{\mathbf{c}_k} = \Big[\frac{\det\big(\frac{\partial^2 F}{\partial\xi_j\partial\xi_k}\big)_{1\leq j,k\leq n}}{\det^2\big(\frac{\partial x_j}{\partial\xi_k}\big)_{1\leq j,k\leq n}}\Big]_{\xi=0}. \tag{12}$$

If φ does not depend on λ we have by saddle point method

$$\int_{\mathfrak{z}_k} \Phi\varphi \approx \Phi(\mathbf{c}_k)\varphi(\mathbf{c}_k)\frac{(2\pi)^{\frac{n}{2}}}{\sqrt{N^n[(-1)^n\, Hess(F)]_{\mathbf{c}_k}}}$$

Under a suitable "non-resonance " condition, κ equals the dimension of the twisted cohomology $H^n_\nabla(X, \Omega^\cdot)$.

Denote by $\varphi_j dx_1 \wedge \cdots \wedge dx_n$ $(1 \leq j \leq \kappa)$ the representative of a basis of $H^n_\nabla(X, \Omega^\cdot)$. The Wronskian W is defined by the determinant $\det Y$ of the fundamental $\kappa \times \kappa$ matrix $Y = (\langle \varphi_j, \mathfrak{z}_k \rangle_{j,k})$.

We have the asymptotic expansion for large N

$$
\begin{aligned}
W \approx & \prod_{k=1}^{\kappa} \{\exp[NF(\mathbf{c}_k)] \prod_{j=1}^{m} f_j^{\lambda'_j}(\mathbf{c}_k) \varphi_j(\mathbf{c}_k)\} \\
& \cdot N^{-\frac{n\kappa}{2}} (2\pi)^{\frac{n\kappa}{2}} (w_0 + \frac{w_1}{N} + \frac{w_2}{N^2} + \cdots)
\end{aligned}
$$

where

$$
w_0 = \prod_{k=1}^{\kappa} \frac{1}{\sqrt{((-1)^n \, HessF)_{\mathbf{c}_k}}}.
$$

We can now pose several questions as follows.

Quest 1 Evaluate $\prod_{k=1}^{\kappa} f_j(\mathbf{c}_k)$.

Quest 2 Evaluate $\prod_{k=1}^{\kappa} (Hess(F))_{\mathbf{c}_k}$.

Quest 3 When $\prod_{k=1}^{\kappa} (Hess(F))_{\mathbf{c}_k}$ vanishes ?

Quest 4 Under which condition all the critical points are real ?

There is an interesting analogy between f_j and the quantity $(\mathrm{Hess}(F))_{\mathbf{c}_k}$ on the one hand and the notion of "norm", "unit" and "differente" in algebraic number theory on the other. In the moduli space for the polynomials $\{f_k\}_{1 \leq k \leq m}$, f_j^{-1} is also regular in X because $f_j(\mathbf{c}_k)$ never vanishes. In this sense f_j is regarded as "unit". However $\mathrm{Hess}(F)$ may vanish sometimes at $\mathbf{c}_k$.

In the sequel for a rational function φ on X the product $\prod_{1 \leq j \leq \kappa} [\varphi]_{\mathbf{c}_j}$ will be called "norm" of φ and be denoted by $\mathcal{N}(\varphi)$. φ is called a unit if and only if $\mathcal{N}(\varphi)$ never vanishes anywhere.

One may conjecture the following :

Ansatz :

$\prod_{k=1}^{\kappa}(\mathrm{Hess}(F))_{\mathbf{c}_k} = \mathcal{N}(\mathrm{Hess}F)$ *is expressed as*

$$\mathcal{N}(\mathrm{Hess}F) = (\mathrm{unit}) \cdot \mathrm{Discr}.$$

It vanishes if and only if a pair of the critical points $\mathbf{c}_k$ *coincides with each other.*

$\prod_{k=1}^{\kappa}(\mathrm{Hess}(F))_{\mathbf{c}_k}$ may play the similar role of "discriminants" as in algebraic number theory.

We shall give a few examples of hyperplane arrangement and circle arrangement illustrating the above facts.

4 hyperplane arrangements

Let f_j $(1 \le j \le n+2)$ be the following linear functions with real coefficients :

$$f_j := x_j \ (1 \le j \le n),$$
$$f_{n+1} := 1 - \sum_{k=1}^{n} x_k, \ f_{n+2} := 1 - \sum_{k=1}^{n} u_k x_k$$

for the parameter $u = (u_1, \ldots, u_n) \in \mathbf{R}^n$ under the condition $(\mathcal{C}_1)$:

$$(\mathcal{C}_1) : u_j \neq u_k \, \{j \neq k\}, \ u_j \notin \{0, 1\}$$

This gives the moduli space of the arrangement of $n+2$ real hyperplanes in general position.

Under $(\mathcal{C}_1)$ it is known that for generic λ such that all $\lambda_j > 0$ one has $\kappa = n+1$, and that one can choose as the representative of a basis of $H_n(X, \mathcal{L}^*)$ the regularization of the compact chambers of the associated real hyperplane arrangements corresponding to the components of the complement of $\bigcup_{j=1}^{m} S_j$ (refer to [1],[9]) :

$$\mathfrak{Re}X = \mathbf{R}^n \cap X.$$

Suppose now that all $\nu_j\,(1 \leq j \leq n+2)$ and $\nu_\infty = \sum_{k=1}^{n+2} \nu_k$ are different from 0 :

$$\nu_\infty \prod_{j=1}^{n+2} \nu_j \neq 0.$$

(11) is equivalent to the system of equations

$$0 = G_j := \frac{\nu_j}{x_j} - \frac{\nu_{n+1}}{f_{n+1}} - \frac{\nu_{n+2} u_j}{f_{n+2}} \quad (1 \leq j \leq n). \tag{13}$$

This system generally gives $n+1$ solutions, namely $n+1$ critical points (real or complex) of $\mathfrak{Re} F$ which we denote by $\mathbf{c}_j\,(1 \leq j \leq n+1)$. It follows from (13)

$$x_j = \nu_j \frac{f_{n+1} f_{n+2}}{\nu_{n+1} f_{n+2} + \nu_{n+2} u_j f_{n+1}} - f_j f_{n+1} f_{n+2} G_j,\ (1 \leq j \leq n) \tag{14}$$

$$1 - f_{n+1} = \sum_{k=1}^{n} \nu_k \frac{f_{n+1} f_{n+2}}{\nu_{n+1} f_{n+2} + \nu_{n+2} u_j f_{n+1}} - \sum_{k=1}^{n} f_{n+1} f_{n+2} f_j G_k, \tag{15}$$

$$1 - f_{n+2} = \sum_{k=1}^{n} \nu_k u_k \frac{f_{n+1} f_{n+2}}{\nu_{n+1} f_{n+2} + \nu_{n+2} u_k f_{n+1}} - \sum_{k=1}^{n} f_{n+1} f_{n+2} u_k f_k G_k$$

$$. \tag{16}$$

For two rational functions φ_1, φ_2 on X we call "congruent" and denote by $\varphi_1 \equiv \varphi_2$ if they have equal values at all $\mathbf{c}_j$.

Hence

$$x_j \equiv \nu_j \frac{f_{n+1} f_{n+2}}{\nu_{n+1} f_{n+2} + \nu_{n+2} u_j f_{n+1}}\,(1 \leq j \leq n), \tag{17}$$

$$1 - f_{n+1} \equiv \sum_{k=1}^{n} \nu_k \frac{f_{n+1} f_{n+2}}{\nu_{n+1} f_{n+2} + \nu_{n+2} u_j f_{n+1}}, \tag{18}$$

$$1 - f_{n+2} \equiv \sum_{k=1}^{n} \nu_k u_k \frac{f_{n+1} f_{n+2}}{\nu_{n+1} f_{n+2} + \nu_{n+2} u_k f_{n+1}}. \tag{19}$$

Introduce the new parameter $t = \frac{f_{n+2}}{f_{n+1}}$ as basic parameter and put

$$\omega_j(t) := \frac{\nu_j t}{\nu_{n+1}t + \nu_{n+2}u_j} \quad (1 \le j \le n).$$

Then

$$x_j \equiv \omega_j(t),$$

i.e., $\omega(t) = (\omega_1(t), \ldots, \omega_n(t))$ represents a rational curve in X interpolating the set of critical points $\{\mathbf{c}_j\,(1 \le j \le n+1)\}$.

Lemma 1 *t satisfies the algebraic equation of $(n+1)$th degree :*

$$\psi(t) := 1 - \frac{1}{t} - \sum_{j=1}^{n} \frac{\nu_j(1-u_j)}{\nu_{n+1}t + \nu_{n+2}u_j} = 0. \tag{20}$$

In particular if $\frac{\nu_j}{\nu_{n+1}}(1-u_j)$ are all positive then all the roots are real and different. Hence $\mathbf{c}_j$ are all real and different.

Proof. In fact from (8), (9) we have

$$\frac{1}{f_{n+1}} \equiv 1 + \sum_{j=1}^{n} \frac{\nu_j t}{\nu_{n+1}t + \nu_{n+2}t},$$
$$\frac{1}{f_{n+2}} \equiv 1 + \sum_{j=1}^{n} \frac{\nu_j u_j}{\nu_{n+1}t + \nu_{n+2}t}.$$

These two equations imply Lemma 1.

Denote by $\overline{\psi}(t)$ the monic polynomial of $(n+1)$th degree which t has the same roots as (20)

$$\nu_{n+1}^n\,\overline{\psi}(t) = t\prod_{j=1}^{n}(\nu_{n+1}t + \nu_{n+2}u_j)\,\psi(t) = \nu_{n+1}^n\,(t-\zeta_1)\cdots(t-\zeta_{n+1}).$$

where ζ_j denote the zeros of $\overline{\psi}(t)$. $\overline{\psi}(t)$ is the characteristic polynomial attached to t such that $\zeta_j = t(\mathbf{c}_j)$.

One has the obvious identity

$$\overline{\psi}'(\zeta_j) = [t\prod_{j=1}^{n}(t+\frac{\nu_{n+2}}{\nu_{n+1}}u_j)]_{\zeta_j}[\psi'(t)]_{\zeta_j}$$

Definition 2 For a rational function φ on X we define the "norm" associated with the system of critical points $\mathbf{c}_j$ $(1 \le j \le n+1)$ as follows :

$$\mathcal{N}(\varphi) := \prod_{j=1}^{n+1}[\varphi]_{\mathbf{c}_j}.$$

We say that φ is "unit" if $\mathcal{N}(\varphi) \neq 0$.

Theorem 3 *The following formulae hold :*

$$\mathcal{N}(\nu_{n+1}t+\nu_{n+2}u_j) = -\nu_{n+2}^n\nu_j u_j(1-u_j)\prod_{k\neq j}(u_k-u_j) \quad (1 \le j \le n),$$
$$\mathcal{N}(t) = (-1)^n\frac{\nu_{n+2}^n\prod_{k=1}^n u_k}{\nu_{n+1}^n},$$
$$\mathcal{N}(\nu_{n+1}t+\nu_{n+2}) = \nu_\infty\nu_{n+2}^n\prod_{k=1}^{n}(1-u_k),$$
$$\mathcal{N}(f_j) = \frac{\nu_j^n}{\nu_\infty^n u_j}\frac{\prod_{k\neq j}(1-u_k)}{\prod_{k\neq j}(u_j-u_k)} \quad (1 \le j \le n),$$
$$\mathcal{N}(f_{n+1}) = (-1)^n\frac{\nu_{n+1}^n}{\nu_\infty^n}\prod_{k=1}^{n}\frac{1-u_k}{u_k},$$
$$\mathcal{N}(f_{n+2}) = \frac{\nu_{n+2}^n}{\nu_\infty^n}\prod_{k=1}^{n}(1-u_k).$$

In particular $f_j(1 \le j \le n+2)$ *are all unit in the above sense.*

Put further

$$G_1^* := -f_{n+1}\,(\sum_{k=1}^{n} f_k G_k\,(1-u_k)),$$
$$G_2^* = f_{n+1}f_{n+2}\sum_{k=1}^{n} f_k\,G_k,$$
$$G_j^* := -f_{n+1}f_{n+2}\,f_jG_j \quad (3 \le j \le n)$$

which are all polynomials. Then under the condition $(\mathcal{C}_1)$ the system of equations (13) is equivalent to the following :

$$G_j^* = 0 \quad (1 \le j \le n) \tag{21}$$

Lemma 4 *We have the Jacobian identities*

(i)

$$\frac{\partial(G_1^*,\dots,G_n^*)}{\partial(x_1,\dots,x_n)} \equiv (-1)^{n-1}(u_1-u_2)(\prod_{j=1}^{n} f_j)(f_{n+1})^n(f_{n+2})^{n-1}\frac{\partial(G_1,\dots,G_n)}{\partial(x_1,\dots,x_n)}.$$

(ii)

$$\frac{\partial(t,G_2^*,\dots,G_n^*)}{\partial(x_1,x_2,\dots,x_n)} \equiv -\frac{u_1-u_2}{f_{n+1}^2}.$$

(iii)

$$\psi'(t)\,\frac{\partial(t,G_2^*,\dots,G_n^*)}{\partial(x_1,x_2,\dots,x_n)} \equiv \frac{\partial(G_1^*,\dots,G_n^*)}{\partial(x_1,\dots,x_n)}$$

Definition 5 Define the discriminant associated with the system of critical points $\mathbf{c}_j$ by

$$\mathrm{Discr} := \prod_{j<k}(\zeta_j-\zeta_k)^2 = (-1)^{\frac{n(n+1)}{2}}\mathcal{N}(\overline{\psi}'(t)).$$

On the other hand the Hessian F is defined by the Jacobian

$$\mathrm{Hess}(F) := \frac{\partial(G_1,\dots,G_n)}{\partial(x_1,\dots,x_n)}.$$

We have the equality

Theorem 6

$$\mathrm{Discr} = \{\prod_{j=1}^{n} \mathcal{N}(f_j)\} \{\mathcal{N}(f_{n+1})\}^{n+2} \{\mathcal{N}(f_{n+2})\}^{n-1} \mathcal{N}(\mathrm{Hess}(F)).$$

Hence a pair of critical points meet each other if and only if $\mathcal{N}(\mathrm{Hess}(F))$ *vanishes.*

5 hypersphere arrangements

Let $n+1$ quadratic polynomials of real coefficients in $x = (x_1, \ldots, x_n)$ be given :

$$f_j(x) := Q(x) + 2\sum_{j=1}^{n} \alpha_{j,k} x_k + \alpha_{j0} \quad (1 \leq j \leq n+1),$$

where $Q(x)$ denotes the quadratic form $\sum_{j=1}^{n} x_j^2$. They define the arrangement of hyperspheres $\mathcal{A}$ consisting of the hyperspheres $S_j : f_j = 0$. The center O_j and the radius r_j ($r_j > 0$) of S_j are equal to

$$O_j : -(\alpha_{j1}, \ldots, \alpha_{jn})$$
$$r_j^2 = -\alpha_{j0} + \sum_{k=1}^{n} \alpha_{jk}^2.$$

We denote the distance between O_j, O_k ($j \neq k$) by ρ_{jk} ($\rho_{jk} > 0$) such that $\rho_{jk}^2 = \sum_{l=1}^{n} (\alpha_{jl} - \alpha_{kl})^2$.

For the multiplicative function

$$\Phi(x) = \prod_{j=1}^{n+1} f_j^{\lambda_j}(x)$$

consider the integral $J(\varphi)$ in §3. For generic exponents λ one can prove that the dimension of $H_{\nabla}^{n}(X, \Omega^{\cdot})$ is equal to $2^{n+1} - 1$. As the representative of a basis one can choose the following nth degree forms

$$\varphi_J dx_1 \wedge \cdots \wedge dx_n,\ \varphi_J := \frac{1}{\prod_{j\in J} f_j}$$

where J ranges over the family of arbitrary (unordered) subsets of indices in $\{1, 2, \ldots, n+1\}$.

Cayley-Menger determinants are defined in the following way and play an important role in the sequel. Denote by $\rho_{*j} = \rho_{j*}$ the radius r_j for $j \in \{1, 2, \ldots, n+1\}$ or 0 for $j = *$.

Definition 7 The determinant

$$B\begin{pmatrix} 0 & J \\ 0 & K \end{pmatrix} := \begin{vmatrix} 0 & 1 & \ldots & 1 \\ 1 & \rho^2_{j_1k_1} & \cdots & \rho_{j_1k_p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \rho^2_{j_pk_1} & \cdots & \rho^2_{j_pk_p} \end{vmatrix}$$

is called "Cayley-Menger determinant" associated with $\mathcal{A}$, where $J = \{j_1, \ldots, j_p\}, K = \{k_1, \ldots, k_p\}$ denote two subsets of the indices in $\{*, 1, \ldots, n+1\}$. In case when $J = K$ we simply denote $B(0\,J)$ instead of $B\begin{pmatrix} 0 & J \\ 0 & K \end{pmatrix}$.

Notice that

$$B(0j\,k) = 2\rho^2_{jk} > 0,\ B(0\star j) = 2r_j^2 > 0.$$

For simplicity we restrict ourselves to the case $n = 2$, so that $\mathcal{A}$ is the arrangement of three circles S_1, S_2, S_3 in $\mathbf{R}^2$. We further assume that r_j are the same simply denoted by r and that $\nu_j = 1$ for all j. One sees that

$$\begin{aligned} &B(0\star jk) = \rho^2_{jk}(\rho^2_{jk} - 4r^2), \\ &B(0123) = \rho^4_{12} + \rho^4_{13} + \rho^4_{23} - 2\rho^2_{12}\rho^2_{13} - 2\rho^2_{12}\rho^2_{23} - 2\rho^2_{13}\rho^2_{23}, \\ &B(0\star 123) = -4r^2 B(0123) - 2\rho^2_{12}\rho^2_{13}\rho^2_{23}. \end{aligned}$$

We assume the following condition of non-degeneracy of $\mathcal{A}$:

$$(\mathcal{C}_2) \quad B(0\star 123) \neq 0,\ B(0 * jk) \neq 0$$

i.e., the triangle $\Delta O_1O_2O_3$ is non-degenerate. Any two circles have no contact point and three circles S_1, S_2, S_3 have no common point.

By taking a suitable choice of coordinates we may assume that

$$\alpha_{31} = \alpha_{32} = \alpha_{22} = 0, \alpha_{21} > 0, \alpha_{12} > 0.$$

so that we have

$$\begin{aligned}
&r^2 = -\alpha_{30} = -\alpha_{20} + \alpha_{21}^2 = -\alpha_{10} + \alpha_{11}^2 + \alpha_{12}^2, \\
&\alpha_{21}^2 = \rho_{23}^2,\ \alpha_{11}^2 + \alpha_{12}^2 = \rho_{13}^2,\ (\alpha_{11} - \alpha_{21})^2 + \alpha_{12}^2 = \rho_{12}^2 \\
&4\alpha_{21}^2\alpha_{12}^2 = -B(0123).
\end{aligned}$$

Hence α_{jk} are completely determined by ρ_{jk}^2, r^2.

Under the condition $(\mathcal{C}_2)$ the system of equations (11) are equivalent to

$$\begin{aligned}
G_1 &:= \frac{x_1 + \alpha_{11}}{f_1} + \frac{x_1 + \alpha_{21}}{f_2} + \frac{x_1}{f_3} = 0, \\
G_2 &:= \frac{x_2 + \alpha_{12}}{f_1} + \frac{x_2}{f_2} + \frac{x_2}{f_3} = 0.
\end{aligned} \tag{22}$$

Generally there exist 7 (real or complex) points in X satisfying (22) denoted by $\{\mathbf{c}_j\,(1 \le j \le 7)\}$. Let $D_j\,(1 \le j \le 3))$ be the open disc surrounded by the circumference $\mathfrak{Re}S_j$.

If

$$(\mathcal{C}_3) : B(0 \star 123) > 0,\ B(0 \star jk) < 0\,(1 \le j < k \le 3)$$

then the intersection $D_1 \cap D_2 \cap D_3$ is not empty. The critical points are all real and contained one by one in each compact chamber i.e., $D_1 \cap D_2 \cap D_3$, $D_1 \cap D_2 - D_3, D_1 \cap D_3 - D_2, D_2 \cap D_3 - D_1, D_1 - D_2 \cap D_3, D_2 \cap - D_1 \cap D_3, D_3 - D_1 \cap D_3$.

We want to find a rational curve $t_2 = \omega(t_1) \in X$ containing all critical points $\mathbf{c}_j$ and a monic polynomial $\overline{\psi}(t_1)$ of degree 7 such that $(t_1, \omega(t_1))$ coincides with all t-coordinates $t(\mathbf{c}_j)$ for any root of $\overline{\psi}(t_1)$. In the sequel we shall call t_1 "basic parameter" and $\overline{\psi}(t_1)$ "characteristic polynomial".

To find out the characteristic polynomials we use Sylvester's elimination method.

Introduce the new polynomials in x

$$\begin{aligned} g_1 &:= f_3(L_{12} - L_{23}) - L_{23}\,(f_1 - f_3), \\ g_2 &:= f_3(L_{12} - L_{13}) - L_{13}\,(f_2 - f_3), \\ g_3 &:= -(L_{12} - L_{13})L_{23}(f_1 - f_3) + (L_{12} - L_{23})L_{13}(f_2 - f_3) \end{aligned}$$

where L_{jk} denote linear functions of x

$$\begin{aligned} L_{12} &: L_{12}(x) = \alpha_{12}x_1 + (-\alpha_{11} + \alpha_{21})x_2 + \alpha_{21}\alpha_{12}, \\ L_{13} &: L_{13}(x) = -\alpha_{12}x_1 + \alpha_{11}x_2, \\ L_{23} &: L_{23}(x) = -\alpha_{21}x_2. \end{aligned}$$

$L_{jk}(x) = 0$ defines the straight line going through O_j, O_k and the triangle $\Delta[O_1, O_2, O_3]$ is defined by $L_{jk} \geq 0$.

Lemma 8 *Under the condition* $(\mathcal{C}_2)$ *the system of equations* (22) *are equivalent to the system*

$$g_1 = g_2 = g_3 = 0. \tag{23}$$

Suppose moreover that $\rho_{12} \neq \rho_{13}$ *then* (23) *is equivalent to the following system*

$$g_2 = g_3 = 0. \tag{24}$$

Introduce the new parameters $t_1 = \frac{f_3}{f_1}, t_2 = \frac{f_3}{f_2}$ and denote $t_\infty = 1 + t_1 + t_2$. We call t_1, t_2 "admissible".

(23) gives the following congruences

$$x_1 \equiv -\frac{\alpha_{11}t_1 + \alpha_{21}t_2}{t_\infty},\ x_2 \equiv -\frac{\alpha_{12}t_1}{t_\infty}. \tag{25}$$

and conversely

$$t_1 \equiv \frac{L_{23}}{L_{12}},\ t_2 \equiv \frac{L_{13}}{L_{12}},\ t_\infty = \frac{\alpha_{21}\alpha_{12}}{L_{12}}. \tag{26}$$

Then (23) can be rewritten using the parameters t_1, t_2 as

$$\tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = 0 \tag{27}$$

respectively where

$$\begin{aligned}
\tilde{g}_1 &:= g_1 \frac{t_\infty^3}{\alpha_{21}\alpha_{12}}, \\
\tilde{g}_2 &:= g_2 \frac{t_\infty^3}{\alpha_{21}\alpha_{12}}, \\
\tilde{g}_3 &:= g_3 \frac{t_\infty^3}{\alpha_{21}^2\alpha_{12}^2}.
\end{aligned}$$

$\tilde{g}_1, \tilde{g}_2, \tilde{g}_3$ are polynimials of third degree in t_1, t_2 as follows

$$\begin{aligned}
\tilde{g}_1 &:= a_0 t_2^2 + a_1 t_2 + a_2, \\
\tilde{g}_2 &:= b_0 t_2^3 + b_1 t_2^2 + b_2 t_2 + b_3, \\
\tilde{g}_3 &:= c_0 t_2^2 + c_1 t_2 + c_2,
\end{aligned}$$

where a_j, b_k, c_l are given by polynomials in t_1 :

$$\begin{aligned}
&a_0 = (r^2 - \rho_{12}^2)t_1 + \rho_{23}^2 - r^2, \\
&a_1 = 2\{r^2 t_1^2 + (\rho_{23}^2 - \rho_{12}^2)t_1 - r^2\}, \\
&a_2 = (t_1 - 1)\{r^2 t_1^2 + (\rho_{13}^2 + 2r^2)t_1 + r^2\}, \\
&b_0 = r^2, \ b_1 = 2r^2 t_1 + \rho_{23}^2 + r^2, \\
&b_2 = (r^2 - \rho_{12}^2)t_1^2 + 2(\rho_{13}^2 - \rho_{12}^2)t_1 - (r^2 + \rho_{23}^2), \\
&b_3 = (\rho_{13}^2 - r^2)t_1^2 - 2r^2 t_1 - r^2, \\
&c_0 = \rho_{12}^2 t_1 - \rho_{23}^2, \\
&c_1 = -\rho_{12}^2 t_1^2 + \rho_{23}^2, \\
&c_2 = \rho_{13}^2 t_1(t_1 - 1).
\end{aligned}$$

Notice that

$$\begin{aligned}
&\tilde{g}_1(t_1, 1) = a_0 + a_1 + a_2 \\
&= r^2 t_1^3 + (\rho_{12}^2 + 3r^2)t_1^2 + 2(\rho_{23}^2 - 2\rho_{12}^2)t_1 + \rho_{23}^2 - 4r^2, && (28) \\
&\tilde{g}_2(t_1, 1) = b_0 + b_1 + b_2 + b_3 = (\rho_{13}^2 - \rho_{12}^2)t_1(t_1 + 2), && (29) \\
&\tilde{g}_3(t_1, 1) = c_0 + c_1 + c_2 = (\rho_{13}^2 - \rho_{12}^2)t_1(t_1 - 1). && (30)
\end{aligned}$$

so that

$$\tilde{g}_2(0,1) = \tilde{g}_3(0,1) = 0. \tag{31}$$

Lemma 9 *Put*

$$\begin{aligned} U : U(t_1) &= b_0(c_1^2 - c_0c_2) - b_1c_0c_1 + b_2c_0^2, \\ V : V(t_1) &= -b_0c_1c_2 + b_1c_0c_2 - b_3c_0^2. \end{aligned}$$

Then the following identity holds :

$$\tilde{g}_{23} := c_0^2\,\tilde{g}_2 - (b_0c_0t_2 + b_1c_0 - b_0c_1)\,\tilde{g}_3 = Ut_2 - V \quad \textit{for arbitrary } t_1, t_2, \tag{32}$$

where

$$U = \frac{\partial \tilde{g}_{23}}{\partial t_2}.$$

If $\tilde{g}_2 = \tilde{g}_3 = 0$ *then* $\tilde{g}_{23} = 0$ *which implies*

$$t_2 \equiv \omega(t_1) \quad \omega(t_1) := \frac{V}{U}.$$

The resultant R *of* $\tilde{g}_2(t_1, t_2)$ *and* $\tilde{g}_3(t_1, t_2)$ *relative to* t_2 *is a polynomial in* t_1 *of degree* 8 *written by Sylvester determinant*

$$R : R(t_1) = \begin{vmatrix} b_0 & b_1 & b_2 & b_3 & \\ & b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & & \\ & c_0 & c_1 & c_2 & \\ & & c_0 & c_1 & c_2 \end{vmatrix}$$

It is related to U, V *and can be described as follows :*

$$\begin{aligned} c_0^2 R &= U^2\,\tilde{g}_{12}(t_1, \frac{V}{U}) \\ &= c_0V^2 + c_1VU + c_2U^2, \end{aligned}$$

where U, V are polynomials of degree 4 which can be written as

$$U = \sum_{j=0}^{4} u_j t_1^{4-j},\ V = \sum_{j=0}^{4} v_j t_1^{4-j}.$$

$$\begin{aligned}
&u_0 = -(\rho_{12}^2 - 4r^2)\rho_{12}^4,\ u_4 = r^2\rho_{23}^4 \\
&v_0 = \rho_{12}^4\{r^2(\rho_{12}^2 + 3\rho_{13}^2) - \rho_2^2\rho_{13}^2\},\ v_4 = r^2\rho_{23}^4.
\end{aligned}$$

Moreover $U - V$ can be evaluated explicitly

$$\begin{aligned}
&U - V = (\rho_{13}^2 - \rho_{12}^2)W^*, \\
&W^* = t_1(w_0 t_1^3 + w_1 t_1^2 + w_2 t_1 + w_3)
\end{aligned}$$

such that

$$\begin{aligned}
&w_0 = \rho_{12}^2(\rho_{12}^2 - 3r^2), \\
&w_1 = -\rho_{12}^2(3\rho_{23}^2 - 2\rho_{12}^2)) + (2\rho_{23}^2 + \rho_{12}^2)r^2, \\
&w_2 = \rho_{23}^2(2\rho_{23}^2 - 3\rho_{12}^2) + (2\rho_{12}^2 + \rho_{23}^2)r^2, \\
&w_3 = \rho_{23}^2(\rho_{23}^2 - 3r^2).
\end{aligned}$$

R is a polynomial in t_1 of degree 8 and in ρ_{jk}^2, r^2.

Lemma 10 (i) *If* $\rho_{12}^2 = \rho_{13}^2$ *then* R *vanishes.*
(ii) $R(0)$ *vanishes.*

Proof. About (i). When $\rho_{12}^2 = \rho_{13}^2$ U coincides with V so that

$$c_0^2 R = (c_0 + c_1 + c_2)U^2 = 0$$

This implies $R = 0$.

About (ii). The identity $U(0) = V(0)$ holds true. Hence

$$-\rho_{23}^2 R(0) = (c_0(0) + c_1(0) + c_2(0))\, U(0) = 0$$

because of (31).

Because of Lemma 10 R has the factor $(\rho_{12}^2 - \rho_{13}^2)t_1$.

As a result

Lemma 11 *R is a polynomial in t_1 of degree 8 and in ρ_{jk}^2, r^2 with the factor $(\rho_{12}^2 - \rho_{13}^2)t_1$ such that*

$$R = \rho_{12}^4 r^2(\rho_{12}^2 - 4r^2)(\rho_{12}^2 - \rho_{13}^2)t_1\overline{\psi}(t_1),$$
$$R \approx -\rho_{23}^4 r^2(\rho_{12}^2 - \rho_{13}^2)(\rho_{23}^2 - 4r^2)t_1 \quad (t_1 \downarrow 0),$$

where $\overline{\psi}(t_1) = \prod_{j=1}^7 (t_1 - \zeta_j)$ is a monic polynomial with with 7 roots ζ_j $(1 \leq j \leq 7)$ such that

$$-\overline{\psi}(0) = \prod_{j=1}^{7} \zeta_j = \frac{\rho_{23}^4(\rho_{23}^2 - 4r^2)}{\rho_{12}^4(\rho_{12}^2 - 4r^2)} = \frac{\rho_{23}^2 B(0 \star 23)}{\rho_{12}^2 B(0 \star 12)}.$$

$\overline{\psi}(t_1)$ is the characteristic polynomial relative to the basic parameter t_1 of the critical points $\mathbf{c}_j$ such that $t_1(\mathbf{c}_j) = \zeta_j$.

Furthermore since

$$U(1) = (\rho_{23}^2 - \rho_{12}^2)^2(\rho_{13}^2 - 4r^2), V(1) = (\rho_{23}^2 - \rho_{13}^2)^2(4r^2 + 2\rho_{13}^2 - 3\rho_{12}^2)$$

we have the formula

$$R(1) = 3(\rho_{12}^2 - \rho_{23}^2)^3(\rho_{13}^2 - \rho_{12}^2)(\rho_{13}^2 - 4r^2)$$

hence

$$\overline{\psi}(1) = \prod_{j=1}^{7}(1 - \zeta_j) = -3\frac{(\rho_{12}^2 - \rho_{23}^2)^3(\rho_{13}^2 - 4r^2)}{\rho_{12}^4 r^2(\rho_{12}^2 - 4r^2)}.$$

Seeing that $\frac{f_1 - f_3}{f_1} = 1 - t_1, \frac{f_2 - f_3}{f_2} = 1 - t_2$ we can conclude

Proposition 12 (i)

$$\mathcal{N}(t_1) = \mathcal{N}(\frac{f_3}{f_1}) = \frac{\rho_{23}^2 B(0 \star 23)}{\rho_{12}^2 B(0 \star 12)}.$$
$$\mathcal{N}(t_2) = \mathcal{N}(\frac{f_3}{f_2}) = \frac{\rho_{13}^2 B(0 \star 13)}{\rho_{12}^2 B(0 \star 12)}.$$

(ii)

$$\mathcal{N}(1-t_1) = \mathcal{N}(\frac{f_1 - f_3}{f_1}) = -3\frac{(\rho_{12}^2 - \rho_{23}^2)^3 B(0\star 13)}{\rho_{12}^2\rho_{13}^2 r^2 B(0\star 12)},$$
$$\mathcal{N}(1-t_2) = \mathcal{N}(\frac{f_2 - f_3}{f_2}) = -3\frac{(\rho_{12}^2 - \rho_{13}^2)^3 B(0\star 23)}{\rho_{12}^2\rho_{23}^2 r^2 B(0\star 12)}.$$

Instead of (t_1, t_2) we now take the new coordinates (t_∞, t_1), t_∞ being the basic parameter. By the substitution $t_2 = t_\infty - t_1 - 1$, $\tilde{g}_2, 2\tilde{g}_3 - \tilde{g}_2$ can be rewritten as

$$\tilde{g}_2^\sharp(t_\infty, t_1) := \tilde{g}_2(t_1, t_\infty - t_1 - 1) = b_0' t_1^3 + b_1' t_1^2 + b_2' t_1 + b_3',$$
$$\tilde{g}_3^\sharp(t_\infty . t_1) := 2\tilde{g}_3(t_1, t_\infty - t_1 - 1) - \tilde{g}_2(t_1, t_\infty - t_1 - 1) = c_0' t_1^2 + c_1' t_1 + c_2',$$

$b_0', b_1', b_2', b_3'; c_0', c_1', c_2',$ denote polynomials in t_∞ as follows :

$$\begin{aligned} b_0' &= \rho_{12}^2, \\ b_1' &= -\rho_{12}^2 t_\infty + \rho_{23}^2 - \rho_{13}^2 + 3\rho_{12}^2, \\ b_2' &= -r^2 t_\infty^2 + 2(-\rho_{12}^2 + \rho_{13}^2 - \rho_{23}^2)t_\infty + (2\rho_{12}^2 - \rho_{13}^2 + 3\rho_{23}^2), \\ b_3' &= (t_\infty - 2)\{r^2 t_\infty^2 + \rho_{23}^2(t_\infty - 1)\}, \end{aligned}$$

and

$$\begin{aligned} c_0' &= c_{00}' t_\infty + c_{01}', \\ c_1' &= c_{10}' t_\infty^2 + c_{11}' t_\infty + c_{12}', \\ c_2' &= c_{20}' t_\infty^3 + c_{21}' t_\infty^2 + c_{22}' t_\infty + c_{23}' \end{aligned}$$

where

$$\begin{aligned} &c_{00}' = \rho_{12}^2,\ c_{01}' = 3(\rho_{23}^2 - \rho_{13}^2 + \rho_{12}^2), \\ &c_{10}' = -(2r^2 + \rho_{12}^2),\ c_{11}' = 4\rho_{13}^2 - 2\rho_{12}^2 - 6\rho_{23}^2,\ c_{12}' = 3(\rho_{12}^2 - \rho_{13}^2 + 3\rho_{23}^2), \\ &c_{20}' = 2r^2,\ c_{21}' = 3\rho_{23}^2 - 4r^2,\ c_{22}' = -9\rho_{23}^2,\ c_{23}' = 6\rho_{23}^2. \end{aligned}$$

Then like Lemma 9 the following Lemma holds.

Lemma 13 *Put*

$$U^\sharp : U^\sharp(t_\infty) = b_0'({c_1'}^2 - c_0'c_1)c_0'c_1' + b_2'c_0',$$
$$V^\sharp : V^\sharp(t_\infty) = -b_0'c_1'c_2' + b_1'c_0'c_2 - b_3'{c_0'}^2.$$

Then

$$0 \equiv U^\sharp t_1 - V^\sharp.$$

i.e., the rational curve $t_1 = \frac{V^\sharp(t_\infty)}{U^\sharp(t_\infty)}$ *gives the interpolating curve. We have*

$$U^\sharp = \sum_{j=0}^{4} u_j' t_\infty^{4-j},$$
$$V^\sharp = \sum_{j=0}^{5} v_j' t_\infty^{5-j}$$

with

$$u_0' = v_0' = r^2\rho_{12}^2(4r^2 - \rho_{12}^2),$$
$$u_1' - v_1' = 2r^2\rho_{12}^2(4r^2 - \rho_{13}^2),$$

so that

$$\frac{V^\sharp}{U^\sharp} \approx t_\infty + \frac{v_1' - u_1'}{u_0'} + O(\frac{1}{t_\infty}) \quad (t_\infty \uparrow \infty)$$

t_∞ being fixed, the resultant $R^\sharp = R^\sharp(t_\infty)$ of $\tilde{g}_1^\sharp, \tilde{g}_3^\sharp$ relative to t_1 is given by

$${c_0'}^2 R^\sharp = c_0' {V^\sharp}^2 + c_1' U^\sharp V^\sharp + c_2' {U^\sharp}^2.$$

As a result

$${c_0'}^2 R^\sharp \approx u_0'\{u_0'(c_{01}' + c_{11}' + c_{21}') + (v_1' - u_1')(2c_{00}' + c_{10}')\} t_\infty^8 (1 + O(\frac{1}{t_\infty})) \qquad (t_\infty \uparrow \infty)$$

Seeing that

$$\begin{aligned}
&c'_{01} + c'_{11} + c'_{21} = -4r^2 + \rho_{12}^2 + \rho_{13}^2, \\
&2c'_{00} + c'_{10} = \rho_{12}^2 - 2r^2
\end{aligned}$$

we have from Lemma 13

$$R^\sharp = \rho_{12}^4 r^4 (\rho_{12}^2 - 4r^2)(\rho_{12}^2 - \rho_{13}^2) t_\infty^8 (1 + O(\frac{1}{t_\infty})).$$

On the other hand (31) shows the equality

$$\tilde{g}_2^\sharp(2,1) = \tilde{g}_3^\sharp(2,1) = 0$$

i.e., the two polynomials $\tilde{g}_2^\sharp(2,t_1)$, $\tilde{g}_3^\sharp(2,t_1)$ have a common zero which means $R^\sharp(2) = 0$. Hence $R^\sharp$ can be described as

$$R^\sharp(t_\infty) = \rho_{12}^2 r^4 (\rho_{12}^2 - 4r^2)(\rho_{12}^2 - \rho_{13})^2 (t_\infty - 2) \prod_{j=1}^{7} (t_\infty - \zeta'_j).$$

where ζ'_j denotes the value $t_\infty(\mathbf{c}_j)$.

Lemma 14 *The following identity holds :*

$$R^\sharp(0) = 54 \rho_{13}^2 \rho_{23}^2 (\rho_{13}^2 - \rho_{12}^2) B(0123).$$

We can evaluate the norm of t_∞ as follows :

Proposition 15

$$\mathcal{N}(t_\infty) = \prod_{j=1}^{7} \zeta'_j = -27 \frac{\rho_{13}^2 \rho_{23}^2 B(0123)}{r^4 B(0 \star 12)}.$$

$\overline{\psi}(t_\infty) = \prod_{j=1}^{7}(t_\infty - \zeta'_j)$ is the characteristic polynomial in t_∞.

The identity (26) derives the formula for $\mathcal{N}(L_{12})$. In the same way by symmetry of isometry the followings hold :

Corollary 16

$$\mathcal{N}(L_{12}) = \frac{1}{2^7 3^3} \frac{r^4 B(0\star 12)}{\rho_{13}^2 \rho_{23}^2} \{-B(0123)\}^{\frac{5}{2}}.$$

$$\mathcal{N}(L_{13}) = \frac{1}{2^7 3^3} \frac{r^4 B(0\star 13)}{\rho_{12}^2 \rho_{23}^2} \{-B(0123)\}^{\frac{5}{2}},$$
$$\mathcal{N}(L_{23}) = \frac{1}{2^7 3^3} \frac{r^4 B(0\star 23)}{\rho_{12}^2 \rho_{13}^2} \{-B(0123)\}^{\frac{5}{2}}.$$

Put $\psi(t_1) = \tilde{g}_3(t_1, \omega(t_1))$ such that $R = \frac{U^2 \psi(t_1)}{c_0^2}$.

Finally we want to discuss a formula related to the norm of "Hessian" of the level function $\mathfrak{Re}F$.

Concerning the derivatives relative to t_1 of $\overline{\psi}(t_1), R(t_1)$ we have

$$\psi'(t_1) \equiv \frac{c_0^2}{U^2} R'(t_1). \tag{33}$$

A direct computation gives the following

Lemma 17

$$\frac{\partial(\tilde{g}_2, \tilde{g}_3)}{\partial(t_1, t_2)} \equiv -r^2 \frac{B(0\star 12)\rho_{12}^2(\rho_{12}^2 - \rho_{13}^2)}{U} t_1 \overline{\psi}'(t_1).$$

Proof. By partial derivation of (32) with respect to t_2

$$U = \frac{\partial \tilde{g}_{23}}{\partial t_2}.$$

On the other hand

$$\tilde{g}_{23}(t_1, \omega(t_1)) = 0$$

By derivation relative to t_1

$$\frac{\partial \tilde{g}_{23}(t_1, \omega(t_1))}{\partial t_1} + \frac{\partial \tilde{g}_{23}(t_1, \omega(t_1))}{\partial t_1} \omega'(t_1) = 0.$$

In the same way by derivation of $\psi(t_1)$ relative to t_1

$$\psi'(t_1) = \frac{\partial \tilde{g}_3(t_1, \omega(t_1))}{\partial t_1} + \frac{\partial \tilde{g}_3(t_1, \omega(t_1))}{\partial t_2}\omega'(t_1).$$

Hence

$$\psi'(t_1) = \frac{\partial(\tilde{g}_3, \tilde{g}_{23})}{\partial(t_1, t_2)} \Big/ \frac{\partial \tilde{g}_{23}}{\partial t_2} = -\frac{c_0^2}{U}\,\frac{\partial(\tilde{g}_2, \tilde{g}_3)}{\partial(t_1, t_2)}. \tag{34}$$

In view of Lemma 11 this implies

$$R'(t_1) \equiv -U(t_1)\frac{\partial(\tilde{g}_2, \tilde{g}_3)}{\partial(t_1, t_2)}$$

which completes Lemma 17 in view of (33).

Lemma 18 *The identity holds*

$$dG_1 \wedge dG_2 \equiv -\frac{t_1 t_2}{1 - t_2}\frac{L_{12}^4}{f_3^4(\alpha_{21}\alpha_{12})^3} d\tilde{g}_2 \wedge d\tilde{g}_3. \tag{35}$$

Proof. Put

$$\begin{aligned} G_{13} &= x_2 G_1 - (x_1 + \alpha_{21})G_2, \\ G_{23} &= (x_2 + \alpha_{12})G_1 - (x_1 + \alpha_{11})G_2, \end{aligned}$$

then

$$dG_{13} \wedge dG_{23} \equiv L_{12} dG_1 \wedge dG_2.$$

Further it holds

$$\begin{aligned} g_2 &= -f_2 f_3 G_{23}, \\ g_3 &= L_{12}\, f_3^2\{-\frac{1 - t_2}{t_1}G_{13} + \frac{1 - t_1}{t_2}G_{23}\}. \end{aligned}$$

so that

$$dg_2 \wedge dg_3 \equiv -\frac{1 - t_2}{t_1 t_2} f_3^4 L_{12} dG_{13} \wedge dG_{23}.$$

From (26)

$$dg_2 \wedge dg_3 \equiv \frac{(\alpha_{21}\alpha_{12})^3}{t_\infty^6} d\tilde{g}_2 \wedge d\tilde{g}_3$$

where $4\alpha_{21}^2\alpha_{12}^2 = -B(0123)$. Summing up these equalities of Jacobian implies Lemma 18.

By definition

$$\text{Hess}(F) = \frac{\partial(G_1, G_2)}{\partial(x_1, x_2)}, \ \frac{\partial(x_1, x_2)}{\partial(t_1, t_2)} = \frac{\sqrt{-B(0123)}}{2t_\infty^3}.$$

By using these equalities one can prove the following :

Proposition 19 *At each critical point* $\mathbf{c}_j$

$$\big[\text{Hess}F\big]_{\mathbf{c}_j} = -\big[\frac{t_1 t_2}{(1-t_2)t_\infty U} \frac{R'(t_1)}{f_3}\big]_{\mathbf{c}_j},$$

such that $\zeta_j = \big[t_1\big]_{\mathbf{c}_j}$ *and* $t_2 = \frac{V}{U}$.

As an immediate consequence of Proposition 18 , Lemma 11 and Lemma 19 we have

Theorem 20 *Suppose that*

$$\mathcal{N}(U - V) \neq 0,$$

then the following equality holds.

$$\mathcal{N}(\text{Hess}F) = (-1)^7 C^7 \frac{\mathcal{N}(t_1^2 t_2)}{\mathcal{N}((U-V)t_\infty)} \frac{\text{Discr}}{\mathcal{N}(f_3)}$$

where Discr, C *denote the discriminant of* $\overline{\psi}(t_1)$ *relative to the basic parameter* t_1 *:*

$$\mathrm{Discr} := \prod_{1\le j<k\le 7} (\zeta_j - \zeta_k)^2 = -\prod_{j=1}^{7} \left[\overline{\psi}'(t_1)\right]_{\zeta_j}.$$

and the constant

$$C = \rho_{12}^2 r^2 B(0\star 12)(\rho_{12}^2 - \rho_{13}^2).$$

Remark $\mathcal{N}(f_3)$ seems to be equal to

$$\frac{1}{2\cdot 3^4}\frac{B(0\star 13)B(0\star 23)B(0\star 123)}{\rho_{12}^2}.$$

The similar formula seems true for $\mathcal{N}(f_1), \mathcal{N}(f_2)$.

6 case of isosceles triangle

The case when $\Delta[O_1O_2O_3]$ is an isosceles triangle is an exceptional one. It is explained in more detail.

Generally we may put

$$R = (\rho_{12}^2 - \rho_{13}^2)R^*,$$
$$U - V = (\rho_{13}^2 - \rho_{12}^2)W^*,$$

where R^*, W^* denote polynomials such that

$$b_0^2 R^* = (b_0 + b_1 + b_2)V^2 + V\{(t_1^2 - t_1)V + (b_1 + 2b_2)W^*\}.$$

.

Suppose now that the equality $\rho_{12}^2 = \rho_{13}^2$ holds.

Then $b_0 + b_1 + b_2 = 0$ and $R, U - V$ both vanish identically because they are divisible by $\rho_{12}^2 - \rho_{13}^2$:

$$\tilde{g}_2 = (t_2 - 1)\tilde{g}_2^*,\ \tilde{g}_3 = (t_2 - 1)\tilde{g}_3^*$$
$$c_0^2\tilde{g}_2^* - (b_0c_0t_2 + b_1c_0 - b_0c_1)\tilde{g}_3^* = U.$$

where

$$\tilde{g}_2^* = b_0^* t_2^2 + b_1^* t_2 + b_2^*,$$

with $b_0^* = r^2,\ b_1^* = 2r^2 t_1 + \rho_{23}^2 + 2r^2, b_2^* = -(\rho_{12}^2 - r^2)t_1^2 + 2r^2 t_1 + r^2,$

$$\tilde{g}_3^* = c_0^* t_2 + c_1 *,$$

with $c_0^* = \rho_{12}^2 t_1 - \rho_{23}^2,\ c_1^* = -\rho_{12}^2 t_1(t_1 - 1)$.

The polynomial $U(t_1) = V(t_1)$ of degree 4 can be written with a monic polynomial $\overline{\psi}_2$

$$\begin{aligned} &U(t_1) = u_0 t_1^4 + u_2 t_1^3 + u_3 t_1^2 + u_2 t_1 + u_4 \\ &= -\rho_{12}^4(\rho_{12}^2 - 4r^2)\overline{\psi}_2(t_1) \end{aligned}$$

where

$$\begin{aligned} &u_0 = -\rho_{12}^4(\rho_{12}^2 - 4r^2), \\ &u_1 = \rho_{12}^2 \rho_{23}^2 (3\rho_{12}^2 - 4r^2), \\ &u_2 = \rho_{23}^2\{-\rho_{12}^2(2\rho_{23}^2 + \rho_{12}^2) + (-4\rho_{12}^2 + \rho_{23}^2)r^2\}, \\ &u_3 = \rho_{23}^4(\rho_{12}^2 + 2r^2), \\ &u_4 = \rho_{23}^4 r^2. \end{aligned}$$

$\overline{\psi}_2(t_1)$ has 4 roots denoted by $\zeta_4, \zeta_5, \zeta_6, \zeta_7 : \overline{\psi}_1(t_1) = \prod_{j=4}^{7}(t_1 - \zeta_j)$.

On the other hand $W^*(t_1)$ has the expression

$$W^* = t_1(w_0 t_1^3 + w_1 t_1^2 + w_2 t_1 + w_3),$$

where

$$\begin{aligned} &w_0 = \rho_{12}^2(\rho_{12}^2 - 3r^2), \\ &w_1 = -\rho_{12}^2(3\rho_{23}^2 - 2\rho_{12}^2) + (2\rho_{23}^2 + \rho_{12}^2)r^2, \\ &w_2 = \rho_{23}^2(2\rho_{23}^2 - 3\rho_{12}^2) + (2\rho_{12}^2 + \rho_{23}^2)r^2, \\ &w_3 = \rho_{23}^2(\rho_{23}^2 - 3r^2)). \end{aligned}$$

Suppose first that $t_2 \neq 1$.

The equation $\tilde{g}_3^*(t_1, t_2) = 0$ can be uniquely solved :

$$t_2 \equiv \frac{V^*}{U^*}$$

where

$$U^* = c_0^* = c_0 = \rho_{12}^2 t_1 - \rho_{23}^2,\ V^* = -c_1^* = \rho_{12}^2 t_1(t_1 - 1).$$

Then the equation $\tilde{g}_2^*(t_1, \frac{V^*}{U^*}) = 0$ relative to t_1 is equivalent to

$$U = V = b_0^*(V^*)^2 + b_1^* V^* U^* + b_2^*(U^*)^2 = 0$$

which have the roots $\zeta_4, \zeta_5, \zeta_6, \zeta_7$. The critical points $\mathbf{c}_j$ $(4 \leq j \leq 7)$ correspond to the t-coordinates $(\zeta_j, \frac{V^*(\zeta_j)}{U^*(\zeta_j)})$.

Suppose next $t_2 = 1$.

Then $\tilde{g}_2 = \tilde{g}_3 = 0$ automatically. According to (28) we may put the polynomial $\overline{\psi}_1(t_1)$ as

$$\begin{aligned} r^2\overline{\psi}_1(t_1) &:= \tilde{g}_1(t_1, 1) \\ &= r^2 t_1^3 + (\rho_{12}^2 + 3r^2)t_1^2 + 2(\rho_{23}^2 - 2\rho_{12}^2)t_1 + \rho_{23}^2 - 4r^2 \end{aligned}$$

and denote the roots of the equation

$$\overline{\psi}_1(t_1) = 0$$

by $\zeta_1, \zeta_2, \zeta_3$. The points $\mathbf{c}_j$ corresponds to the t-coordinates $(\zeta_j, 1)$.

The critical points are divided into two parts. Three of them corresponding to $t_1 = \{\zeta_1, \zeta_2, \zeta_3\}$, is contained in the mid-line of the triangle $\Delta[O_1, O_2, O_3]$ defined by : $t_2 = 1$, while the remaining ones corresponds to $t_1 = \zeta_4, \zeta_5, \zeta_6, \zeta_7$ lie outside the mid-line.

Lemma 21 *We have the identification*

$$(t_1^2 - t_1)V + (b_1 + 2b_2)W^* = b_0^2 t_1 \overline{\psi}_1(t_1)$$

such that

$$R^* = t_1 \overline{\psi}_1(t_1) \overline{\psi}_2(t_1).$$

$\overline{\psi}_1(t_1)$ *has three roots denoted by* $\zeta_1, \zeta_2, \zeta_3$.

The characteristic polynomial $\overline{\psi}(t_1)$ is equal to the product of two factors of $\overline{\psi}_1, \overline{\psi}_2$:

$$\overline{\psi}(t_1) = \overline{\psi}_1(t_1)\overline{\psi}_2(t_1) = \prod_{j-1}^{7}(t_1 - \zeta_j).$$

We can show that

Theorem 22

$$\begin{aligned}
\mathcal{N}(f_1) &= \prod_{j=1}^{7}[f_1]_{\mathbf{c}_j} \\
&= \frac{r^2}{2\cdot 3^4}\frac{B(0\star 12)\,B(0\star 13)\,B(0\star 123)}{\rho_{23}^2}, \\
\mathcal{N}(f_2) &= \prod_{j=1}^{7}[f_2]_{\mathbf{c}_j} \\
&= \frac{r^2}{2\cdot 3^4}\frac{B(0\star 23)\,B(0\star 12)\,B(0\star 123)}{\rho_{13}^2}, \\
\mathcal{N}(f_3) &= \prod_{j=1}^{7}[f_3]_{\mathbf{c}_j} \\
&= \frac{r^2}{2\cdot 3^4}\frac{B(0\star 23)\,B(0\star 13)\,B(0\star 123)}{\rho_{12}^2}.
\end{aligned}$$

Theorem 23

$$\mathcal{N}(\mathrm{Hess}F) = (-1)^7 C^{*7}\frac{\mathcal{N}(t_1^2 t_2)}{\mathcal{N}(W^*\, t_\infty)}\frac{\mathrm{Discr}^*}{\mathcal{N}(f_3)}$$

where W^* is related with the equality

$$(b_1 + 2b_2)W^* = t_1\{b_0^2\overline{\psi}_1(t_1) + \rho_{12}^4(\rho_{12}^2 - 4r^2)(t_1 - 1)\overline{\psi}_2(t_1)\}.$$

and with the constant

$$C^* = \rho_{12}^2 r^2 B(0\star 12).$$

Discr^* *means the discriminant of the polynomial* $\overline{\psi}(t_1) = \overline{\psi}_1(t_1)\overline{\psi}_2(t_1)$.

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