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Application of the intersection theory to the connection problem related with differential equations

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Introduction

- Connection formulas related with rigid systems of Fuchsian differential equations (the one satisfied by the generalized hypergeometric function ${}_{n+1}F_n$ and the Even-Odd family in Simpson's list) and related with Lauricella's F_D, F_A, F_B (n variable version of Appell's F_1, F_2, F_3).
- Intersection numbers for twisted cycles.

1 rigid systems of Fuchsian differential equations

Rigid local system

of irreducible rigid Fuchsian differential systems with 3 singularities on \mathbb{P}^1

order	2	3	4	5	6	7	8	9	10	11	12	13	14
#	1	1	3	5	13	20	45	74	142	212	421	588	1004

of irreducible rigid Fuchsian differential systems on \mathbb{P}^1

order	2	3	4	5	6	7	8	9	10	11	12	13	14
#	1	2	6	11	28	44	96	157	306	441	857	1117	2032

Simpson's list

	rank	spectral type
GHDE	n	$1^n ; 1^n ; n - 1, 1$
Even family	$2m$	$1^{2m} ; m, m - 1, 1 ; m, m$
Odd family	$2m + 1$	$1^{2m+1} ; m, m, 1 ; m + 1, m$
Extra case	6	$1^6 ; 2^3 ; 4, 2$

1.1 The differential equation ${}_{n+1}E_n$

The generalized hypergeometric function ${}_{n+1}F_n$ is the analytic continuation of the generalized hypergeometric series

$${}_{n+1}F_n \left(\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{n+1} \\ \beta_1, \dots, \beta_n \end{matrix} ; x \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_{n+1})_k}{(\beta_1)_k \cdots (\beta_n)_k k!} x^k.$$

This satisfies

$$({}_{n+1}E_n) \left\{ \prod_{1 \leq i \leq n+1} (\theta_x + \beta_i - 1) - x \prod_{1 \leq i \leq n+1} (\theta_x + \alpha_i) \right\} F = 0,$$

where $\beta_{n+1} = 1$ and $\theta_x = x d/dx$.

- The rank of ${}_{n+1}E_n$ is $n + 1$.

- The characteristic exponents are

$$1 - \beta_1, 1 - \beta_2, \dots, 1 - \beta_n, 0, \quad \text{at } x = 0,$$

$$0, 1, \dots, n - 1, \sum_{i=1}^n \beta_i - \sum_{i=1}^{n+1} \alpha_i \quad \text{at } x = 1,$$

$$\alpha_1, \alpha_2, \dots, \alpha_{n+1} \quad \text{at } x = \infty.$$

- Connection formulas are

$$f_i^{(\infty)}(z) = \sum_{j=1}^{n+1} \prod_{s \neq i} \frac{\Gamma(\alpha_i - \alpha_s + 1)}{\Gamma(\beta_j - \alpha_s)} \prod_{s \neq j} \frac{\Gamma(\beta_j - \beta_s)}{\Gamma(\alpha_i - \beta_s + 1)} \times f_j^{(0)}(z),$$

$$f_1^{(1)}(z) = \sum_{j=1}^{n+1} \prod_{s \neq i} \frac{\Gamma(1 + \sum_{s=1}^n \beta_s - \sum_{s=1}^{n+1} \alpha_s) \prod_{\substack{1 \leq s \leq n+1 \\ s \neq i}} \Gamma(\beta_j - \beta_s)}{\prod_{1 \leq s \leq n+1} \Gamma(\beta_j - \alpha_s)} \times f_j^{(0)}(z),$$

where $f_i^{(\infty)}(z) = (-z)^{-\alpha_i} (1 + O(z^{-1}))$, $f_j^{(0)}(z) = (-z)^{1-\beta_j} (1 + O(z))$ and

$$f_1^{(1)}(z) = (1 - z)^{\sum_{i=1}^n \beta_i - \sum_{i=1}^{n+1} \alpha_i} (1 + O(1 - z)).$$

Derivation by use of the intersection numbers

$H_n^{\text{lf}}(T, \mathcal{L})$ or $H_n(T, \mathcal{L})$, where \mathcal{L} is determined by

$$u(t) = \prod_{i=1}^n t_i^{\alpha_{i+1} - \beta_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\beta_i - \alpha_i - 1}, \quad (\beta_{n+1} = 1, t_0 = 1, t_{n+1} = z)$$

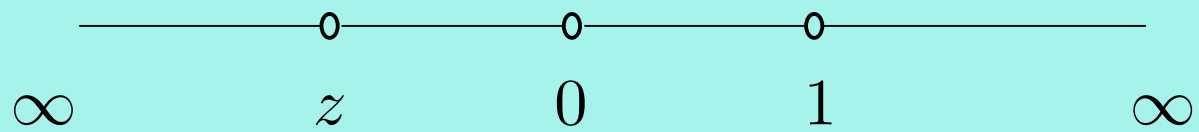
on

$$T = \mathbb{C}^n \setminus \bigcup_{i=1}^n \{t_i = 0\} \cup \bigcup_{i=1}^{n+1} \{t_i - t_{i-1} = 0\}.$$

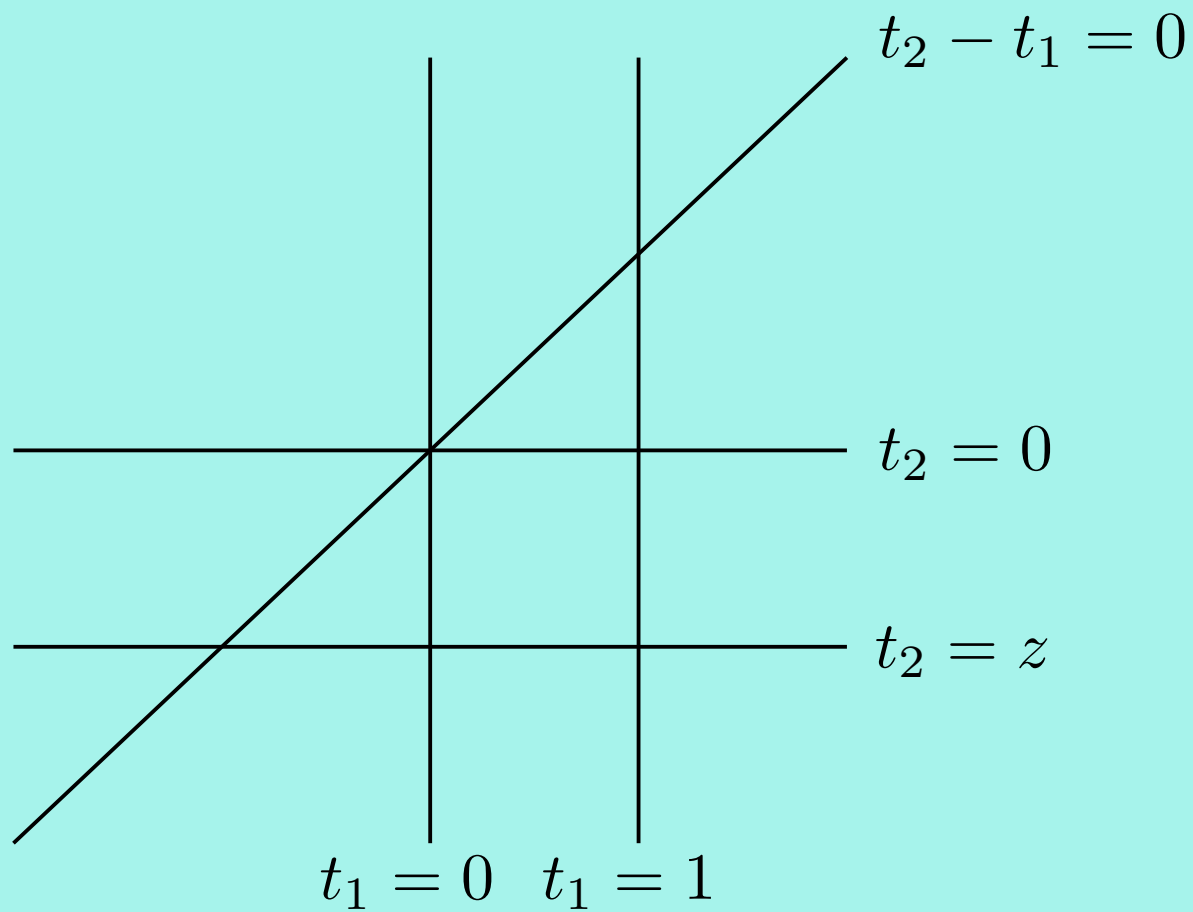
- $\text{rank} H_n^{\text{lf}}(T, \mathcal{L}) = \text{rank} H_n(T, \mathcal{L}) = n + 1.$

In what follows, z is fixed to be $\infty < z < 0$.

$$n = 1$$



$$n = 2$$



- **Integral representations of $f_i^{(0)}(z)$ and $f_i^{(\infty)}(z)$:**

$$\int_{D_i^{(0)}} u_{D_i^{(0)}}(t) dt_1 \cdots dt_n = \prod_{\substack{1 \leq s \leq n+1 \\ s \neq i}} B(\alpha_s - \beta_i + 1, \beta_s - \alpha_s) \times f_i^{(0)}(z),$$

$$\int_{D_i^{(\infty)}} u_{D_i^{(\infty)}}(t) dt_1 \cdots dt_n = \prod_{\substack{1 \leq s \leq n+1 \\ s \neq i}} B(\alpha_i - \beta_s + 1, \beta_s - \alpha_s) \times f_i^{(\infty)}(z),$$

where $u_D(t) = \prod_i (\epsilon_i f_i(t))^{\alpha_i}$ with $\epsilon_i = \pm$ being determined so that $\epsilon_i f_i(t) > 0$ on D for $u(t) = \prod_i f_i(t)^{\alpha_i}$.

• **Bases of $H_n^{\text{lf}}(T, \mathcal{L})$:**

$$\left\{ D_1^{(0)}, D_2^{(0)}, \dots, D_{n+1}^{(0)} \mid D_i^{(0)} = \left(\begin{array}{l} \infty < z < t_n < \dots < t_i < 0 \\ 1 < t_1 < \dots < t_{i-1} < \infty \end{array} \right) \right\},$$

$$\left\{ D_1^{(\infty)}, D_2^{(\infty)}, \dots, D_{n+1}^{(\infty)} \mid D_i^{(\infty)} = \left(\begin{array}{l} \infty < t_i < \dots < t_n < z \\ 0 < t_{i-1} < \dots < t_1 < 1 \end{array} \right) \right\}.$$

$\implies \exists c_{ij}$ such that

$$D_i^{(\infty)} = \sum_{1 \leq j \leq n+1} c_{ij} D_j^{(0)}.$$

Master decomposition formula for cycles:

If $\{C_1, \dots, C_\nu\}$ is a basis of $H_n^{\text{lf}}(T, \mathcal{L})$,

$$C = \sum_{1 \leq j, k \leq \nu} C \bullet C_k \left((C_p \bullet C_q)_{1 \leq p, q \leq \nu}^{-1} \right)_{kj} C_j$$

for $\forall C \in H_n^{\text{lf}}(T, \mathcal{L})$.

\implies

$$D_i^{(\infty)} = \sum_{1 \leq j \leq n+1} c_{ij} D_j^{(0)},$$

where

$$c_{ij} = \sum_{k=1}^{n+1} D_i^{(\infty)} \bullet D_k^{(0)} \left((D_p^{(0)} \bullet D_q^{(0)})_{1 \leq p, q \leq n+1}^{-1} \right)_{kj} \cdot$$

Intersection numbers

The *intersection form* $\bullet : H_m^{\text{lf}}(T, \mathcal{L}) \times H_m^{\text{lf}}(T, \mathcal{L}) \longrightarrow \mathbb{C}$ is the Hermitian form defined by

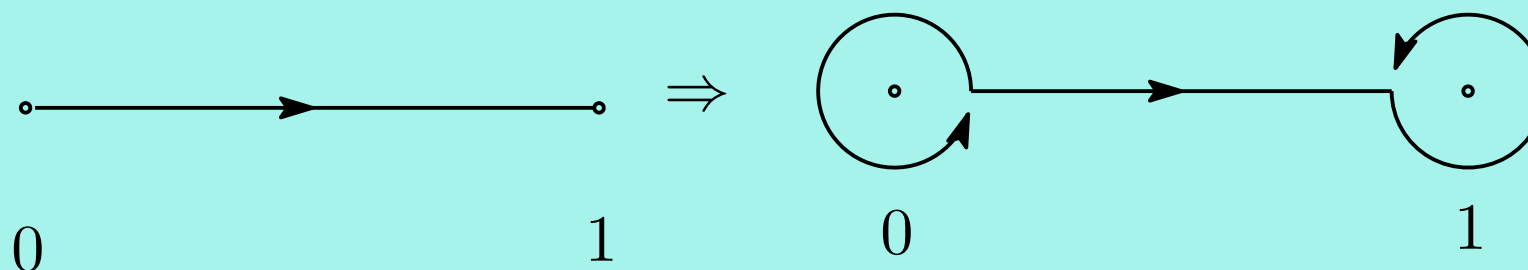
$$(C, C') \longmapsto C \bullet C' = \sum_{\rho, \sigma} a_{\rho} \overline{a'_{\sigma}} \sum_{t \in \rho \cap \sigma} I_t(\rho, \sigma) v_{\rho}(t) \overline{v'_{\sigma}(t)} / |u|^2,$$

for $C, C' \in H_m^{\text{lf}}(T, \mathcal{L})$, if $\text{reg } C = \sum_{\rho} a_{\rho} \rho \otimes v_{\rho}$, $C' = \sum_{\sigma} a'_{\sigma} \sigma \otimes v'_{\sigma}$, where $a_{\rho}, a'_{\sigma} \in \mathbb{C}$, each ρ or σ is an m -simplex, v_{ρ} or v'_{σ} a section of \mathcal{L} on ρ or σ , $\overline{}$ the complex conjugation, $I_t(\rho, \sigma)$ the topological intersection number of ρ and σ at t and the map $\text{reg} : H_m^{\text{lf}}(T, \mathcal{L}) \longrightarrow H_m(T, \mathcal{L})$ is defined as an inverse of the natural map $\iota : H_m(T, \mathcal{L}) \longrightarrow H_m^{\text{lf}}(T, \mathcal{L})$.

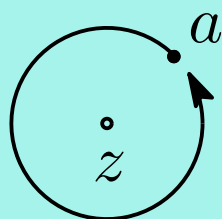
The value $C \bullet C'$ of the intersection form for $C, C' \in H_m^{\text{lf}}(T, \mathcal{L})$ is called the *intersection number* of C and C' .

An example of regularization. $T = \mathbb{C} \setminus \{0, 1\}$, $u(t) = t^\alpha(1-t)^\beta$.

$$\overrightarrow{(0, 1)} \Rightarrow \text{reg } \overrightarrow{(0, 1)} = \left\{ \frac{1}{d_\alpha} S(0; \epsilon) + [\epsilon, 1 - \epsilon] - \frac{1}{d_\beta} S(1; 1 - \epsilon) \right\}$$



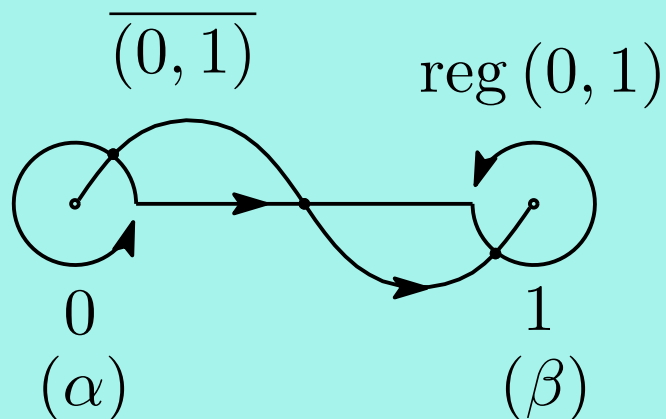
Here $d_a = e(a) - 1$, $e(a) = \exp(2\pi\sqrt{-1}a)$. The symbol $S(z; a)$ stands for the positively oriented circle centered at the point z with starting and ending point a , ϵ is a small positive number and the argument of each factor of $u(t)$ on the oriented circle $S(0; \epsilon)$ or $S(1; 1 - \epsilon)$ is defined so that $\arg t$ takes value from 0 to 2π on $S(0; \epsilon)$, and $\arg(1 - t)$ from 0 to 2π on $S(1 - \epsilon; 1)$.



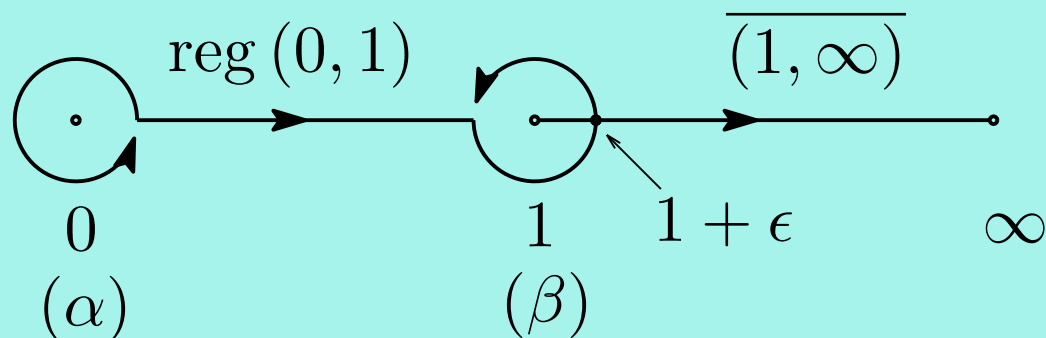
Examples of intersection numbers. $T = \mathbb{C} \setminus \{0, 1\}$, $u(t) = t^\alpha(1-t)^\beta$.

$$\begin{aligned} \overrightarrow{(0, 1)} \bullet \overrightarrow{(0, 1)} &= -\frac{1}{d_\alpha} - 1 + \frac{-1}{d_\beta} \\ &= -\frac{d_{\alpha+\beta}}{d_\alpha d_\beta} = -\frac{s(\alpha + \beta)}{s(\alpha)s(\beta)}, \end{aligned}$$

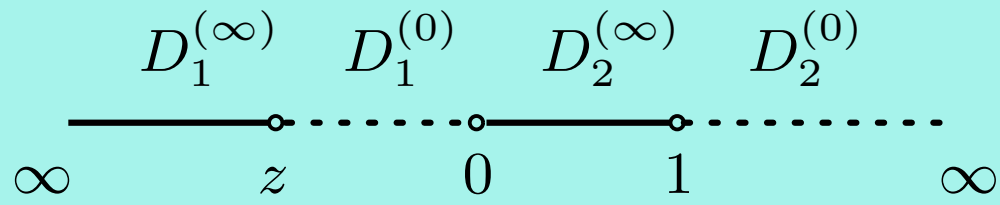
where $s(a) = \sin(\pi a)$.



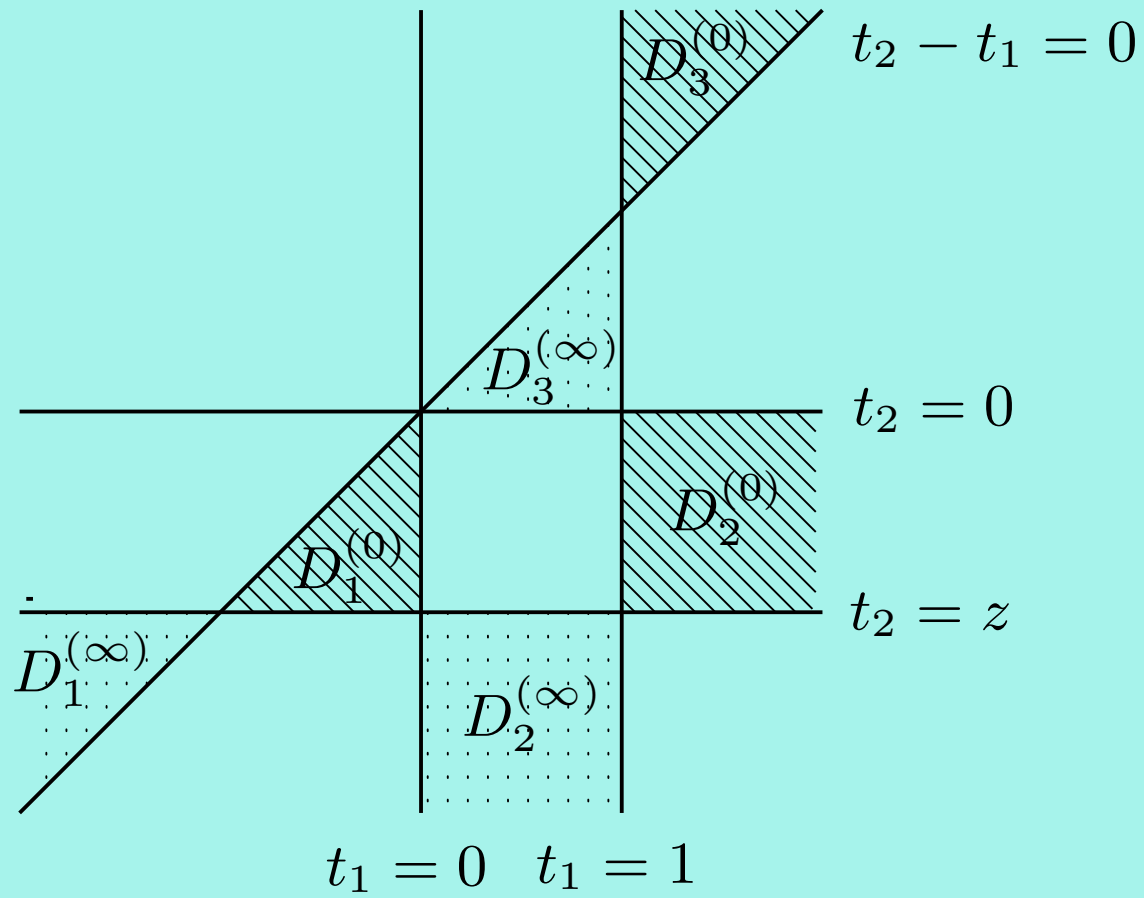
$$\overrightarrow{(0, 1)} \bullet \overrightarrow{(1, \infty)} = \frac{e(\beta/2)}{e(\beta) - 1}.$$



$n = 1$



$n = 2$



$$D_i^{(0)} \bullet D_j^{(0)} = \delta_{ij} \left(\frac{\sqrt{-1}}{2} \right)^n \prod_{\substack{1 \leq s \leq n+1 \\ s \neq j}} \frac{\sin(\beta_s - \beta_j)}{\sin(\beta_s - \alpha_s) \sin(\alpha_s - \beta_j)},$$

$$D_i^{(\infty)} \bullet D_j^{(0)} = \left(\frac{\sqrt{-1}}{2} \right)^n \frac{1}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \leq s \leq n+1 \\ s \neq i, j}} \frac{1}{\sin(\beta_s - \alpha_s)}.$$

$$\Rightarrow c_{ij} = \frac{D_i^{(\infty)} \bullet D_j^{(0)}}{D_j^{(0)} \bullet D_j^{(0)}} = \frac{\sin(\beta_i - \alpha_i)}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \leq s \leq n+1 \\ s \neq j}} \frac{\sin(\alpha_s - \beta_j)}{\sin(\beta_s - \beta_j)}$$

$$\Rightarrow f_i^{(\infty)}(z) = \sum_{j=1}^{n+1} \prod_{s \neq i} \frac{\Gamma(\alpha_i - \alpha_s + 1)}{\Gamma(\beta_j - \alpha_s)} \prod_{s \neq j} \frac{\Gamma(\beta_j - \beta_s)}{\Gamma(\alpha_i - \beta_s + 1)} \times f_j^{(0)}(z)$$

1.2 The Even family and the Odd family

- The characteristic exponents of the Even-family:

$$0, 1, \dots, m-1, a_1, a_1 + 1, \dots, a_1 + m - 2, a_2 \quad \text{at } z = 0,$$

$$0, 1, \dots, m-1, b, b + 1, \dots, b + m - 1, \quad \text{at } z = 1,$$

$$c_1, c_2, \dots, c_{2m}, \quad \text{at } z = \infty.$$

- A connection formula of the Even-Odd family:

$$f_1^{(0)}(z) = \sum_{j=1}^n \frac{\Gamma(1 - a_1 + a_2, 1 + a_2)}{\Gamma(1 - b - c_j, 1 - c_j)} \prod_{\substack{1 \leq s \leq n \\ s \neq j}} \frac{\Gamma(c_s - c_j)}{\Gamma(2 - a_1 - b - c_j - c_s)} \times f_j^{(\infty)}(z),$$

where $f_1^{(0)}(z) = (-z)^{a_2} (1 + O(z))$, $f_j^{(\infty)}(z) = (-z)^{-c_j} (1 + O(z^{-1}))$ and $n = 2m$ in the case of the Even family and $n = 2m + 1$ in the case of the Odd family.

2 Lauricella's differential equations

Lauricella's hypergeometric functions F_D, F_A, F_B are the analytic continuation of Lauricella's hypergeometric series

$$\begin{aligned} &F_D(\alpha, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n) \\ &= \sum_{m_1, \dots, m_n \geq 0} \frac{(\alpha)_{m_1 + \dots + m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1 + \dots + m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}, \\ &F_A(\alpha, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) \\ &= \sum_{m_1, \dots, m_n \geq 0} \frac{(\alpha)_{m_1 + \dots + m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}, \\ &F_B(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n) \\ &= \sum_{m_1, \dots, m_n \geq 0} \frac{(\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1 + \dots + m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}, \end{aligned}$$

where $(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1)$ and $(\alpha)_0 = 1$.

- When $n = 2$, F_D, F_A, F_B are Appell's F_1, F_2, F_3 .
- When $n = 1$, all F_D, F_A, F_B are Gauss HF ${}_2F_1$.

Lauricella's F_D, F_A, F_B satisfy

$$(E_D) \begin{cases} \theta_{x_i}(\theta_{x_1} + \cdots + \theta_{x_n} + \gamma - 1)F = x_i(\theta_{x_i} + \beta_i)(\theta_{x_1} + \cdots + \theta_{x_n} + \alpha)F, \\ x_i(\theta_{x_i} + \beta_i)\theta_{x_j}F = x_j(\theta_{x_j} + \beta_j)\theta_{x_i}F, \end{cases}$$

$$(E_A) \quad \theta_{x_i}(\theta_{x_i} + \gamma_i - 1)F = x_i(\theta_{x_i} + \beta_i)(\theta_{x_1} + \cdots + \theta_{x_n} + \alpha)F,$$

$$(E_B) \quad x_i(\theta_{x_i} + \alpha_i)(\theta_{x_i} + \beta_i)F = \theta_{x_i}(\theta_{x_1} + \cdots + \theta_{x_n} + \gamma - 1)F,$$

where $1 \leq i, j \leq n$ and $\theta_{x_j} = x_j \partial / \partial x_j$.

- Each rank of E_D, E_A, E_B is $n + 1, 2^n, 2^n$.
- When $n = 2$, E_D, E_A, E_B are E_1, E_2, E_3 .
- When $n = 1$, all E_D, E_A, E_B are GHE ${}_2E_1$.

2.1 Lauricella's E_D

- $f_{n+1}^{(1)}(x) = F_D(\alpha, \beta_1, \dots, \beta_n, 1 + \alpha - \gamma + \sum_{i=1}^n \beta_i; 1 - x_1, \dots, 1 - x_n)$ is the holomorphic solution around $(1, \dots, 1)$.
- $\{f_j^{(0)}(x) \mid j = 1, \dots, n + 1\}$ is a fundamental set of solutions around $(0, \dots, 0)$ for $0 < x_1 < \dots < x_n < 1$. Here $f_{n+1}^{(0)}(x) = F_D(\alpha, \beta_1, \dots, \beta_n, \gamma; x_1, \dots, x_n)$ and $f_j^{(0)}(x) = x_j^{1-\gamma+\sum_{i=j+1}^n \beta_i} \prod_{i=j+1}^n x_i^{-\beta_i} \times F_{D,j} \left(\underline{1 - \gamma + \sum_{i=j}^n \beta_i}, \beta_1, \dots, \beta_{j-1}, 1 + \alpha - \gamma, \beta_{j+1}, \dots, \beta_n, \underline{2 - \gamma + \sum_{i=j+1}^n \beta_i}; \frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_j}{x_{j+1}}, \dots, \frac{x_j}{x_n} \right)$, and $F_{D,j}(\underline{\alpha}, \beta_1, \dots, \beta_n, \underline{\gamma}; x_1, \dots, x_n) = \sum_{m_1, \dots, m_n \geq 0} \frac{(\alpha)_{-m_1 - \dots - m_{j-1} + m_j + \dots + m_n} (\beta_1)_{m_1} \dots (\beta_n)_{m_n}}{(\gamma)_{-m_1 - \dots - m_{j-1} + m_j + \dots + m_n} m_1! \dots m_n!} x_1^{m_1} \dots x_n^{m_n}$, where $(A)_m = \Gamma(A + m)/\Gamma(A)$ for any integer $m \in \mathbb{Z}$.

- A connection formula is

$$\begin{aligned}
& f_{n+1}^{(1)}(x) \\
&= \sum_{j=1}^n \frac{\Gamma(-1 + \gamma - \sum_{i=j+1}^n \beta_i, 1 - \gamma + \sum_{i=j}^n \beta_i, 1 + \alpha - \gamma + \sum_{i=1}^n \beta_i)}{\Gamma(\alpha, \beta_j, 1 - \gamma + \sum_{i=1}^n \beta_i)} \\
&\times f_j^{(0)}(x) \\
&+ \frac{\Gamma(1 - \gamma, 1 + \alpha - \gamma + \sum_{i=1}^n \beta_i)}{\Gamma(1 + \alpha - \gamma, 1 - \gamma + \sum_{i=1}^n \beta_i)} \times f_{n+1}^{(0)}(x),
\end{aligned}$$

where $\Gamma(A_1, \dots, A_l) = \prod_{j=1}^l \Gamma(A_j)$.

Derivation by use of the intersection numbers:

$H_1^{\text{lf}}(T, \mathcal{L})$ or $H_1(T, \mathcal{L})$, where \mathcal{L} is determined by

$$\begin{aligned} u(t) &= t^{\beta_1 + \cdots + \beta_n - \gamma} (t - 1)^{\gamma - \alpha - 1} \prod_{j=1}^n (t - x_j)^{-\beta_j} \\ &= t^{\mu_0} (t - 1)^{\mu_{n+1}} \prod_{j=1}^n (t - x_j)^{\mu_j} \end{aligned}$$

on $T = \mathbb{C} \setminus \{x_0, x_1, \dots, x_{n+1}\}$.

• $\text{rank} H_1^{\text{lf}}(T, \mathcal{L}) = \text{rank} H_1(T, \mathcal{L}) = n + 1$.

In what follows, $x_0 < x_1 < \cdots < x_{n+1} = 1$ are fixed.

•Integral representations:

For $l = 1, \dots, n + 1$,

$$\int_{(\{x_0, \dots, x_{l-1}\}, x_l)} u_{(x_{l-1}, x_l)}(t) dt = (\text{const.}) \times f_l^{(0)}(x)$$

and $\int_{-\infty}^0 u_{(-\infty, 0)}(t) dt = (\text{const.}) \times f_{n+1}^{(1)}(x).$

$\{ (\{x_0, \dots, x_{l-1}\}, x_l) \mid l = 1, \dots, n + 1 \}$: a basis of $H_1^{\text{lf}}(T_x, \mathcal{L})$,

where

$$\begin{aligned}
 & (\{x_0, \dots, x_{l-1}\}, x_l) \\
 &= \frac{1}{d_{00tno01\dots l-1}} S(x_0; x_{l-1} + \epsilon) \otimes u_{(x_{l-1}, x_l)}(t) + \overrightarrow{[x_{l-1} + \epsilon, x_l]} \otimes u_{(x_{l-1}, x_l)}(t)
 \end{aligned}$$

$$= \frac{1}{d_{01\dots l-1}} \left\{ \begin{array}{c} \text{A circle with a counter-clockwise arrow. Inside the circle, there are dots representing points } x_0, \dots, x_{l-1}. \end{array} \right\} \otimes u_{(x_{l-1}, x_l)}(t)$$

$$+ \left\{ \begin{array}{c} \text{A horizontal arrow pointing from } x_{l-1} + \epsilon \text{ to } x_l. \end{array} \right\} \otimes u_{(x_{l-1}, x_l)}(t),$$

where $\mu_{ij\dots k} = \mu_i + \mu_j + \dots + \mu_k$ and $d_{ij\dots k} = e(\mu_{ij\dots k}) - 1$.

$\{ (\{x_0, \dots, x_{l-1}\}, x_l) \mid l = 1, \dots, n + 1 \}$: a basis of $H_1^{\text{lf}}(T_x, \mathcal{L})$

\Downarrow

$$\exists q_l \in \mathbb{C} \quad \text{s.t.} \quad (-\infty, 0) = \sum_{l=1}^{n+1} q_l (\{x_0, \dots, x_{l-1}\}, x_l).$$

$\{ (\{x_0, \dots, x_{l-1}\}, x_l) \mid l = 1, \dots, n+1 \}$: a basis of $H_1^{\text{lf}}(T_x, \mathcal{L})$

\Downarrow

$$\exists q_l \in \mathbb{C} \quad \text{s.t.} \quad (-\infty, 0) = \sum_{l=1}^{n+1} q_l (\{x_0, \dots, x_{l-1}\}, x_l).$$

• Intersection numbers:

$$(-\infty, 0) \bullet (\{x_0, \dots, x_{m-1}\}, x_m) = \frac{1}{\langle e_{01\dots m-1} \rangle},$$

$$(\{x_0, \dots, x_{l-1}\}, x_l) \bullet (\{z_0, \dots, z_{m-1}\}, z_m) = -\delta_{l,m} \frac{d_{01\dots m}}{d_{01\dots m-1} d_m}.$$

$\{ (\{x_0, \dots, x_{l-1}\}, x_l) \mid l = 1, \dots, n+1 \}$: a basis of $H_1^{\text{lf}}(T_x, \mathcal{L})$

\Downarrow

$$\exists q_l \in \mathbb{C} \quad \text{s.t.} \quad (-\infty, 0) \otimes u_{0-\epsilon}(t) = \sum_{l=1}^{n+1} q_l (\{x_0, \dots, x_{l-1}\}, x_l).$$

• Intersection numbers:

$$(-\infty, 0) \bullet (\{x_0, \dots, x_{m-1}\}, x_m) = \frac{1}{\langle e_{01\dots m-1} \rangle},$$

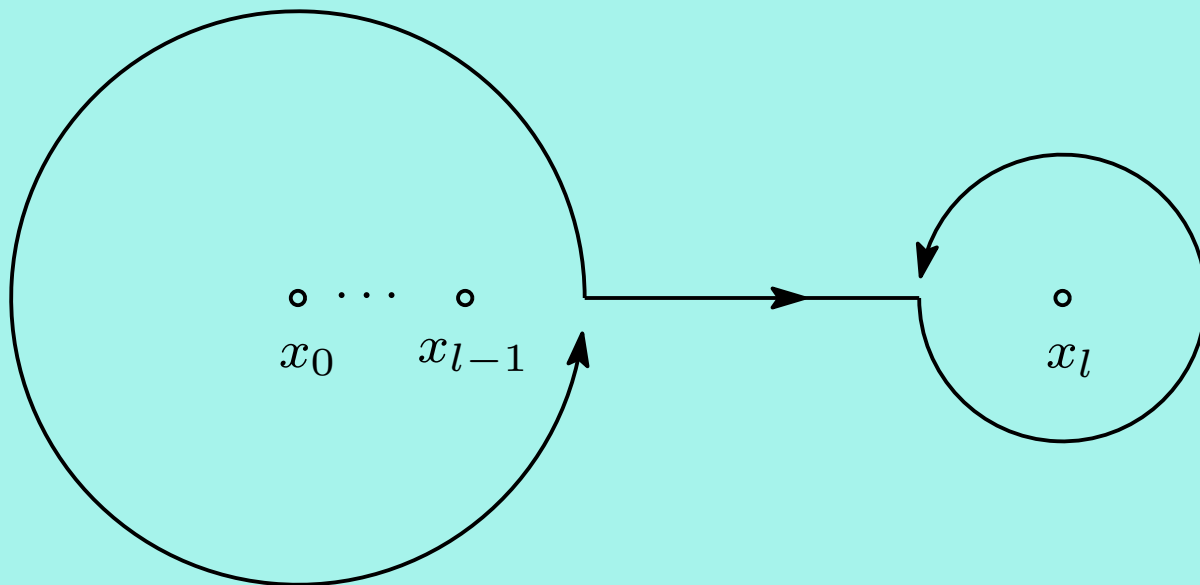
$$(\{x_0, \dots, x_{l-1}\}, x_l) \bullet (\{z_0, \dots, z_{m-1}\}, z_m) = -\delta_{l,m} \frac{d_{01\dots m}}{d_{01\dots m-1} d_m}.$$

\Downarrow

$$q_l = -\frac{\langle e_l \rangle}{\langle e_{01\dots l} \rangle}$$

The regularization or the compactification of $(\{x_0, \dots, x_{l-1}\}, x_l)$:

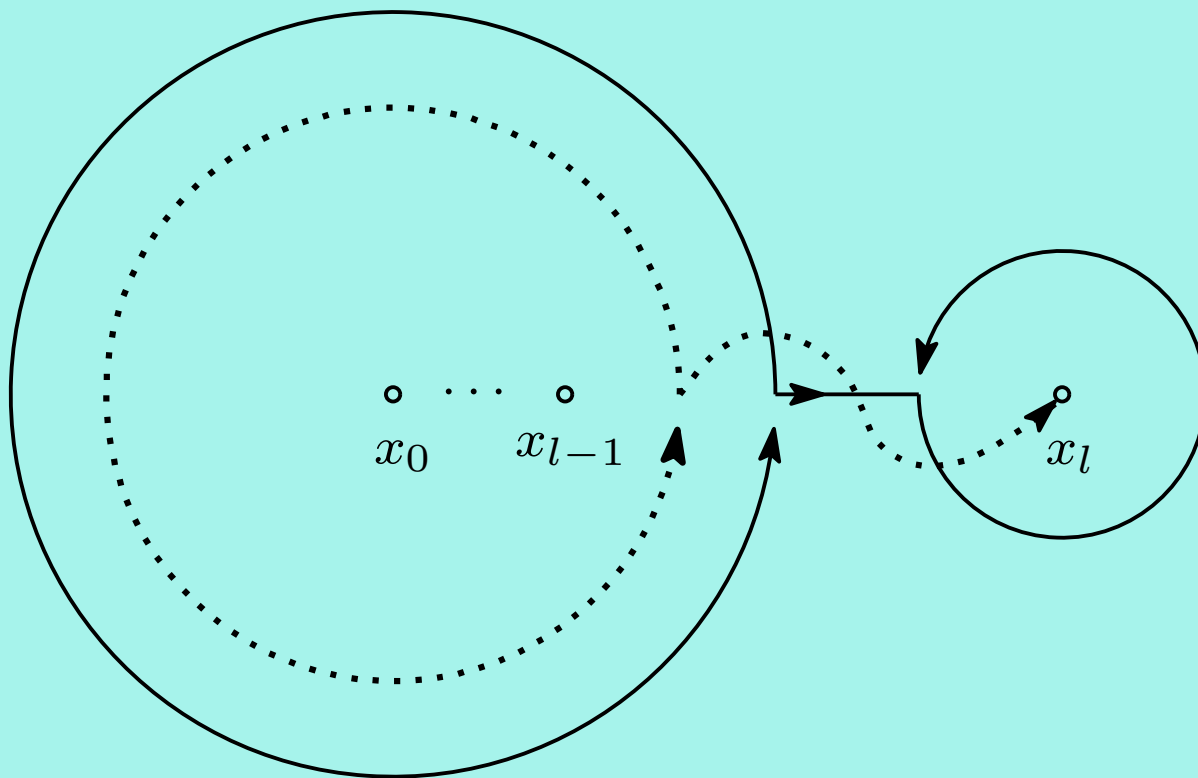
$$\text{reg}(\{x_0, \dots, x_{l-1}\}, x_l) = \frac{1}{d_{01\dots l-1}} S(x_0; x_{l-1} + \epsilon) \otimes u_{x_{l-1} + \epsilon}(t) \\ + \overrightarrow{[x_{l-1} + \epsilon, x_l - \epsilon]} \otimes u_{x_{l-1} + \epsilon}(t) - \frac{1}{d_l} S(x_l; x_l - \epsilon) \otimes u_{x_l - \epsilon}(t).$$



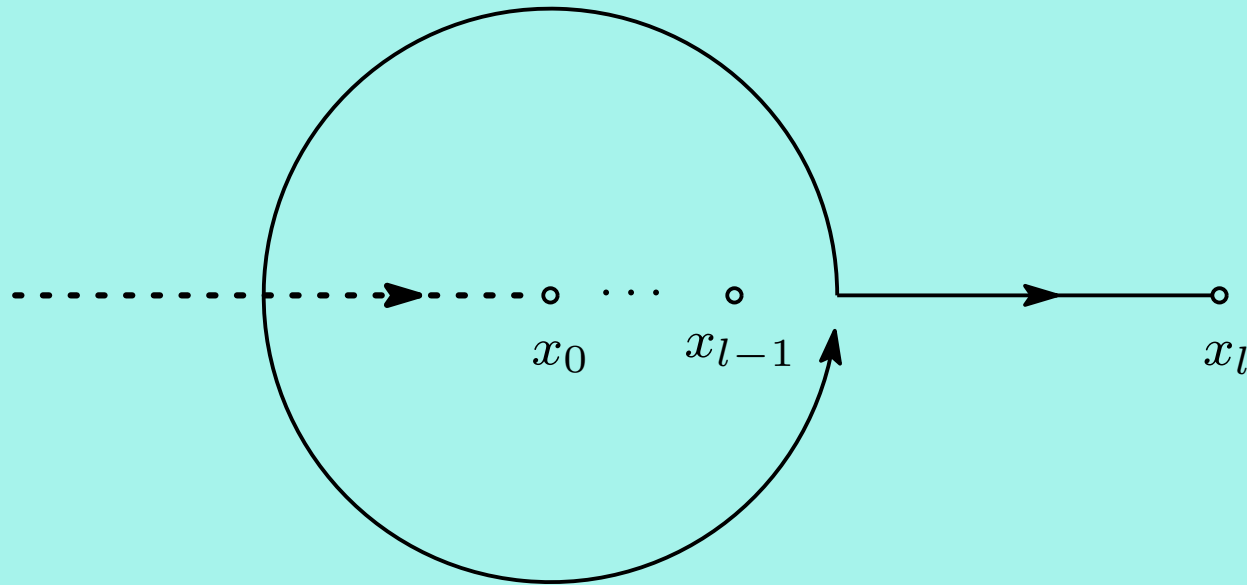
The cycle $\text{reg}(\{x_0, \dots, x_{l-1}\}, x_l)$

The self-intersection number of $(\{x_0, \dots, x_{l-1}\}, x_l)$:

$$\begin{aligned} & (\{x_0, \dots, x_{l-1}\}, x_l) \bullet (\{x_0, \dots, x_{l-1}\}, x_l) \\ &= -\frac{1}{d_{01\dots l-1}} - 1 + \frac{-1}{d_l} = -\frac{d_{01\dots l}}{d_{01\dots l-1}d_l}. \end{aligned}$$



$$\{(-\infty, 0) \otimes u_{0-\epsilon}\} \bullet (\{x_0, \dots, x_{l-1}\}, x_l) = \frac{1}{\langle e_{01\dots l-1} \rangle},$$



and moreover

$$(\{x_0, \dots, x_{l-1}\}, x_l) \bullet (\{x_0, \dots, x_{m-1}\}, x_m) = -\delta_{l,m} \frac{d_{01\dots l}}{d_{01\dots l-1} d_l}.$$

2.2 Lauricella's E_A

$$\begin{aligned}
& \prod_{i=1}^n (-x_i)^{-\beta_i} F_B \left(\begin{matrix} 1 + \beta_1 - \gamma_1, \dots, 1 + \beta_n - \gamma_n; \beta_1, \dots, \beta_n \\ 1 - \alpha + \sum_{i=1}^n \beta_i \end{matrix} ; \frac{1}{x_1}, \dots, \frac{1}{x_n} \right) \\
&= \sum_{r=0}^n \sum_{\substack{i_1 < \dots < i_r \\ j_1 < \dots < j_{n-r}}} \frac{\Gamma(1 - \alpha + \sum_{i=1}^n \beta_i)}{\Gamma(1 - \alpha + \sum_{p=1}^r (\gamma_{i_p} - 1))} \prod_{p=1}^r \frac{\Gamma(\gamma_{i_p} - 1)}{\Gamma(\beta_{i_p})} \prod_{q=1}^{n-r} \frac{\Gamma(\gamma_{j_q} - 1)}{\Gamma(1 + \beta_{j_q} - \gamma_{j_q})} \\
&\times \prod_{p=1}^r (-x_{i_p})^{1 - c_{i_p}} \\
&\times F_A \left(\begin{matrix} \alpha - \sum_{p=1}^r (\gamma_{i_p} - 1); 1 + \beta_{i_1} - \gamma_{i_1}, \dots, 1 + \beta_{i_r} - \gamma_{i_r}, \beta_{j_1}, \dots, \beta_{j_{n-r}} \\ 2 - \gamma_{i_1}, \dots, 2 - \gamma_{i_p}, \gamma_{j_1}, \dots, \gamma_{j_{n-r}} \\ x_{i_1}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_q} \end{matrix} \right),
\end{aligned}$$

where $\{i_1 < \dots < i_r\} \cup \{j_1 < \dots < j_{n-r}\} = \{1, \dots, n\}$.

- For any cycle C ,

$$\int_C \left(1 - \sum_{j=1}^n x_j t_j\right)^{-\alpha} \prod_{1 \leq i \leq n} t_i^{\beta_i - 1} (1 - t_i)^{\gamma_i - \beta_i - 1} dt_1 \cdots dt_n$$

gives a solution of E_A (Mimachi-Noumi (2016)).

- $$\prod_{j=1}^n \frac{\Gamma(\beta_j)\Gamma(\gamma_j - \beta_j)}{\Gamma(\gamma_j)} F_A \left(\begin{matrix} \alpha, \beta_1, \dots, \beta_n \\ \gamma_1, \dots, \gamma_n \end{matrix} ; x_1, \dots, x_n \right)$$

$$= \int_{(0,1)^n} \left(1 - \sum_{j=1}^n x_j t_j\right)^{-\alpha} \prod_{1 \leq i \leq n} t_i^{\beta_i - 1} (1 - t_i)^{\gamma_i - \beta_i - 1} dt_1 \cdots dt_n.$$

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Grazie mille!

Fine