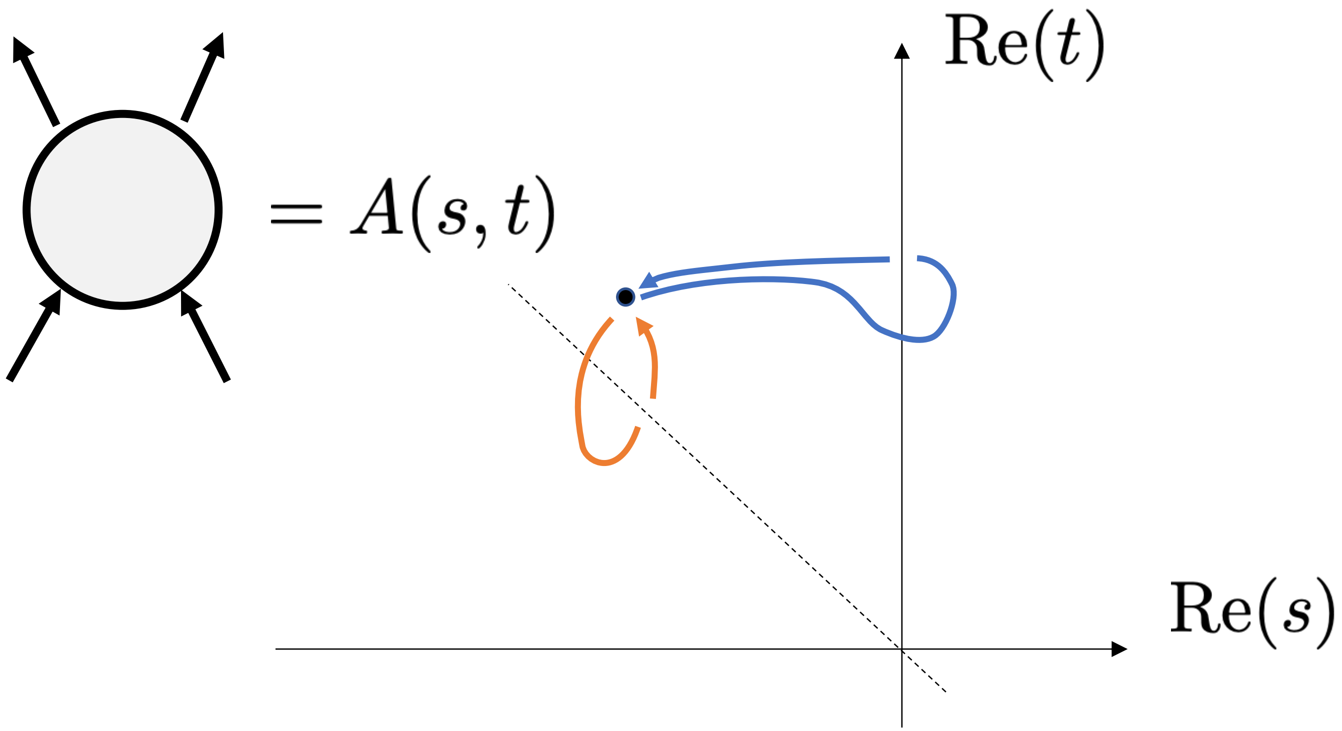
Status of Intersection Theory and Feynman Integrals

Sebastian Mizera

Scattering amplitudes are functions of kinematic variables, e.g., $A(s,t)$

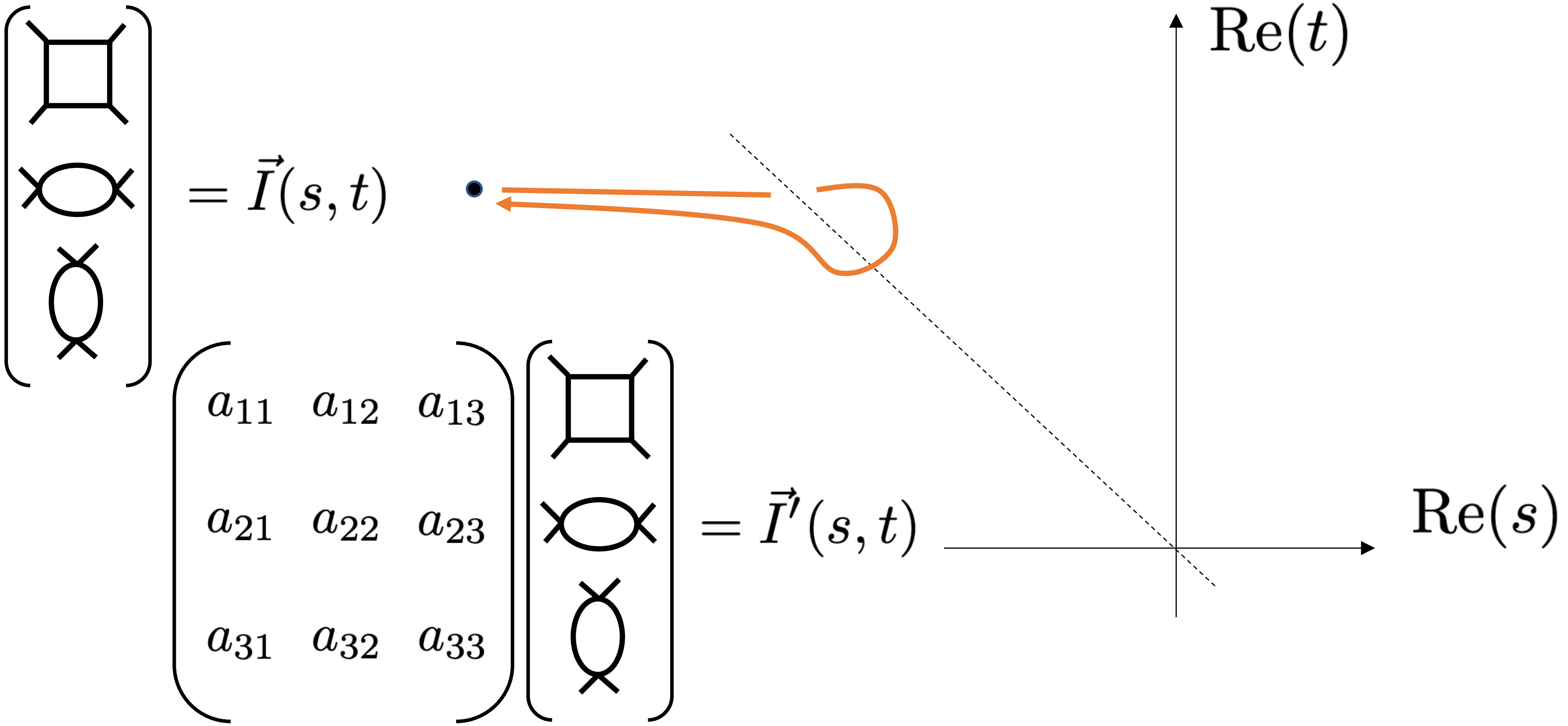


We would like to understand analytic properties of such functions, in particular branch cut structure of the kinematic space, discontinuities, etc.

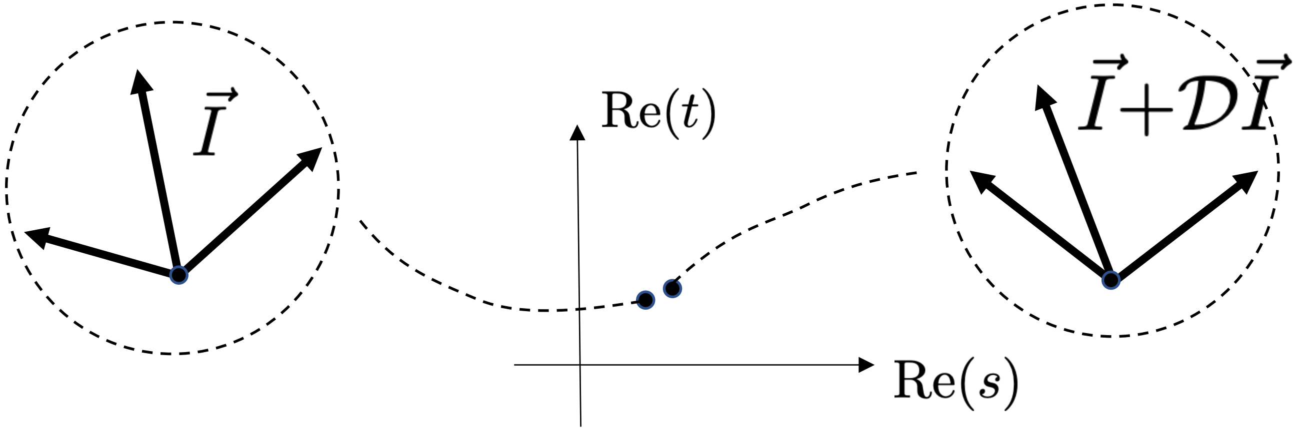
We still don’t know a general answer to such questions [Eden et al. ‘60s]

Some simplifications:

- Scattering amplitudes in perturbation theory (fixed number of loops L)
- Individual families of scalar Feynman integrals (common set of propagators)
- Dimensional regularization (space-time dimension $D = 4 - 2\varepsilon, \quad \varepsilon \ll 1$)

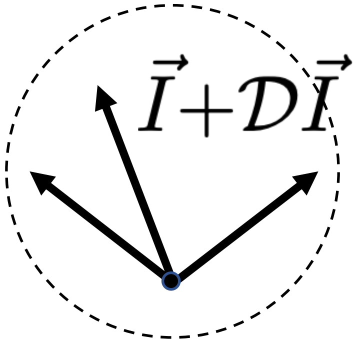


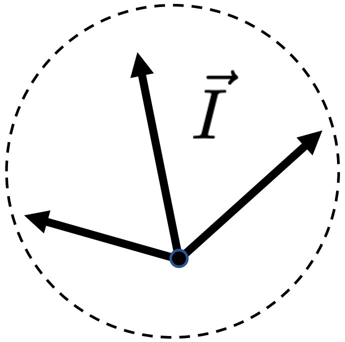
Hence we should really think of Feynman integrals as sections of a flat vector bundle over the kinematic space, locally $(s,t) \times V$ with $\vec{I} \in V$

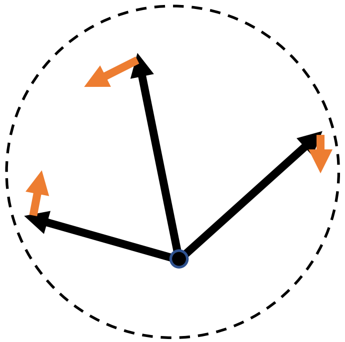


$$\mathcal{D} = ds\,\frac{\partial}{\partial s} + dt\,\frac{\partial}{\partial t}$$

If we knew V and its dual V^* then we can use linear algebra to find the rotation matrix:

$\mathcal{D}\vec{I} =$ 

$-$ 

$=$ 
$= \underbrace{\langle \mathcal{D}\vec{I} | \vec{I}^\vee \rangle}_{\mathbf{\Omega}}\, \vec{I}$

Infinitesimally governed by differential equations

[rich literature: Kotikov, Remiddi, Gehrmann, Henn,...]

$$(\mathcal{D} - \mathbf{\Omega})\vec{I} = 0$$

where $\mathbf{\Omega}$ is a $(\dim V)\times(\dim V)$ matrix-valued one-form subject to integrability constraints:

$$\mathcal{D}\mathbf{\Omega} - \mathbf{\Omega} \wedge \mathbf{\Omega} = 0$$

Typically a polynomial in ε:

$$\mathbf{\Omega} = \sum_{k=0}^{k_{\max}} \varepsilon^k \, \mathbf{\Omega}_{(k)}$$

[Henn, Smirnov, …]
[talks by Henn, Herrmann]

To understand this better we need to address the questions:

- What is the vector space V ?
- What is the dual vector space V^*?
- What is the scalar product $V \times V^* \to \mathbb{C}$?

[Mastrolia, SM ‘18]
[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, SM ‘19]

Let us briefly review the definition of a single Feynman integral,

$$I_i = \int_\Gamma e^{\varepsilon W} \varphi_i$$

W is universal for a family of Feynman integrals and can have different meanings:

- Loop-momentum representation: $W = \log(\text{momenta in the } -2\varepsilon \text{ dimensions})$
- Baikov representation: $W = \log(\text{Baikov polynomial})$
- Feynman parametrization: $W = \log(\mathcal{F}+\mathcal{U})$,

$$\varphi_{n_1,n_2,\ldots,n_P} = \frac{1}{(\mathcal{F}+\mathcal{U})^2} \bigwedge_{a=1}^{P} \frac{dz_a}{z_a^{1-n_a}}$$

$$\Gamma = \mathbb{R}_+^P$$

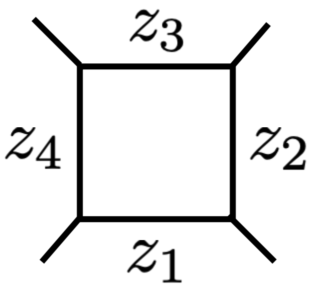
$n_a \in \mathbb{Z}$

$P = (\#\text{ of propagators})$

Symanzik polynomials

[this version popularized by Lee, Pomeransky ‘13]

Example:

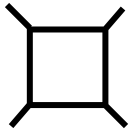


$$P = 4$$

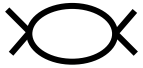
$$\Gamma = \mathbb{R}^4_+$$

$$\mathcal{F} = s\, z_1 z_2 + t\, z_2 z_4$$

$$\mathcal{U} = z_1 + z_2 + z_3 + z_4$$

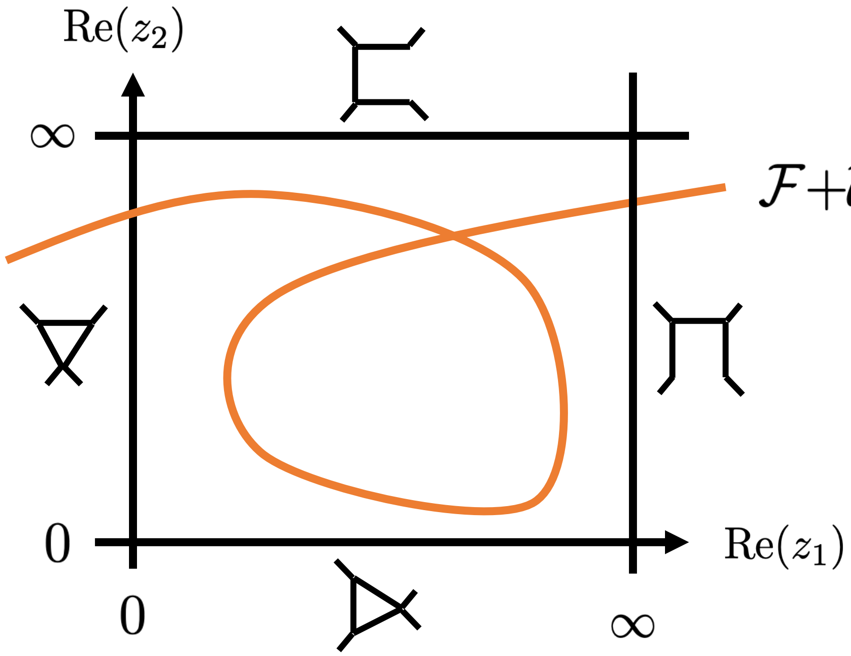


$$\varphi_{1,1,1,1} = \frac{1}{(\mathcal{F}+\mathcal{U})^2} d^4 z$$



$$\varphi_{1,0,1,0} = \frac{1}{(\mathcal{F}+\mathcal{U})^2} \frac{d^4 z}{z_2 z_4}$$

Defined on the moduli space of Riemannian metrics on a graph:



The idea of Aomoto, Gelfand: treat such integrals as pairings between twisted homology and cohomology classes

[Aomoto, Gelfand 70-80's]
[talks by Aomoto, Mimachi, Yoshida, Matsubara-Heo]

$$\langle \Gamma \otimes e^{\varepsilon W} | \varphi \rangle = \int_\Gamma e^{\varepsilon W} \varphi$$

Broadly speaking, twisted cohomology is the space of integrands φ up to integration-by-parts:

$$\int_\Gamma e^{\varepsilon W} \varphi = \int_\Gamma e^{\varepsilon W} \varphi + d\left(e^{\varepsilon W} \xi\right) = \int_\Gamma e^{\varepsilon W} \left(\varphi + \underbrace{(d + dW \wedge)}_{\nabla_{dW}} \xi\right)$$

This is *almost* what we want, except for boundary terms at $\{z_a = 0, \infty\}$

Feynman integrals in dimensional regularization should be defined with a cohomology twisted along $\{\mathcal{F}+\mathcal{U}=0\}$ and relative to $\{z_a=0,\infty\}$

[Matsumoto '18]
[talk by Caron-Huot]

However, we can simplify our life a bit by regulating the "relative" boundaries

$$n_a \to n_a + \varepsilon\delta_a$$

so that $W=\log(\mathcal{F}+\mathcal{U})+\sum_{a=1}^{P}\delta_a\log z_a$ and take $\delta_a\to 0$ at the end

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, SM '19]
[SM, Pokraka '19]

Hence we obtain our model for the vector space V:

$$\langle \underbrace{\mathbb{R}_+^P \otimes e^{\varepsilon W}}_{\text{fixed}} \,|\, \varphi_{n_1,n_2,\ldots,n_P} \rangle = \int_{\mathbb{R}_+^P} e^{\varepsilon W} \varphi_{n_1,n_2,\ldots,n_P}$$

Feynman integrals are prescribed by twisted cohomology classes:

$$[\varphi] : \; \varphi \sim \varphi + \nabla_{dW} \xi \qquad \in \; H^P_{dW} := H^P(\underbrace{(\mathbb{C}^*)^P - \{\mathcal{F} + \mathcal{U} = 0\}}_{M}, \nabla_{dW})$$

specific representative: “twisted form”

This motivates the identification $V \cong H^P_{dW}$

Vanishing theorem: H^k_{dW} non-trivial only for $k = P$ [Aomoto '75]

(assuming genericity conditions on the twisting)

In particular, the Euler characteristic equals

$$\chi(M) = \sum_k (-1)^k \dim H^k_{dW} = (-1)^P \dim H^P_{dW}$$

And therefore $\dim V = |\chi(M)|$

[Mastrolia, SM '18]
[by other techniques also
Bitoun, Bogner, Klausen, Panzer '17]

This result is significant because Euler characteristics can be computed in many different ways

In particular, treating $\mathrm{Re}(W)$ as a Morse function we find

$$\chi(M) = \sum_k (-1)^k (\#\text{ of critical points with index } k)$$
$$= (-1)^P (\#\text{ of critical points})$$

Since all critical points are saddles ($k{=}P$) for holomorphic W

[talks by Mastrolia, Frellesvig, Mandal, Laporta]

Different ways: Triangulations, Newton polytopes, finite fields,
Chern-Schwartz-MacPherson classes, …

[Aluffi, Marcolli, … 00-10’s]

[talk by Aluffi]

often in D=4 for “graph hypersurface” $\{\mathcal{U} = 0\}$

What is $\dim V$ actually computing? By definition rank of the period matrix:

$$\dim V = \mathrm{rank} \begin{pmatrix} \boxed{\int_{\Gamma_1} e^{\varepsilon W} \varphi_1 \quad \int_{\Gamma_1} e^{\varepsilon W} \varphi_2 \quad \cdots} \\ \int_{\Gamma_2} e^{\varepsilon W} \varphi_1 \quad \int_{\Gamma_2} e^{\varepsilon W} \varphi_2 \quad \cdots \\ \vdots \qquad\qquad \vdots \qquad\qquad \ddots \end{pmatrix} \rightarrow$$

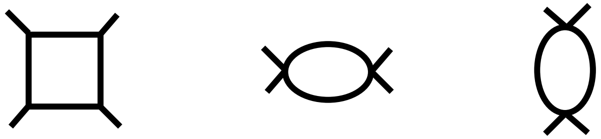
fixed

$\Gamma_1 = \mathbb{R}_+^P$

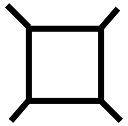
Hence the number of linearly-independent Feynman integrals might in principle be *smaller* than $\dim V$ (cf. sector symmetries)

Note for mathematicians: often term “master integrals” is used for basis of Feynman integrals, though it has different meanings

- The basis of Feynman integrals in box topology is $|\chi(M)| = 3$, e.g.,



- Requiring support on the maximal cut surface (top sector) gives 1 basis integral, e.g.,



(obtained by counting critical points of $W(\delta_a = 0)$ [Lee, Pomeransky ‘13])

To summarize, our model for V is given by the cohomology group $H^P_{dW} \ni \varphi_+$

We will take the dual vector space V* to be $H^P_{-dW} \ni \varphi_-$

Scalar product $H^P_{-dW} \times H^P_{dW} \to \mathbb{C}$ is given by intersection numbers:

$$\langle \varphi_- | \varphi_+ \rangle_{dW} = \int_M \varphi_- \wedge \varphi_+^c$$

[Cho, Matsumoto '95]
[Matsumoto '98]
[Deligne, Mostow '86]
[Saito '83]

differential form with compact support

Before spending some time understanding what this formula means, let us recall why we need it

Differential equations for Feynman integrals: [Mastrolia, SM '18]

$$\mathcal{D} \int_{\mathbb{R}_+^P} e^{\varepsilon W} \varphi_i = \int_{\mathbb{R}_+^P} e^{\varepsilon W} \left(\mathcal{D} + \varepsilon \mathcal{D}W\right) \varphi_i = \sum_{j=1}^{|\chi(M)|} \underbrace{\langle \varphi_j^\vee | \left(\mathcal{D} + \varepsilon \mathcal{D}W\right) \varphi_i \rangle_{dW}}_{} \int_{\mathbb{R}_+^P} e^{\varepsilon W} \varphi_j$$

This is $(\mathcal{D} - \mathbf{\Omega})\vec{I} = 0$ with the connection matrix $\mathbf{\Omega}_{ij} = \langle \varphi_j^\vee | \left(\mathcal{D} + \varepsilon \mathcal{D}W\right) \varphi_i \rangle_{dW}$

[number-theoretic version: talks by Britto, Brown]

Similarly, any integration-by-parts identity becomes:

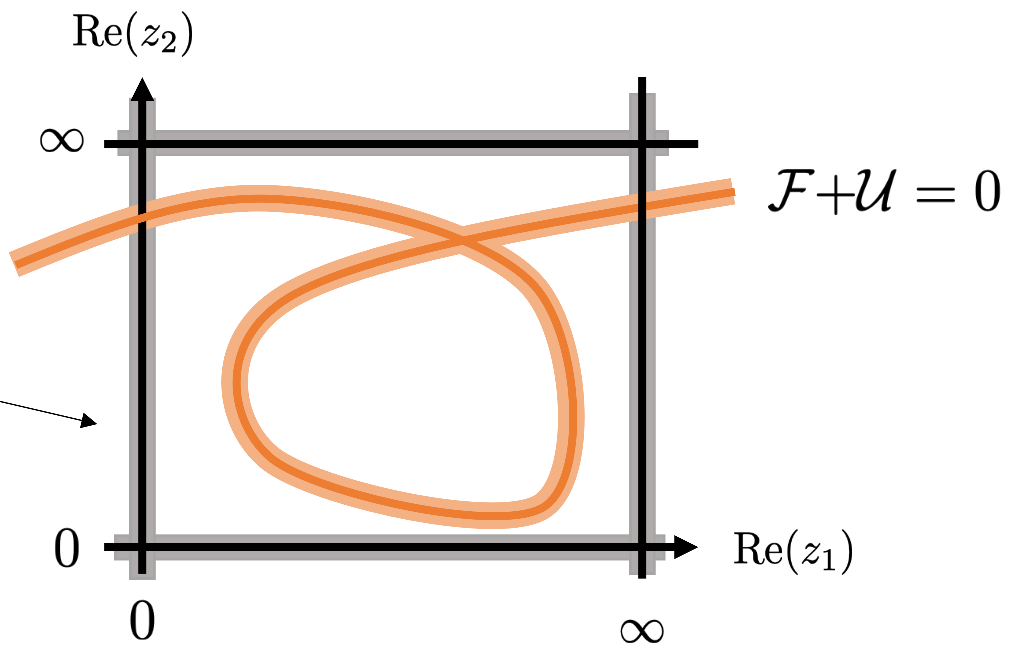
$$\int_{\mathbb{R}_+^P} e^{\varepsilon W} \varphi_i = \sum_{j=1}^{|\chi(M)|} \langle \varphi_j^\vee | \varphi_i \rangle_{dW} \int_{\mathbb{R}_+^P} e^{\varepsilon W} \varphi_j$$

(If we didn't know dual forms $\varphi_j^\vee$, use $\varphi_j^\vee = \sum_{k=1}^{|\chi(M)|} \mathbf{C}_{jk}^{-1} \vartheta_k$ with $\mathbf{C}_{kj} = \langle \vartheta_k | \varphi_j \rangle_{dW}$)

any auxiliary basis

Since $\varphi_- \wedge \varphi_+ = \varphi_- \wedge \varphi_+^c = 0$ in the bulk of M, the integral has to *localize* on the boundaries $\{\mathcal{F}+\mathcal{U}=0\}$ and $\{z_a = 0, \infty\}$

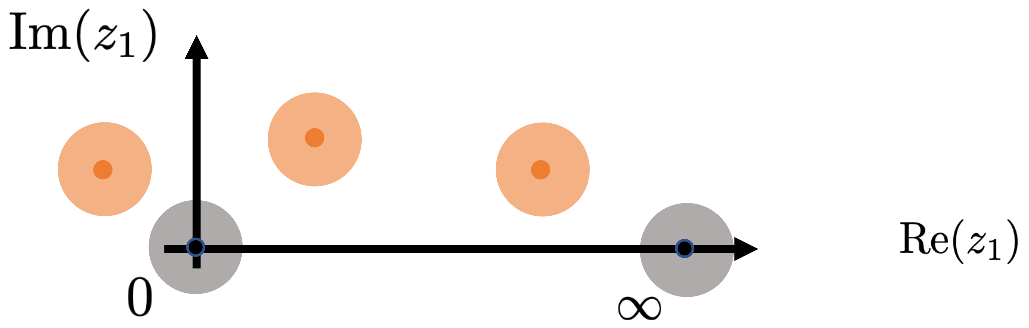
$$\langle \varphi_- | \varphi_+ \rangle_{dW} = \int_{\partial M} \varphi_- \wedge \varphi_+^c$$



That's what we wanted because intersection number should be easier to compute than the full integral

Explicit formulae for intersection numbers:

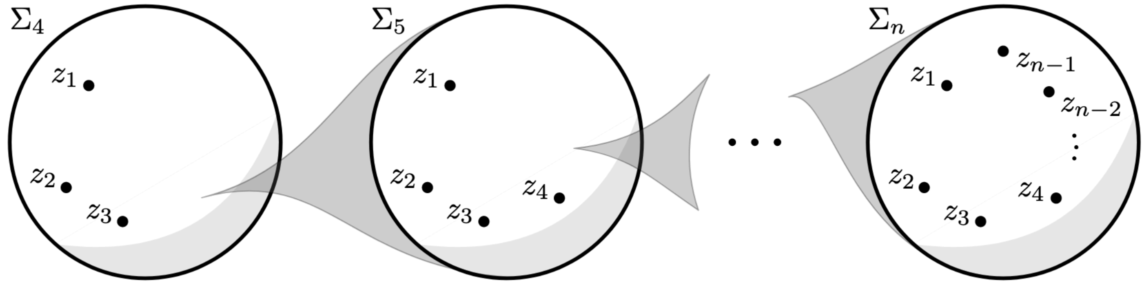
- $\dim_{\mathbb{C}} M = 1$: $\quad \langle \varphi_- | \varphi_+ \rangle_{dW} = \sum_{p \in \partial M} \mathrm{Res}_{z_1 = p} \left(\varphi_- \nabla_{dW}^{-1} \varphi_+ \right)$ [Cho, Matsumoto '95]



- $\dim_{\mathbb{C}} M > 1$: Fiber M into many one-dimensional spaces and apply the above formula recursively

[SM '19]

W on each fiber is promoted to a matrix

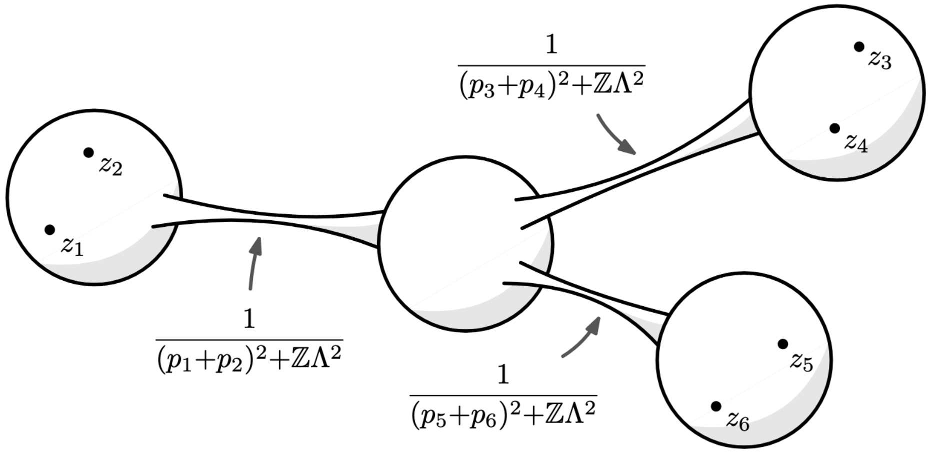


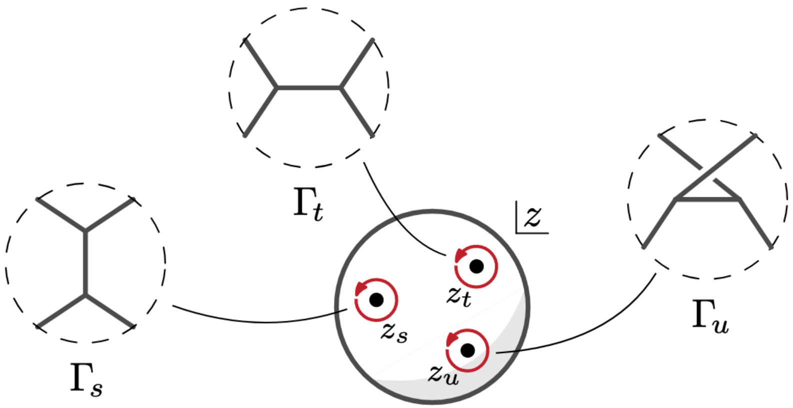
Has some limitations, but currently the most efficient technique

[FGMMMM '19]

[talks by Mastrolia, Frellesvig, Mandal]

Aside: Intersection numbers computed on the moduli space of genus-g Riemann surfaces with n punctures, $\mathcal{M}_{g=0,n}$, give a new definition of tree-level scattering amplitudes (loop-level integrands) of quantum field theories *after* summing over all Feynman diagrams



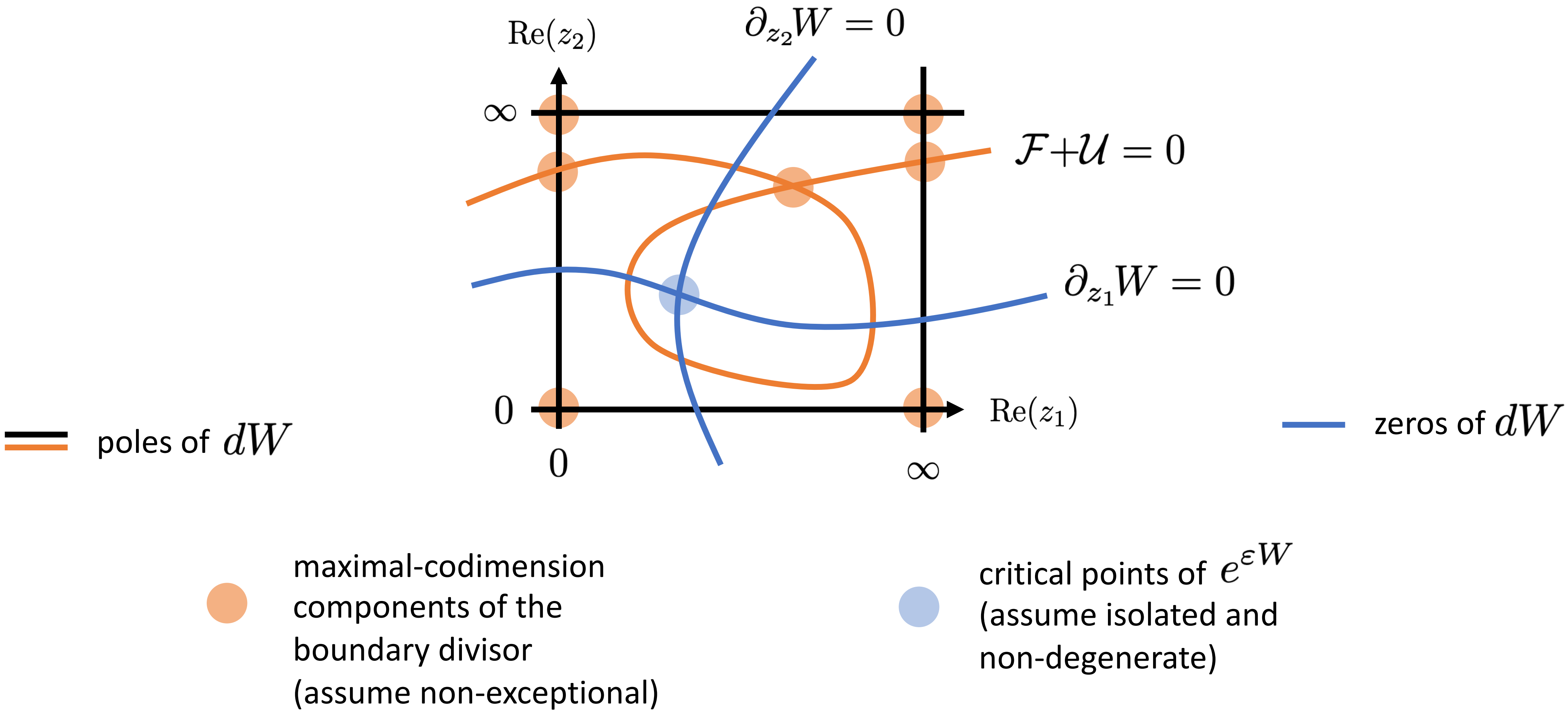


Seems to unify many aspects of scattering amplitudes, such as KLT and BCJ relations, scattering equations, color-kinematics duality, …

[SM '17-19]

[talk by Weinzierl]

In this talk we'll pursue an alternative approach by expanding intersection numbers in $\mathcal{E}$, which localize on two distinct sets of points:



Localization in the two limits [SM ’17-19]
(recall $\mathrm{Res}_p = \oint_{|z_1-p_1|=\epsilon} \oint_{|z_2-p_2|=\epsilon} \cdots \oint_{|z_P-p_P|=\epsilon}$)

$$\langle \varphi_- | \varphi_+ \rangle_{dW} = \sum_{\substack{\text{boundary} \\ \text{points } p \\ =\cap_{i=1}^P H_i}} \frac{\mathrm{Res}_p(\varphi_-)\,\mathrm{Res}_p(\varphi_+)}{\prod_{i=1}^P \mathrm{Res}_{H_i}(dW)} + \mathcal{O}(\varepsilon)$$

independent of ε, e.g., for logarithmic forms [Matsumoto ‘98]

$$\langle \varphi_- | \varphi_+ \rangle_{dW} = \sum_{\substack{\text{critical} \\ \text{points } p \\ =\cap_{i=1}^P \{\partial_i W=0\}}} \mathrm{Res}_p \left(\frac{\varphi_- \, \varphi_+}{d^P z \prod_{i=1}^P \partial_i W} \right) + \mathcal{O}(\varepsilon^{-1})$$

higher residue pairings [Saito 80’s]

Physically the second formula seems completely crazy!
It computes scalar products between Feynman integrals in $D = 4-2\varepsilon \to -\infty$

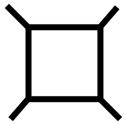
However, we can apply it to computing the connection matrix: [SM, Pokraka '19]

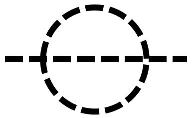
$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_{(0)} + \varepsilon\, \boldsymbol{\Omega}_{(1)} + \ldots + \varepsilon^{k_{\max}} \boldsymbol{\Omega}_{(k_{\max})}$$

expansion in ε

expansion in ε^{-1}

Both expansions truncate and give *exact* answers if the matrix is a polynomial.
Demonstrated on simple examples as proof-of-principle:

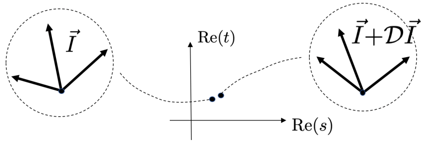




Therefore regions of the loop-momentum space that were previously thought to dominate the $D \to \pm\infty$ asymptotics turns out to know everything about $D \to 4$!

Summary

Hence we should really think of Feynman integrals as sections of a flat vector bundle over the kinematic space, locally $(s,t) \times V$ with $\vec{I} \in V$



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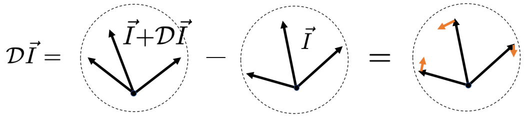

$$= \underbrace{\langle \mathcal{D}\vec{I} | \vec{I}^{\vee} \rangle}_{\boldsymbol{\Omega}}\, \vec{I}$$

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[Mastrolia, SM '18]
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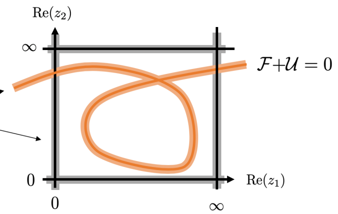

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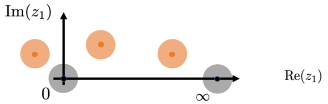

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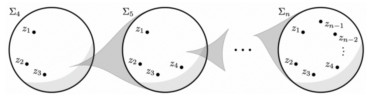

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[talks by Mastrolia, Frellesvig, Mandal]

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expansion in ε expansion in ε^{-1}

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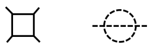

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What next?

- Practical computations: efficient implementations of algorithms computing intersection numbers (recursion relations, $\varepsilon^{\pm 1}$-expansion). Should exploit the combinatorics of Feynman integrals, as opposed to treating them like generic hypergeometric integrals
- More conceptual: relative twisted cohomologies, their intersection pairings, relations to complex Morse theory
- Construction of Poincaré-dual bases of Feynman integrals. Geometric criterion for a “good” basis giving differential equations in a canonical form?
- Combining individual graph topologies to full scattering amplitudes, perhaps using moduli spaces of curves

Grazie!