On the Application of Intersection Theory to Feynman Integrals: The univariate case

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December 18, 2019

The work on Intersection Theory and Feynman Integrals,
that has been centered in Padova.

Three publications so far:

Pierpaolo Mastrolia and Sebastian Mizera,
Feynman Integrals and Intersection Theory,
JHEP **1902** (2019) 139 [arXiv:1810.03818],

HF, F. Gasparotto, S. Laporta, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera,
Decomposition of Feynman integrals on the maximal cut by intersection numbers,
JHEP **1905** (2019) 153 [arXiv:1901.11510],

HF, F. Gasparotto, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera,
Vector Space of Feynman Integrals and Multivariate Intersection Numbers,
PhysRevLett. **123** (2019) 201602 [arXiv:1907.02000].


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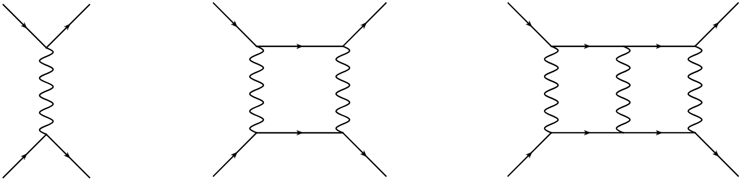


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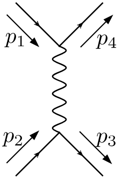
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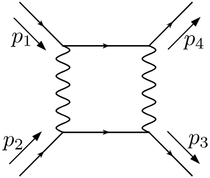
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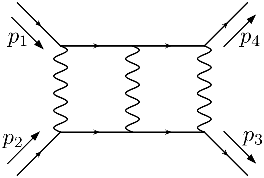
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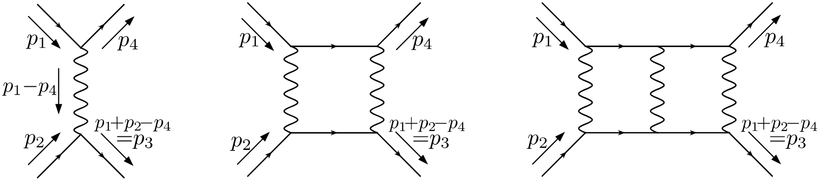
Physics Introduction

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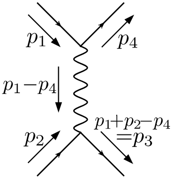
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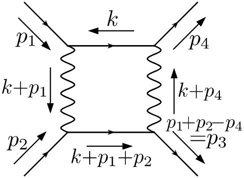
$p_1 + p_2 - p_4$
$= p_3$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$p_1 + p_2 - p_4$
$= p_3$
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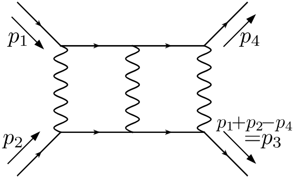
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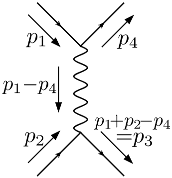

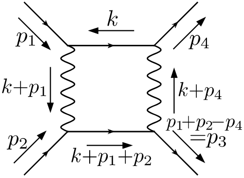

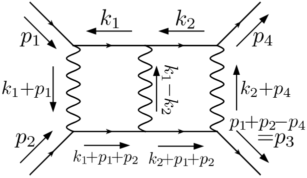

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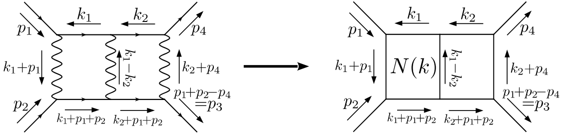
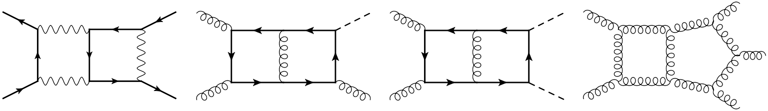


A Feynman integral:

$$I_{a_1 \cdots a_P; \cdots a_n} = \int \frac{\mathrm{d}^d k_i}{\pi^{d/2}} \cdots \int \frac{\mathrm{d}^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) D_2^{a_2}(k) \cdots D_P^{a_P}(k)}$$

The Ds are propagators of the form $D_i = (k+p)^2 - m^2$,
$d = 4 - 2\epsilon$ is the space-time dimensionality,
k and p are d-dimensional momenta (internal and external),
$N(k) = \prod_{i=P+1}^{n} D_i^{a_i}(k)$ is a numerator function,
P is the number of propagators,
L and E are the numbers of loops and (independent) legs,
$n = L(L+1)/2 + EL$ is the number of independent scalar products,
a_i are integer powers.



For state-of-the art two-loop scattering amplitude calculations
Feynman diagrams $\rightarrow \mathcal{O}(10000)$ Feynman integrals

Linear relations bring this down to $\mathcal{O}(100)$ *master integrals*

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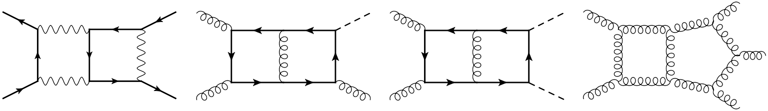

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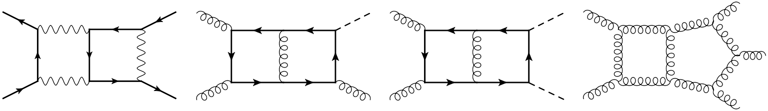
Linear relations may be derived using IBP (integration by part) identities

$$\int \frac{\mathrm{d}^d k}{\pi^{d/2}} \frac{\partial}{\partial k^\mu} \frac{q^\mu N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} = 0$$

Systematic by Laporta's algorithm $\Rightarrow$ Solve a huge linear system.
[Chetyrkin, Tkachov (1981); Laporta (2000)]



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The linear relations are often informally referred to as IBPs as well.

The linear relations form a vector space

$$I = \sum_{i \in \text{masters}} c_i I_i$$

Subsectors are sub-spaces

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Not all vector spaces are *inner product spaces*

$$\langle v| = \sum_i \langle v v_j^* \rangle \left(C^{-1}\right)_{ji} \langle v_i| \qquad \text{with} \qquad C_{ij} = \langle v_i v_j^* \rangle$$

$$= \sum_i c_i \, \langle v_i| \qquad \qquad \left(c_i = \langle v v_i^* \rangle \ \text{ if } \ C_{ij} = \delta_{ij}\right)$$

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If only there were a way to define an inner product for Feynman integrals...

Baikov representation

$$I = \int \frac{\mathrm{d}^d k_1}{\pi^{d/2}} \cdots \int \frac{\mathrm{d}^d k_L}{\pi^{d/2}} \frac{N(k)}{D_1^{a_1}(k) \cdots D_P^{a_P}(k)} \; = \; K \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^\gamma(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

The x_i are Baikov variables, $\mathcal{B}$ is the Baikov Polynomial, $\mathcal{C} = \{\mathcal{B} > 0\}$.

$$n = L(L{+}1)/2{+}EL \qquad\qquad \gamma = (d{-}E{-}L{-}1)/2$$

P. Baikov: *Nucl. Instrum. Meth.A* **389** (1997) 347–349, [hep-ph/9611449]

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The loop-by-loop version of Baikov representation can often decrease n

$$I = \tilde{K} \int_{\mathcal{C}} \mathrm{d}^{\tilde{n}} x \, \frac{\left(\prod_{j=1}^{2L-1} \mathcal{B}_j^{\gamma_j}(x) \right) N(x)}{x_1^{a_1} \cdots x_P^{a_P}}$$

HF and C. Papadopoulos, *JHEP* **04** (2017) 083, [arXiv:1701.07356]

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HF and C. Papadopoulos, *JHEP* **04** (2017) 083, [arXiv:1701.07356]

Baikov representation is suitable for *generalized unitarity cuts*

$\int \mathrm{d}x \to \oint \mathrm{d}x$. Preserve linear relations.

J. Bosma, M. Søgaard, Y. Zhang, *JHEP* **08** (2017) 051, [arXiv:1704.04255]

$$I = \int_{\mathcal{C}} \mathrm{d}^n x \, \frac{\mathcal{B}^{\gamma}(x) N(x)}{x_1^{a_1} \cdots x_P^{a_P}} = \int_{\mathcal{C}} u \phi$$

$u = \mathcal{B}^{\gamma}$ is a multivalued function in $\{x\}$

$\phi = \frac{N(x)}{x_1^{a_1} \cdots x_P^{a_P}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a form

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$\omega = \mathrm{d}\log(u)$ is *the twist*

$\langle \phi | \mathcal{C}]_\omega$ is a pairing of a *twisted cycle* ($\mathcal{C}$) and a *twisted cocycle* (ϕ) (equivalence classes of contours and integrands respectively)

K. Aomoto and M. Kita, *Theory of Hypergeometric Functions*,
P. Mastrolia and S. Mizera, *Feynman Integrals and Intersection Theory*, JHEP **1902** (2019) 139

dim of the set of ϕs, is the number of master integrals.

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From there follows the criterion:
nr. of master integrals $=$ nr. of solutions to "$\omega = 0$"

R. Lee and A. Pomeransky, *JHEP* **11** (2013) 165, [arXiv:1308.6676].

The *intersection number* $\langle \phi | \xi \rangle$ is a pairing of a twisted cocycle ϕ with a *dual* twisted cocycle ξ

Lives up to all criteria for being a scalar product.

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When there is one integration variable z (ϕ and ξ are one-forms)

$$\langle\phi|\xi\rangle_\omega = \sum_{p\in\mathcal{P}} \mathrm{Res}_{z=p}(\psi_p \xi) \qquad (\mathrm{d}+\omega)\psi_p = \phi$$

$\mathcal{P}$ is the set of poles of ω.

$(\mathrm{d}+\omega)\psi_p = \phi$ can be solved with a series ansatz around $z = p$
$\psi_p = \sum_i \psi_p^{(i)}(z-p)^i$. The exact expressions are not needed for the Res.

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References:

K. Cho and K. Matsumoto, *Intersection theory for twisted cohomologies and twisted Riemann's period relations*, Nagoya Math. J. **139** (1995) 67-86

K. Matsumoto, *Intersection numbers for logarithmic k-forms*, Osaka J. Math. **35** (1998) no. 4 873-893

S. Mizera, *Scattering Amplitudes from Intersection Theory*, Phys. Rev. Lett. **120** (2018) no. 14 141602

Summary:

$$I = \sum_{i \in \text{masters}} c_i I_i \quad \Leftrightarrow \quad \langle \phi | \mathcal{C}] = \sum_i c_i \langle \phi_i | \mathcal{C}]$$

with $I = \int_{\mathcal{C}} u\phi$. u is multivalued function,
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solve with series ansatz

Theory

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HF, F. Gasparotto, S. Laporta, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera, *Decomposition of Feynman integrals on the maximal cut by intersection numbers*, JHEP **1905** (2019) 153

HF, F. Gasparotto, M. Mandal, P. Mastrolia, L. Mattiazzi, S. Mizera, *Vector Space of Feynman Integrals and Multivariate Intersection Numbers*, PhysRevLett. **123** (2019) 201602.

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Gauss' Hypergeometric Function

$$ {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 z^{b-1}(1-z)^{c-b-1}(1-xz)^{-a}\mathrm{d}z $$

K. Aomoto and M. Kita, *Theory of Hypergeometric Functions*,
M. Yoshida, *Hypergeometric Funtions, My Love*.

Example: ${}_2F_1$

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Linear relations are known as contiguity relations

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Università degli Studi di Padova

MARIE CURIE ACTIONS

Example: ${}_2F_1$

$$ {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 z^{b-1}(1-z)^{c-b-1}(1-xz)^{-a}\mathrm{d}z $$

$$ {}_2F_1(a,b{+}1,c{+}1;x) = c_1\ {}_2F_1(a,b{+}1,c;x) + c_2\ {}_2F_1(a,b,c;x) $$

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in agreement with known results.

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Also works for the Lauricella F_D-function

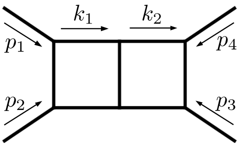
$$ F_D(a,b_1,\cdots,b_m,c;x_1,\ldots,x_m) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int z^{a-1}(1-z)^{c-a-1}\prod_i^m (1-x_i z)^{-b_i}\,\mathrm{d}z $$

Università degli Studi di Padova

Marie Curie Actions

Example: double box

Massless double box:



$$D_1 = k_1^2, \qquad D_3 = (k_1-p_1-p_2)^2, \qquad D_5 = (k_2-p_1-p_2)^2, \qquad D_7 = k_2^2,$$

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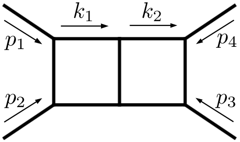

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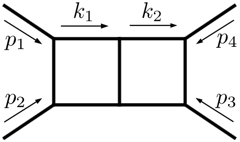

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$$\omega = \left(\frac{d-6}{2z} + \frac{4-d}{2(z+s)} + \frac{d-5}{z-t} \right) \mathrm{d}z \qquad \Rightarrow \qquad \nu = 2$$

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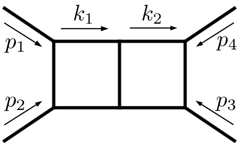

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We want to reduce

$$I_{1111111;-2} = c_1 I_{1111111;0} + c_2 I_{1111111;-1} + \text{lower}$$

$$\phi = z^2 \, \mathrm{d}z\,, \quad \phi_1 = 1 \, \mathrm{d}z\,, \quad \phi_2 = z \, \mathrm{d}z\,, \quad \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s} \right) \mathrm{d}z\,, \quad \xi_2 = \left(\tfrac{1}{z+s} - \tfrac{1}{z-t} \right) \mathrm{d}z\,,$$

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Using $\langle\phi|\xi\rangle = \sum_{p\in\mathcal{P}} \mathrm{Res}_{z=p}(\psi_p\xi)$ with $(\mathrm{d}+\omega)\psi_p = \phi$, we get

$$\langle\phi|\xi_1\rangle = \tfrac{s(4(d-5)t^2 - 3(d-4)(3d-14)s^2 - 4(d-5)(2d-9)st)}{4(d-5)(d-4)(d-3)}\,,$$

$$\langle\phi|\xi_2\rangle = \tfrac{s(s+t)(3(d-4)(3d-14)s + 2(d-6)(d-5)t)}{4(d-5)(d-4)(d-3)}\,,$$

$$\langle\phi_1|\xi_1\rangle = \tfrac{-s}{d-5}\,, \qquad \langle\phi_1|\xi_2\rangle = \tfrac{s+t}{d-5}\,,$$

$$\langle\phi_2|\xi_1\rangle = \tfrac{s((3d-14)s + 2(d-5)t)}{2(d-5)(d-4)}\,, \qquad \langle\phi_2|\xi_2\rangle = \tfrac{-(3d-14)s(s+t)}{2(d-5)(d-4)}\,.$$

Example: double box

$$c_i = \langle \phi | \xi_j \rangle (\mathbf{C}^{-1})_{ji} \qquad \text{with} \qquad \mathbf{C}_{ij} = \langle \phi_i | \xi_j \rangle$$

$$\phi = z^2 \, \mathrm{d}z \,, \quad \phi_1 = 1 \, \mathrm{d}z \,, \quad \phi_2 = z \, \mathrm{d}z \,, \quad \xi_1 = \left(\tfrac{1}{z} - \tfrac{1}{z+s} \right) \mathrm{d}z \,, \quad \xi_2 = \left(\tfrac{1}{z+s} - \tfrac{1}{z-t} \right) \mathrm{d}z \,,$$

We need 6 intersection numbers: $\Big\{ \langle \phi | \xi_1 \rangle, \langle \phi | \xi_2 \rangle, \langle \phi_1 | \xi_1 \rangle, \langle \phi_1 | \xi_2 \rangle, \langle \phi_2 | \xi_1 \rangle, \langle \phi_2 | \xi_2 \rangle \Big\}$

Using $\langle \phi | \xi \rangle = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p}(\psi_p \xi)$ with $(\mathrm{d} + \omega)\psi_p = \phi$, we get

$$\langle \phi | \xi_1 \rangle = \frac{s(4(d-5)t^2 - 3(d-4)(3d-14)s^2 - 4(d-5)(2d-9)st)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi | \xi_2 \rangle = \frac{s(s+t)(3(d-4)(3d-14)s + 2(d-6)(d-5)t)}{4(d-5)(d-4)(d-3)} \,,$$

$$\langle \phi_1 | \xi_1 \rangle = \frac{-s}{d-5} \,, \qquad \langle \phi_1 | \xi_2 \rangle = \frac{s+t}{d-5} \,,$$

$$\langle \phi_2 | \xi_1 \rangle = \frac{s((3d-14)s + 2(d-5)t)}{2(d-5)(d-4)} \,, \qquad \langle \phi_2 | \xi_2 \rangle = \frac{-(3d-14)s(s+t)}{2(d-5)(d-4)} \,.$$

$$I_{1111111;-2} = c_0 I_{1111111;0} + c_1 I_{1111111;-1} + \text{lower} \qquad c_0 = \frac{(d-4)st}{2(d-3)} \,, \quad c_1 = \frac{2t - 3(d-4)s}{2(d-3)} \,,$$

in agreement with the public codes.

Further examples

We did $\mathcal{O}(30)$ examples in arXiv:1901.11510

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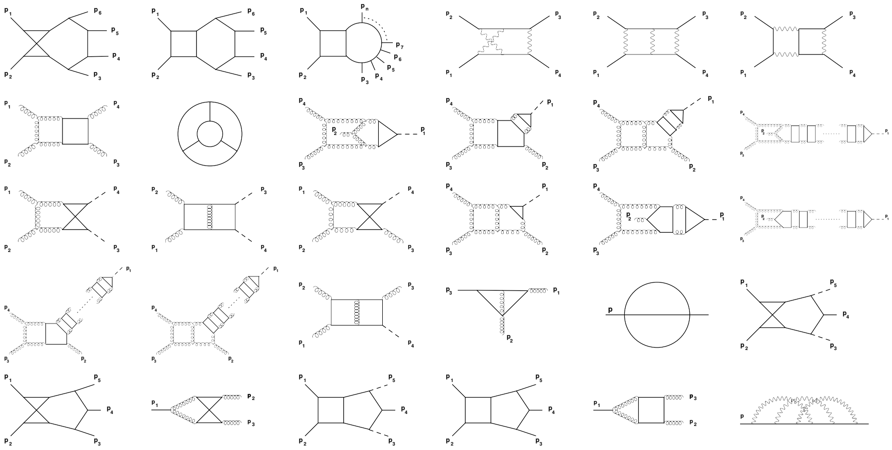

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but now $\langle\phi|\xi\rangle$ is a *multivariate intersection number*!

See Manoj' talk tomorrow!

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Differential equations:

$$\partial_s I = \int_{\mathcal{C}} \partial_s(u\phi) = \int_{\mathcal{C}} u\tilde{\phi}$$

[Kotikov (1991), Remiddi (1997), Henn (2013)]

Dimension shift relations:

$$I_{d\to d\pm 2n} = \int_{\mathcal{C}} (u\phi)_{d\to d\pm 2n} = \int_{\mathcal{C}} u\tilde{\tilde{\phi}}$$

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"IBPs without IBPs"

Perspectives

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We will hear a lot more of these subjects in the coming days!

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Thank you for attending our conference,
and thank you for listening!

Hjalte Frellesvig

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