From positive geometries to a coaction on hypergeometric functions

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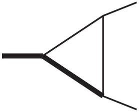
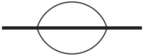
The simplest Feynman diagrams are generalized hypergeometric functions.

Examples:

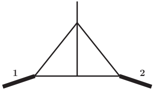


$$\sim \quad {}_2F_1\left(1, 1+\epsilon; 2-\epsilon; \frac{p^2}{m^2}\right)$$



$$\sim \quad {}_2F_1\left(1+2\epsilon, 1+\epsilon, 1-\epsilon, \frac{p^2}{m^2}\right)$$

$$\sim \quad F_4\left(1+\epsilon, 1, 1-\epsilon, 1-\epsilon; \frac{m_1^2}{m_2^2}, \frac{p^2}{m_2^2}\right)$$



$$\sim \quad {}_3F_2\left(1-\epsilon, 1, 1-2\epsilon; 1+\epsilon, 2-2\epsilon; 1-\frac{p_1^2}{p_2^2}\right)$$

Here ϵ is the parameter of dimensional regularization: the integrals are evaluated in $4-2\epsilon$ dimensions of spacetime, or more generally $n-2\epsilon$ where n is even.

- Feynman integrals are usually computed in the Laurent expansion in ϵ. For all of the simplest Feynman integrals, this expansion is given in terms of multiple polylogarithms (MPLs).

- There is a well-known coaction on multiple polylogarithms. [Goncharov, Brown]

- We've conjectured a corresponding coaction on Feynman diagrams. [Abreu, RB, Duhr, Gardi]

- Natural compatibility of coaction with discontinuities and differential equations is potentially useful for computation.

- Now: we propose a corresponding coaction on hypergeometric functions, of the form

$$\Delta\left(\int_\gamma \omega\right) = \sum_{i,j} c_{ij} \int_\gamma \omega_i \otimes \int_{\gamma_j} \omega$$

  where the coefficients c_{ij} can be derived from intersection numbers.

An algebra H is a ring (addition group & multiplication),
which has a multiplicative unit,
and which is also a vector space.

Example: $n \times n$ matrices with entries in a field K.

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A bialgebra is an algebra H with two more maps, the coproduct $\Delta : H \to H \otimes H$, and the counit $\varepsilon : H \to \mathbb{Q}$, satisfying the following axioms.

- Coassociativity: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$
- Δ and ε are algebra homomorphisms:
  $\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b) \;\; \text{and} \;\; \varepsilon(a \cdot b) = \varepsilon(a) \cdot \varepsilon(b)$
- The counit and the coproduct are related by $(\varepsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$

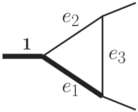
Coaction generalizes the coproduct to a tensor product of two different spaces.

If H is a Hopf algebra, then a H (right-) comodule is a vector space A with a map $\rho : A \to A \otimes H$ such that

$$(\rho \otimes \mathrm{id})\rho = (\mathrm{id} \otimes \Delta)\rho \quad \text{and} \quad (\mathrm{id} \otimes \varepsilon)\rho = \mathrm{id}\,.$$

Here Δ is a coproduct on H. ρ is a coaction on A.
In this presentation, we continue to use Δ to denote our coactions of bialgebras.

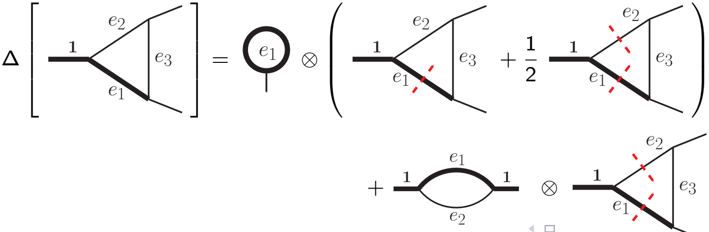
Feynman diagram example with ${}_2F_1$



$$= \frac{e^{\gamma_E \epsilon}\Gamma(1+\epsilon)}{\epsilon(1-\epsilon)}(m^2)^{-1-\epsilon}\,{}_2F_1\left(1, 1+\epsilon; 2-\epsilon; \frac{p^2}{m^2}\right)$$

$$= \frac{1}{p^2}\left[\frac{\log\left(\frac{m^2}{m^2-p^2}\right)}{\epsilon} + \mathrm{Li}_2\left(\frac{p^2}{m^2}\right) + \log^2\left(1-\frac{p^2}{m^2}\right) + \log(m^2)\log\left(1-\frac{p^2}{m^2}\right)\right] + \mathcal{O}(\epsilon)$$

$$\begin{aligned}
\Delta(\log z) &= 1\otimes \log z + \log z \otimes 1\\
\Delta(\log^2 z) &= 1\otimes \log^2 z + 2\log z \otimes \log z + \log^2 z \otimes 1\\
\Delta(\mathrm{Li}_2(z)) &= 1\otimes \mathrm{Li}_2(z) + \mathrm{Li}_2(z)\otimes 1 + \mathrm{Li}_1(z)\otimes \log z
\end{aligned}$$



How to construct coactions for Feynman integrals?

$$\Delta\left(\int_\gamma \omega\right) = \sum_{i,j} c_{ij} \int_\gamma \omega_i \otimes \int_{\gamma_j} \omega$$

- Our starting point is the integral $\int_\gamma \omega$, so ω is a cohomology class and γ is a homology class. (What are the co/homology groups?)
- Take $\{\omega_i\}$ to be a basis of the cohomology group, and $\{\gamma_j\}$ to be a basis of the homology group.
- Coaction on MPLs takes a similar form.
- Factors in second entry are simpler/similar: Strictly speaking, second entry is a single-valued/nonmotivic version, e.g. MPLs are taken modulo $i\pi$.

  What are the integrands ω_i and contours γ_j, and the coefficients c_{ij}?

Multiple polylogarithms (MPL)

A large class of iterated integrals are described by multiple polylogarithms:

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

Examples:

$$G(0; z) = \log z, \quad G(a; z) = \log\left(1 - \frac{z}{a}\right)$$

$$G(\vec{a}_n; z) = \frac{1}{n!} \log^n \left(1 - \frac{z}{a}\right), \quad G(\vec{0}_{n-1}, a; z) = -\mathrm{Li}_n\left(\frac{z}{a}\right)$$

n is the *transcendental weight*.

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n is the *transcendental weight*.

There is a coaction on MPLs, graded by weight, which maps MPLs to combinations of simpler functions (lower weight). [Goncharov; Duhr]

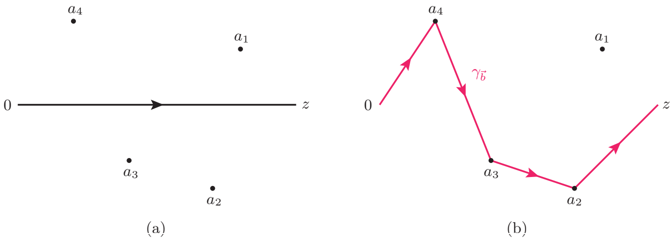
$$\begin{aligned}
\Delta(\log x \log y) &= 1 \otimes (\log x \log y) + \log x \otimes \log y + \log y \otimes \log x + (\log x \log y) \otimes 1 \\
\Delta(\mathrm{Li}_n(z)) &= 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\log^k z}{k!}
\end{aligned}$$

MPLs and their coaction have become extremely useful in recent years!

Contour integrals

The coaction is a pairing of contours and integrands.

$$\Delta(G(\vec{a}; z)) = \sum_{\vec{b}\subseteq\vec{a}} G(\vec{b}; z) \otimes G_{\vec{b}}(\vec{a}; z)$$



Contour (b) encircles a subset of residues in a given order.

A graphical coaction for Feynman integrals

The coaction on a basis of 1-loop graphs defined by pinching and cutting subsets of propagators — when evaluated by Feynman rules in dimensional regularization, and expanded order by order in ϵ — agrees with the coaction on MPLs.

[Abreu, RB, Duhr, Gardi]

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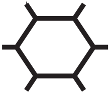
Use this basis:



$$J_n = \frac{ie^{\gamma_E \epsilon}}{\pi^{D_n/2}} \int d^{D_n} k \prod_{j=1}^{n} \frac{1}{(k-q_j)^2 - m_j^2}$$

$$D_n = \begin{cases} n - 2\epsilon\,, & \text{for } n \text{ even}\,, \\ n + 1 - 2\epsilon\,, & \text{for } n \text{ odd}\,. \end{cases}$$

e.g. tadpoles and bubbles in $2-2\epsilon$ dimensions,
triangles and boxes in $4-2\epsilon$ dimensions, etc.

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[Abreu, RB, Duhr, Gardi]

These coactions capture useful properties.

Discontinuities and cuts act from the front:

$$\Delta \, \mathrm{Disc} = (\mathrm{Disc} \otimes 1) \, \Delta$$

Differential operators act from the rear:

$$\Delta \, \partial = (1 \otimes \partial) \, \Delta$$

$\Delta\Big[$ 

$\Big] =$ 

$\otimes$ 

.

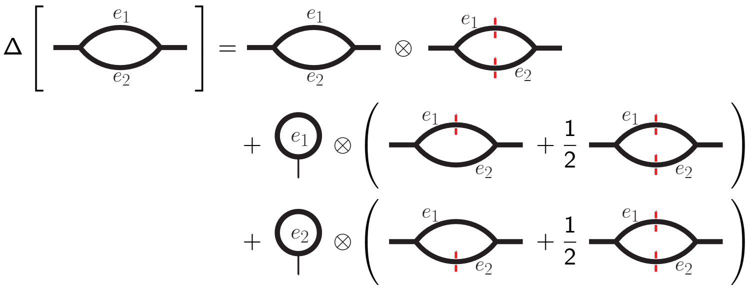
$$= -\frac{e^{\gamma_E \epsilon}\Gamma(1+\epsilon)\left(m^2\right)^{-\epsilon}}{\epsilon}$$

$$= \frac{e^{\gamma_E \epsilon}\left(-m^2\right)^{-\epsilon}}{\Gamma(1-\epsilon)}\,.$$

$$\Delta\left[(m^2)^{-\epsilon}\right] = (m^2)^{-\epsilon} \otimes (m^2)^{-\epsilon}\,,$$

$$\Delta\left[e^{\gamma_E \epsilon}\Gamma(1+\epsilon)\right] = \left(e^{\gamma_E \epsilon}\Gamma(1+\epsilon)\right) \otimes \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)}\,,$$

Examples of the graphical algebra



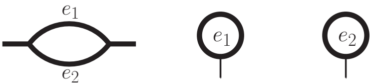
$$\Delta\left(\int_{\Gamma_\emptyset}\omega_{12}\right) = \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12}+\int\omega_1\otimes\left(\int_{\Gamma_1}\omega_{12}+\frac{1}{2}\int_{\Gamma_{12}}\omega_{12}\right)+\cdots$$

$$= \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12}+\int_{\Gamma_\emptyset}\omega_1\otimes\int_{-\frac{1}{2}\Gamma_{1\infty}}\omega_{12}+\int_{\Gamma_\emptyset}\omega_2\otimes\int_{-\frac{1}{2}\Gamma_{2\infty}}\omega_{12}$$

The odd-shaped integrals have poles at infinity.

$$\Delta\left(\int_{\Gamma_\emptyset}\omega_{12}\right) = \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12} + \int\omega_1\otimes\left(\int_{\Gamma_1}\omega_{12}+\frac{1}{2}\int_{\Gamma_{12}}\omega_{12}\right)+\cdots$$
$$= \int_{\Gamma_\emptyset}\omega_{12}\otimes\int_{\Gamma_{12}}\omega_{12} + \int_{\Gamma_\emptyset}\omega_1\otimes\int_{-\frac{1}{2}\Gamma_{1\infty}}\omega_{12} + \int_{\Gamma_\emptyset}\omega_2\otimes\int_{-\frac{1}{2}\Gamma_{2\infty}}\omega_{12}$$

Basis of integrands and corresponding contours:



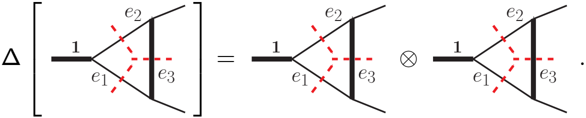
ω_j :	ω_{12}	ω_1	ω_2
γ_j :	Γ_{12}	$-\frac{1}{2}\Gamma_{1\infty}$	$-\frac{1}{2}\Gamma_{2\infty}$
	$\mathcal{C}_{12}$	$\mathcal{C}_1+\frac{1}{2}\mathcal{C}_{12}$	$\mathcal{C}_2+\frac{1}{2}\mathcal{C}_{12}$

At leading order in ϵ, they satisfy

$$\int_{\gamma_i}\omega_j \sim \delta_{ij}.$$

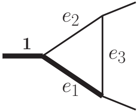
Examples of the graphical algebra

Maximal cuts are grouplike.



Beyond one loop, we have some examples where graphical operations correspond to the MPL coaction, but a general treatment remains to be found. Moreover, there are multiloop integrals without MPL expansions. In view of Brown's motivic Galois coaction, we do expect generalizations.

Feynman diagram example with ${}_2F_1$



$$= \frac{e^{\gamma_E \epsilon}\Gamma(1+\epsilon)}{\epsilon(1-\epsilon)}(m^2)^{-1-\epsilon}\,{}_2F_1\left(1, 1+\epsilon; 2-\epsilon; \frac{p^2}{m^2}\right)$$

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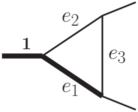

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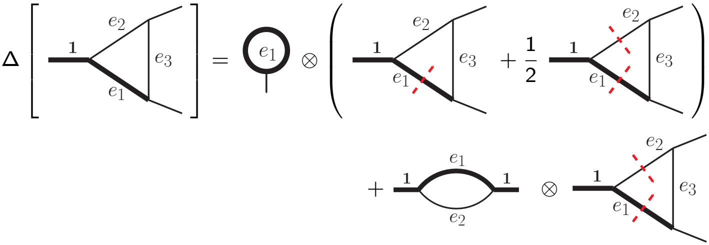

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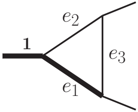

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There is a coaction on ${}_2F_1$ that agrees with the coaction on MPLs to all orders in ϵ!

At first, we tried to select just the right integrands ω_i and "dual" contours γ_j so that there were no coefficients: $\Delta\left(\int_\gamma \omega\right) = \sum_i \int_\gamma \omega_i \otimes \int_{\gamma_i} \omega$

We got pretty far with ${}_{p+1}F_p$ and Appell F_1, F_2, F_3, F_4 by guesswork...

...but some issues with F_4 led us to hunt for math results online, where we found that we'd been starting to get at intersection numbers, and where we found many helpful papers on hypergeometric functions from Japan.

${}_2F_1$

Gauss's original hypergeometric function. Series definition:

$$ {}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}\,, \qquad \text{where } (x)_n \equiv \frac{\Gamma(x+n)}{\Gamma(x)}. $$

Euler's integral representation:

$$ {}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 du\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} $$

Let

$$ \int_\gamma \omega \equiv \int_0^1 du\, u^{n_0+a_0\epsilon}(1-u)^{n_1+a_1\epsilon}(1-ux)^{n_x+a_x\epsilon} $$

with $a_0, a_1, a_x \in \mathbb{C}^*$ are generic, and $n_i \in \mathbb{Z}$.

Principal example: ${}_2F_1$

$$\int_\gamma \omega \equiv \int_0^1 du\, u^{n_0+a_0\epsilon}(1-u)^{n_1+a_1\epsilon}(1-ux)^{n_x+a_x\epsilon}$$
$$\omega \equiv \Phi\varphi$$

$\varphi = u^{n_0}(1-u)^{n_1}(1-ux)^{n_x}\, du$ is a single-valued differential form.

$\Phi = u^{a_0\epsilon}(1-u)^{a_1\epsilon}(1-ux)^{a_x\epsilon}$ is a multi-valued function,
from which we construct the twist 1-form $d\log\Phi$:

$$d\log\Phi = a_0\frac{du}{u} - a_1\frac{du}{1-u} - xa_x\frac{du}{1-ux}$$

Then Stokes' theorem implies $\int_\gamma \Phi\varphi = \int_\gamma \Phi(\varphi + \nabla_\Phi \xi)$, so φ is a twisted cohomology class.

Contours γ also get a "twist" carrying the information of the choice of branch of Φ.

Twisted co/homology and intersection numbers

$$\Delta\left(\int_\gamma \omega\right) = \sum_{i,j} c_{ij} \int_\gamma \omega_i \otimes \int_{\gamma_j} \omega$$

- Our starting point is the integral $\int_\gamma \omega$, so ω is a cohomology class and γ is a homology class.
- What are the twisted co/homology groups?

  They can be defined systematically when $\omega = d\mathbf{u} \prod_k P_k(\mathbf{u})^{\alpha_k}$, where the $P_k(\mathbf{u})$ are polynomials in the integration variables $\mathbf{u}$, and when γ has its boundary on the zeros of $\prod_k P_k(\mathbf{u})$.

  [Aomoto-Kita book]
- Then, take $\{\omega_i\}$ to be a basis of the twisted cohomology group, and $\{\gamma_j\}$ to be a basis of the twisted homology group.

  In our examples where the $P_k(\mathbf{u})$ can be combined to a suitable number of positive geometries, this can be done in a natural way.

- Positive geometries have corresponding *canonical* differential forms having logarithmic singularities and unit residues precisely at their boundaries. [Arkani-Hamed, Bai, Lam]

  This class of integrals includes all known examples of Feynman diagrams with MPL expansions.

- If the $P_k(\mathbf{u})$ are mostly linear, there is a natural choice of the basis of integration contours $\{\gamma_j\}$, and it is straightforward to construct their canonical forms. Differential forms $\bigwedge_i d\log y_i(\mathbf{u})$ are called dlog forms.

- If an integration contour γ_j is a positive geometry, construct its canonical form $\Omega(\gamma_j)$.

- We compute the cohomology intersection matrix $\langle \varphi_i, \Omega(\gamma_j) \rangle_\Phi$.

Intersection numbers

- The intersection number pairing of cohomology classes (forms) is defined by

$$\langle \varphi_i, \psi_j \rangle_\Phi = \frac{1}{(2\pi i)^2} \int \iota_\Phi(\varphi_i) \wedge \psi_j \,,$$

  where ι_Φ denotes a compactification away from the branch points.

- Residue formulas are more practical for computation. For dlog forms in one variable, we have

$$\langle \varphi_i, \psi_j \rangle_\Phi = \sum_{u_p} \frac{\text{Res}_{u=u_p}\, \varphi_i \;\; \text{Res}_{u=u_p}\, \psi_j}{\text{Res}_{u=u_p}\, d\log \Phi} \,,$$

  where the u_p are the poles of $d\log\Phi$. Produces the simple expressions seen above. [Matsumoto, Mizera; more general versions by Frellesvig, Gasparotto, Mandal, Mastrolia, Mattiazzi, Mizera]

- Invert the matrix of intersection numbers to read off the coefficients c_{ij} that we seek for the coaction formula.

To find bases of co/homology $\{\varphi_i\}$ and $\{\gamma_j\}$:

- Compute the dimension r.

- Try a set of r forms $\{\varphi_i\}$ and use intersection numbers to check linear independence.

- Try a set of r cycles $\{\gamma_j\}$. Construct their canonical forms, and use intersection numbers to check linear independence.

Intersection theory for bases of co/homology

To find bases of co/homology $\{\varphi_i\}$ and $\{\gamma_j\}$:

- Compute the dimension r.

  Given by Morse theory/Euler characteristic. In simple cases, r is the number of solutions to $d \log \Phi = 0$.

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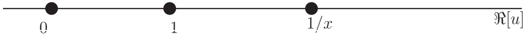
  When the factors $P_k(\mathbf{u})$ are linear, a natural basis is given by "bounded chambers."

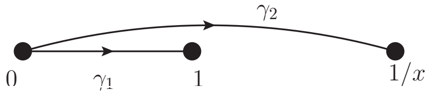
Principal example: ${}_2F_1$

$$\Phi = u^{a_0\epsilon}(1-u)^{a_1\epsilon}(1-ux)^{a_x\epsilon}$$
$$d\log\Phi = a_0\frac{du}{u} - a_1\frac{du}{1-u} - xa_x\frac{du}{1-ux}$$

- The equation $d\log\Phi = 0$ has 2 solutions.
- We also see two bounded chambers in real coordinates;



Choose two independent contours with boundaries at the branch points, for example $\gamma_1 = [0,1]$, $\gamma_2 = [0,1/x]$.



From these contours, construct their associated "canonical" dlog forms

$$\psi_1 = d\log\frac{u-1}{u}, \qquad \psi_2 = d\log\frac{u-1/x}{u},$$

- It is always possible to use these same forms as the basis of twisted cohomology: $\varphi_1 = \psi_1$, $\varphi_2 = \psi_2$.

Principal example: ${}_2F_1$

With

$$\varphi_1 = \psi_1 = d\log\frac{u-1}{u}\,, \qquad \varphi_2 = \psi_2 = d\log\frac{u-1/x}{u}\,,$$

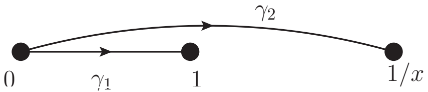
the intersection matrix is

$$\langle\varphi_1,\psi_1\rangle_\Phi = \frac{1}{a_0\epsilon} + \frac{1}{a_1\epsilon} \qquad \langle\varphi_1,\psi_2\rangle_\Phi = \frac{1}{a_0\epsilon}$$

$$\langle\varphi_2,\psi_1\rangle_\Phi = \frac{1}{a_0\epsilon} \qquad \langle\varphi_2,\psi_2\rangle_\Phi = \frac{1}{a_0\epsilon} + \frac{1}{a_x\epsilon}$$

which can be inverted to produce a coaction formula.

In this case with dlog forms and linear polynomials, it's no coincidence that you can just read off overlaps as intersections.



Construction generalizes to ${}_{p+1}F_p$ and Lauricella $F_D^{(n)}$.
All of our other examples work in a similar way.

- Choose the same basis of homology, $\gamma_1 = [0,1], \quad \gamma_2 = [0, 1/x]$.
- Choose a different basis of cohomology to minimize overlap of singularities:

$$\varphi_1 = d\log(1-u)\,, \qquad \varphi_2 = d\log(1-xu)\,,$$

- Compute cohomology intersection matrix.

int	φ_1	φ_2
γ_1	$\frac{1}{a_1\epsilon}$	0
γ_2	0	$\frac{1}{a_x\epsilon}$

- The resulting coaction formula has just two terms:

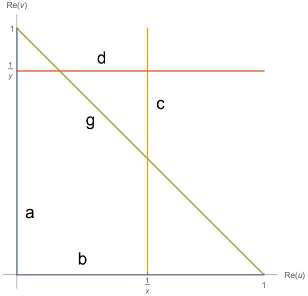
$$\Delta\left({}_2F_1(\alpha,\beta;\gamma;x)\right) = {}_2F_1(1+a\epsilon, b\epsilon; 1+c\epsilon; x) \otimes {}_2F_1(\alpha,\beta;\gamma;x)$$
$$- \frac{b\epsilon}{1+c\epsilon}\, {}_2F_1(1+a\epsilon, 1+b\epsilon; 2+c\epsilon; x)$$
$$\otimes \frac{\Gamma(1-\beta)\Gamma(\gamma)}{\Gamma(1-\beta+\alpha)\Gamma(\gamma-\alpha)} x^{1-\alpha}\, {}_2F_1\left(\alpha, 1+\alpha-\gamma; 1-\beta+\alpha; \frac{1}{x}\right)$$

where $\alpha = n_\alpha + a\epsilon$, $\beta = n_\beta + b\epsilon$ and $\gamma = n_\gamma + c\epsilon$.

Appell F_3 coaction

$$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')}$$

$$\int_0^1 dv \int_0^{1-v} du\, u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'}$$



Dimension of (nontrivial) homology = number of bounded chambers = 4.
Choose $\gamma_{abg} = \gamma, \gamma_{bcg}, \gamma_{cdg}, \gamma_{adg}$.

Appell F_3 coaction

Canonical forms for 2-simplex in 2 dimensions [Matsumoto]

$$d\log\frac{P_1}{P_2}\wedge d\log\frac{P_2}{P_3}\,.$$

Basis of integrands:

$$\varphi_{ab} = d\log u\wedge d\log v\,,\qquad \varphi_{bc} = d\log(1-ux)\wedge d\log v\,,$$
$$\varphi_{cd} = d\log(1-ux)\wedge d\log(1-vy)\,,\qquad \varphi_{da} = d\log u\wedge d\log(1-vy)\,.$$

The cohomology intersection matrix is

$\langle\Omega(\gamma_i),\varphi_j\rangle$	φ_{ab}	φ_{bc}	φ_{cd}	φ_{ad}
γ_{abg}	$\frac{1}{ab\epsilon^2}$	0	0	0
γ_{bcg}	0	$\frac{1}{bc\epsilon^2}$	0	0
γ_{cdg}	0	0	$\frac{1}{cd\epsilon^2}$	0
γ_{adg}	0	0	0	$\frac{1}{ad\epsilon^2}$

Coaction formula:

$$\begin{aligned}\Delta\left(\int_\gamma\Phi\varphi\right) \;=\;& ab\epsilon^2\int_\gamma\Phi\varphi_{ab}\otimes\int_{\gamma_{abg}}\Phi\varphi + bc\epsilon^2\int_\gamma\Phi\varphi_{bc}\otimes\int_{\gamma_{bcg}}\Phi\varphi\\ &+cd\epsilon^2\int_\gamma\Phi\varphi_{cd}\otimes\int_{\gamma_{cdg}}\Phi\varphi + ad\epsilon^2\int_\gamma\Phi\varphi_{ad}\otimes\int_{\gamma_{adg}}\Phi\varphi.\end{aligned}$$

Construction generalizes to Lauricella $F_B^{(n)}$.

$$\frac{\Gamma(1-\gamma)\Gamma(1-\gamma')\Gamma(\gamma+\gamma'-\alpha-1)}{\Gamma(1-\alpha)} F_4(\alpha,\beta,\gamma,\gamma';x(1-y),y(1-x))$$
$$= \int_{\gamma_1} t_1^{\beta-\gamma} t_2^{\beta-\gamma'} L(t_1,t_2)^{\gamma+\gamma'-\alpha-2} Q(t_1,t_2,x,y)^{-\beta} dt_1 dt_2\,,$$

where

$$L(t_1,t_2) = 1 - t_1 - t_2, \quad Q(t_1,t_2,x_1,x_2) = t_1 t_2 - x(1-y)t_2 - y(1-x)t_1\,,$$

and the integration region γ_1 is a twisted version of the region bounded by $L(t_1,t_2)$ and $Q(t_1,t_2,x_1,x_2)$.

Dimension of co/homology is 4 (= no. critical points of Φ).

Use bases given by Goto & Matsumoto to write dlog forms on positive geometries, and to evaluate the integrals in second entry of coaction.

Summary

- We propose a coaction on generalized hypergeometric functions,

$$\Delta\left(\int_\gamma \omega\right) = \sum_{i,j} c_{ij} \int_\gamma \omega_i \otimes \int_{\gamma_j} \omega$$

  where the coefficients c_{ij} can be derived from intersection numbers, provided $\omega = d\mathbf{u} \prod_k P_k(\mathbf{u})^{\alpha_k}$ and γ has its boundary on $\prod_k P_k(\mathbf{u}) = 0$.

- When the exponents α_k are expanded around integer values, we claim that this coaction is compatible with the coaction on the MPLs in the Laurent expansion. Checked to several orders for ${}_2F_1$, Appell F_1, F_2, F_3, F_4. Generalizes to ${}_{p+1}F_p$, Lauricella F_A, F_B, F_D.

- A version of this coaction has recently been proven for ${}_2F_1$ and Lauricella F_D [Brown, Dupont 1907.06603].

- Note that Feynman integrals typically have degeneracies in the exponents – not allowed by the mathematics, but we have been able to take these limits as needed, preserving the general structure.

- Our coaction supports our examples of Feynman-diagrammatic coaction to all orders, despite having some exponents with integer values.