Maximal cuts and Wick Rotations

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DIFA-INFN, Bologna

Padua, December 18-20, 2019

- generalities & maxcuts ;
- the 1-loop Bhabha box and its (vanishing) maxcut;
- the Wick rotation for the Bhabha box;
- the equal mass sunrise (once more);
- the Bhabha planar double box maxcut (7 δ's);
- the 6 δ's subtopology;
- (¿ the 5 δ's subtopology ?)

If a Feynman propagator is written as

$$\frac{-i}{D - i\epsilon} = \mathcal{P}\left(\frac{-i}{D}\right) + \pi\ \delta(D)\ ,$$

the corresponding *cut* propagator is

$$\delta(D)\ .$$

• Given any Feynman graph (fully scalar and without squared propagators, for simplicity) the corresponding *maximally cut* graph (or maxcut for short) is obtained by replacing all the propagators by the corresponding *cut* propagators.

• Feynman amplitudes satisfy non-homogeneous equations; it is known that the corresponding maximally cut amplitudes satisfy the corresponding homogeneous equation.

• It is also known (from *Euler*) that the solutions of non-homogeneous equations can be written as the integral (repeated, when needed) of a suitable (and simple) expression built in terms of the homogeneous solutions;

• in the differential equation approach to the evaluation of Feynman graph amplitudes, the homogeneous solutions, *i.e.* the it maximally cut graphs, are the key ingredient for writing the solutions of the equations and therefore for characterizing their analytic properties.

The simplest approach to maxcuts is trying the direct evaluation by using the very definition of the maximally cut amplitude as an integral on the components of the loop momenta.

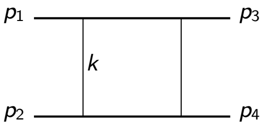
That naïve approach was used – and worked – for the *imaginary parts* of the 1-loop self-mass, of the 2-loop *sunrise* as well as the 3-loop *banana* graphs (P.Amedeo, L.Tancredi 2016)
(in all those amplitudes, imaginary parts and mxcuts are strictly related);

but in (almost) all the other cases, when integrating directly in the loop components the naïve approach gives vanishing results.

Indeed, in (almost) all the other maxcuts there are *too many* δ-functions and *too few* overconstrained integration loop components,

• and the multi-dimensional integration region shrinks to an empty domain.

As an example, consider the 1-loop Bhabha box



The process is $p_1 + p_2 \to p_3 + p_4$,

$$
\begin{aligned}
\texttt{with kinematics}\ p_1^2 = p_2^2 = p_3^2 = p_4^2 &= -m^2 \ , \\
p_1 &= (E, p, 0, 0) \ , \\
p_2 &= (E, -p, 0, 0) \ , \\
p_3 &= (E, p_x, p_y, 0) \ . \\
s &= -(p_1 + p_2)^2 = 4E^2 = 4(p^2 + m^2) \ , \\
t &= -(p_1 - p_3)^2 = -2p(p - p_x) \ , \\
s + t - 4m^2 &= 2p(p + p_x) \ ,
\end{aligned}
$$

The maxcut, in the (usual) d-continuous dimensions, is

$$B(s,t) = \int \mathcal{D}^d k \; \delta(D_1)\; \delta(D_2)\; \delta(D_3)\; \delta(D_4)\;,$$

with

$$\begin{aligned} D_1 &= k^2\;, \\ D_2 &= (p_1+k)^2 + m^2 = +2p_1 k + k^2\;, \\ D_3 &= (p_2-k)^2 + m^2 = -2p_2 k + k^2\;, \\ D_4 &= (p_1 - p_3 + k)^2 = -t + 2p_1 k - 2p_3 k + k^2\;. \end{aligned}$$

Quite in general ($i \neq j$!)

$$\delta(D_i)\; \delta(D_j) = \delta(D_i)\; \delta(D_j + aD_i)\;,$$

where a is any real quantity; as an example, in particular

$$\delta(D_1)\; \delta(D_2) = \delta(D_1)\; \delta(D_2 - D_1) = \delta(D_1)\; \delta(2p_1 k) = \delta(D_1)\; \delta(D_2')$$

which amounts to $\delta(D_2) \to \delta(D_2')$, with $D_2' = D_2 - D_1 = 2p_1 k$.

By *repeated* use of the δ identities, and by introducing the vector components, one arrives at

$$B(s,t) = \int dk_0 \; dk_x \; dk_y \; \Omega(d-3) \int_0^\infty K^{d-4} \; dK \; \delta(D_1') \; \delta(D_2') \; \delta(D_3') \; \delta(D_4') \; ,$$

where:

• $\Omega(n)$ is the n-dimensional solid angle;

• the physical external vectors do not span the z-component, so that k_z can be lumped within the $(d-3)$ (*Euclidean*) continuous regularizing components of k,

• and the new argument of the δ's are

$$\begin{aligned} D_1' &= k_y^2 + K^2 \; , \\ D_2' &= 4pk_x \; , \\ D_3' &= 2Ek_0 \; , \\ D_4' &= -t - 2p_y k_y \; . \end{aligned}$$

It is apparent that

$$D_1' = k_y^2 + K^2$$

cannot vanish; therefore

$$B(s,t) = 0 \ ,$$

which is a solution of the homogeneous equation for the Bhabha amplitude, but obviously of no interest.

Proposal:

- give a Minkowski metric to the $(d-3)$ regularizing components of $\mathcal{D}^d k$.

- That is immediately achieved by a Wick rotation (or perhaps counter-rotation), which gives an overall factor $(-i)^{(d-3)}$ (irrelevant for the maxcut, as the maxcut is required to satisfy a homogeneous equation).

As already observed, those regularizing components do not mix with the physical external vectors; the only effect of that change of the metric is to modify D_1'

$$D_1' = k_y^2 + K^2 \quad \rightarrow \quad D_1' = k_y^2 - K^2 \ .$$

All the integrations are (anyhow) trivial, the result is

$$B(s,t) = \frac{1}{4}\Omega(d-3)\left[\frac{-t(s-4m^2)}{4(s+t-4m^2)}\right]^{\frac{d-4}{2}}\left(-\frac{1}{t\sqrt{s(s-4m^2)}}\right) ,$$

which is positive, as expected, because t is spacelike (and negative).

A short comment can be appropriate here.

• As already recalled, any propagator can be written as

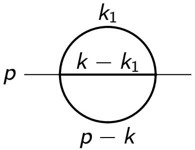
$$-i/(D-i\epsilon) = -i\mathcal{P}(1/D) + \pi\delta(D) ;$$

when that is done, the the 4 propagators of the 1-loop Bhabha box become the sum of 16 terms, the product of the 4 δ's being just one of the 16 terms.

• The amplitude, which is the sum of the 16 teerms, does not change if the $(d-3)$ regularizing dimensions are propely Wick-rotated, but the values of the single terms can change (and *do* change);

• in particular, the product of the 4 δ's vanishes with the usual Euclidean regularization but gives the above non vanishing result when the Minkowskian regularizing variables are used.

As a less simple example, consider the 2-loop equal mass sunrise of external momentum p of arbitrary components (p_0, p_x)
(p can be timelike or spacelike, but in any case it spans only two dimensions, say energy and x-direction).



The corresponding maxcut is

$$S(p_0^2 - p_x^2) = \int \mathcal{D}^d k \; \mathcal{D}^d k_1 \; \delta(D_1) \; \delta(D_2) \; \delta(D_3) \; ,$$

where

$$\begin{aligned} D_1 &= k_1^2 + m^2 \; , \\ D_2 &= (k - k_1)^2 + m^2 \; , \\ D_3 &= (p - k)^2 + m^2 \; , \end{aligned}$$

The cuts of the sunrise (both the unitarity cut and the maxcut) do *not* vanish even with the usual (Euclidean) d continuous regularization;
what happens with the Wick-rotated regularization?

The components of the loop momentum k are k_0, k_x and $(d-2)$ regularizing Minkowskian components lumped into K, so that

$$\int \mathcal{D}^d k = \int dk_0 \; dk_x \; \Omega(d-2) \int_0^\infty K^{d-3} dK \; ,$$

and $$k^2 = -k_0^2 + k_x^2 - K^2$$

(the same applies to k_1).

It can be convenient to integrate k_1 first.
Define $s = -k^2$, (when k is spacelike s is negative and when k is timelike s is positive) and

$$S(-k^2) = \int \mathcal{D}^d k_1 \; \delta(D_1) \; \delta(D_2) \; ,$$

one finds:

- if $-\infty < s < 0$

$$S(s) = \frac{1}{2} 2^{-(d-2)} \Omega(d-1)(4m^2 - s)^{\frac{d-2}{2}} \times \frac{1}{\sqrt{-s}\sqrt{4m^2 - s}} \; ;$$

- if $0 < s < 4m^2$

$$S(s) = \frac{1}{2} 2^{-(d-2)} \Omega(d-1) \frac{\sin\left(\pi \frac{d-2}{2}\right)}{\cos\left(\pi \frac{d-2}{2}\right)} (4m^2 - s)^{\frac{d-2}{2}} \times \frac{1}{\sqrt{s}\sqrt{4m^2 - s}} \; ;$$

- if $4m^2 < s < \infty$

$$S(s) = \frac{1}{2} 2^{-(d-2)} \Omega(d-1) \frac{1}{\cos\left(\pi \frac{d-2}{2}\right)} (s - 4m^2)^{\frac{d-2}{2}} \times \frac{1}{\sqrt{s}\sqrt{s - 4m^2}} \; ;$$

The sunrise maxcut now reads

$$S(p_0^2 - p_x^2) = \int \mathcal{D}^d k \; S(-k^2)\delta(D_3) \; ;$$

the explicit integration (in all the integration variables *but one*) is elementary (even if non totally trivial, as for instance the implementation of the positive constraints in terms of allowed integration regions) and the result depends (of course) on the value of $p_0^2 - p_x^2 = E^2$.
In the range and $3m < E < \infty$, for instance, the result reads

$$\begin{aligned} S(W^2) = {} & \frac{1}{16}\Omega(d-1)\Omega(d-2)B\left(\frac{d-2}{2},\frac{1}{2}-\frac{d-2}{2}\right)\frac{1}{\cos\left(\pi\frac{d-2}{2}\right)}E^{2-d} \\ & \times \int_{(E+m)^2}^{\infty} \frac{db}{\sqrt{b}} \; \frac{\left[((E+m)^2-b)((E-m)^2-b)(b-4m^2)\right]^{\frac{d-2}{2}}}{\sqrt{((E+m)^2-b)((E-m)^2-b)(b-4m^2)}} \\ & + \frac{1}{16}\Omega(d-1)\Omega(d-2)B\left(\frac{1}{2}-\frac{d-2}{2},\frac{1}{2}\right)\frac{1}{\cos\left(\pi\frac{d-2}{2}\right)}E^{2-d} \\ & \times \int_{(E-m)^2}^{(E+m)^2} \frac{db}{\sqrt{b}} \; \frac{\left[((E+m)^2-b)(b-(E-m)^2)(b-4m^2)\right]^{\frac{d-2}{2}}}{\sqrt{((E+m)^2-b)(b-(E-m)^2)(b-4m^2)}} \end{aligned}$$

`to be continued in the next slide ...`

$$
\begin{aligned}
&+\frac{1}{16}\Omega(d-1)\Omega(d-2)B\left(\frac{d-2}{2},\frac{1}{2}-\frac{d-2}{2}\right)\frac{1}{\cos\left(\pi\frac{d-2}{2}\right)}E^{2-d}\\
&\quad\times\int_{4m^2}^{(E-m)^2}\frac{db}{\sqrt{b}}\;\frac{\left[((E+m)^2-b)((E-m)^2-b)(b-4m^2)\right]^{\frac{d-2}{2}}}{\sqrt{((E+m)^2-b)((E-m)^2-b)(b-4m^2)}}\\
&+\frac{1}{16}\Omega(d-1)\Omega(d-2)B\left(\frac{d-2}{2},\frac{1}{2}-\frac{d-2}{2}\right)\frac{\sin\left(\pi\frac{d-2}{2}\right)}{\cos\left(\pi\frac{d-2}{2}\right)}E^{2-d}\\
&\quad\times\int_{0}^{4m^2}\frac{db}{\sqrt{b}}\;\frac{\left[((E+m)^2-b)((E-m)^2-b)(4m^2+b)\right]^{\frac{d-2}{2}}}{\sqrt{((E+m)^2-b)((E-m)^2-b)(4m^2+b)}}\\
&+\frac{1}{16}\Omega(d-1)\Omega(d-2)B\left(\frac{d-2}{2},\frac{1}{2}-\frac{d-2}{2}\right)E^{2-d}\\
&\quad\times\int_{-\infty}^{0}\frac{db}{\sqrt{-b}}\;\frac{\left[((E+m)^2-b)((E-m)^2-b)(4m^2-b)\right]^{\frac{d-2}{2}}}{\sqrt{((E+m)^2-b)((E-m)^2-b)(4m^2-b)}}\;,
\end{aligned}
$$

where $B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the Euler's Beta function .

Analog results (*i.e.* sums of 5 somewhat similar terms) hold for the ranges $m < E < 3m$ and $0 < E < m$, as well for spacelike value of the external momentum.

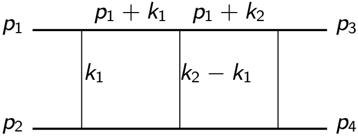
• According to the general discussion of the properties of the maxcuts, the previous sum of 5 terms is a solution of the homogeneous equation for the sunrise amplitude;

• on the other hand, as the 5 contributions above differ for the integration region and the "angular" factors, it is almost natural to consider the possibility that the 5 terms satisfy separately the equation, so that they would provide with 5 tentatively different solutions;

• but it is known that the equal mass sunrise has only two Master Integrals, hence only two of the solutions, at most, can be independent.
Indeed, it is not difficult to identify the 2 terms corresponding to the two solutions (known for a long time; look at the red symbol $\times$).

• The remaining 3 terms must therefore be a combination of those two (or, in other words, different integral representations of the two solutions).

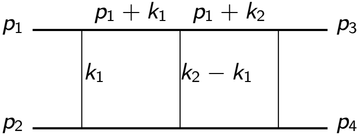
As a further example we consider the maxcut of the planar double-box Bhabha graph (and then of its "subtopologies" as well)



The graph has been already evaluated analytically (J.M. Henn and V.A.Smirnov, 2013), so that, strictly speaking, the knowledge of a few maxcuts adds only some more or less irrelevant marginal details to their results;

the hope is that an elementary approach to the planar maxcut, when successfull, might give some directions for the treatment of the non-planar graph and other two loop amplitudes.

The planar double-box Bhabha graph



has (of course) the same external kinematics of the 1-loop single-box graphs – but two loop momenta, k_1 and k_2.

Following closely the 1-loop case, one could tentatively write the loop momentum k_1 as

$$k_1 = (k_{10}, k_{1x}, k_{1y}, K_1),$$

i.e. in terms of the 4 integration variables $(k_{10}, k_{1x}, k_{1y}, K_1)$, which give

$$k_1^2 = -k_{10}^2 + k_{1x}^2 + k_{1y}^2 - K_1^2;$$

similarly, the second loop variable could be written as $k_2 = (k_{20}, k_{2x}, k_{2y}, K_2)$, which give

$$k_2^2 = -k_{20}^2 + k_{2x}^2 + k_{2y}^2 - K_2^2 \ .$$

The regularizing (and Wick-rotated) components K_1 and K_2 are not multiplied by the external vectors p_i (which do not span the space of the regularizing components;
but in the 2-loop case (as matter of fact also in the previously discussed sunrise) the scalar product $(k_1 \cdot k_2)$ is also present in one of the propagators.
One could write that scalar product as

$$(k_1 \cdot k_2) = -k_{10}k_{20} + k_{1x}k_{2x} + k_{1y}k_{2y} - K_1 K_2 \cos\theta,$$

where θ is, say, the angle between the regularizing vectors with moduli K_1, K_2;
but it is perhaps more convenient to take the direction of K_2 as the z-axis of k_1, so that

$$(k_{10}, k_{1x}, k_{1y}, K_{1z}, K_1),$$

which give
$$k_1^2 = -k_{10}^2 + k_{1x}^2 + k_{1y}^2 - K_{1z}^2 - K_1^2 \ ,$$
$$(k_1 \cdot k_2) = -k_{10}k_{20} + k_{1x}k_{2x} + k_{1y}k_{2y} - K_{1z}K_2 \ ,$$

and
$$\int \mathcal{D}^d k_1 = \int dk_{10}\ dk_{1x}\ dk_{1y}\ dK_{1z}\ \Omega(d-4) \int_0^\infty K^{d-5} dK \ ,$$

while
$$\int \mathcal{D}^d k_2 = \int dk_{20}\ dk_{2x}\ dk_{2y}\ \Omega(d-3) \int_0^\infty K_2^{d-4} dK_2 \ ,$$

i.e. keeping unchanged the components of k_2.

At this point, the calculation of the double box maxcut becomes the (multidimensional) integration of the product of 7 δ-functions (corresponding to the 7 propagators) in 9 phase-space integration variables.

All the integrations are essentially elementary, provided that
- they are carefully performed in the proper order;
- the positivity constraints are also properly accounted for.

It is convenient to integrate k_1 first; the result is essentially the 1-loop Bhabha box (the four "external" electron lines are all on the mass-shell), but with the would-be (spacelike) Mandelstam variable t now becoming k_2^2, where the loop-momentum k_2 can be also timelike.

The integration of the 4 δ-functions of the k_1 loop gives

$$\int \mathcal{D}^d k_1 \; \delta(D_1)\; \delta(D_2)\; \delta(D_3)\; \delta(D_4) = 2^{-d}\; \Omega(d-3)\; \frac{1}{Ep}\; \left(k_2^4\right)^{-\frac{1}{2}+\frac{d-4}{2}}$$

$$\times\Big[\quad \theta(k_{2y}^2 - K_2^2)\left(k_{2y}^2 - K_2^2\right)^{-\frac{d-4}{2}}$$

$$+\,\theta(K_2^2 - k_{2y}^2)\frac{1}{\cos\left(\pi\frac{d-4}{2}\right)}\left(K_2^2 - k_{2y}^2\right)^{-\frac{d-4}{2}}\Big]\;;$$

the first term corresponds (of course) to the previous result for the 1 loop Bhabha maxcut, where the momentum transfer t is spacelike.

The calculation of the maxcut continues with the (almost obvious) integration in k_{20}, k_{2y} and K_2 of the 3 remaining δ functions, and a final integration in k_{2x} is left.

The k_{2x} integration requires only some care in working out the integration intervals which satisfy the positivity constraints, and the final result for the 7-cut Bhabha, expanded up to the next to leading order in $(d-4)$, can be written as

$$B_7(s,t) = B_{7a}(s,t) + B_{7b}(s,t)\ ,$$

$$B_{7a}(s,t) = \frac{1}{256}\Omega^2(d-3)\frac{1}{E^2p^3(p-p_x)}\left(\frac{1}{d-4} + \ln\frac{p^2(p-px)}{p+p_x}\right)\ ,$$

$$B_{7b}(s,t) = \frac{1}{256}\frac{\Omega^2(d-3)}{\cos\left(\pi\frac{d-4}{2}\right)}\frac{1}{E^2p^3(p-p_x)}\left(\frac{1}{d-4} + \ln\frac{p(p-px)}{2}\right)\ .$$

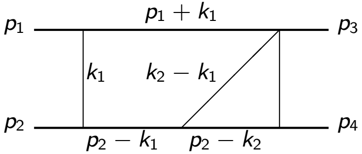
The two terms are (slightly) different, not only in the angular factors.

Indeed, the above maxcut satisfies a second order (homogeneous) differential equation (both on s and t).
The equation on t, for instance, is (courtesy of L. Mattiazzi)

$$
\begin{aligned}
t(s+t-4m^2)\frac{d^2}{dt^2}B_7(s,t) \\
+\left[3t+2(s-4m^2)-\frac{1}{2}(s-4m^2)(d-4)\right]\frac{d}{dt}B_7(s,t) \\
+\left[1+\frac{s-4m^2}{2t}\left((d-4)-(d-4)^2\right)\right]B_7(s,t)=0\ ;
\end{aligned}
$$

it is easy to check that the equation is satisfied (up to next-to-leading order in $(d-4)$) by the two above terms $B_{7a}(s,t)$ and $B_{7b}(s,t)$,
– separately.

Consider now the maxcut of the subtopology of the double box planar Bhabha in which one electron propagator is missing



The k_2 loop now correspond to a vertex amplitude for the process $(p_4 - p_2 + k_1) \to (k_1 - p_2) + p_4$.
It is convenient to evaluate the k_2 loop first; the resulting vertex amplitude (dropping an overall solid angle factor) can be written as

$$V(-m^2, V_2, V_3) = -\frac{1}{2}\frac{\pi}{d-4}\frac{\cos\left(\pi\frac{d-4}{2}\right)}{\sin\left(\pi\frac{d-4}{2}\right)} \times \Theta\big(R_2(-m^2, V_2, V_3)\big)$$
$$\times\, m^{d-4}\,\left(V_2^2\right)^{\frac{d-4}{2}}\big(R_2(-m^2, V_2, V_3)\big)^{-\frac{1}{2}+\frac{d-4}{2}}\ ,$$

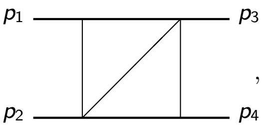
where $-m^2 = p_4^2$, $V_2 = (p_4 - p_2 + k_1)^2,\ V_3 = (k_1 - p_2)^2$,
and $R_2(V_1, V_2, V_3) = V_1^2 + V_2^2 + V_3^2 - 2V_1V_2 - 2V_2V_3 - 2V_1V_3$.

By integrating also in k_1, and dropping again any overall d dependent factor, the final result for the 6-cut Bhabha subtopology, expanded up to the next to leading order in $(d-4)$ is

$$
\begin{aligned}
B_6(s,t) = & \frac{1}{\sqrt{s(s-4m^2)}\sqrt{t(t-4m^2)}} \\
& \times \Bigg[-\frac{1}{d-4} \\
& \quad + \ln(2m) + \frac{1}{2}\ln\frac{s+t-4m^2}{s-4m^2} - \ln(-t) - \frac{1}{2}\ln(4m^2-t) \Bigg] \,,
\end{aligned}
$$

(which satisfies the homogeneous equation in s for the maxcut up to first order in $(d-4)$).

The next maxcut to consider should be the subtopology with 5 propagator



,

which has several (4 ?) Master Integrals, and which appears in the inhomogeneous parts of the equations of the previous amplitudes ....

- but unfortunately I am not (not yet, hopefully) able to give the results.

- conclusion:

• Generalized cuts and maxcuts are an important first step in evaluating Feynman graph amplitudes;

• the Wick (counter)-rotation of the continuous d regularizing dimensions can be one more useful tool when dealing with maxcuts.