Module Intersection
a powerful method for IBP reduction

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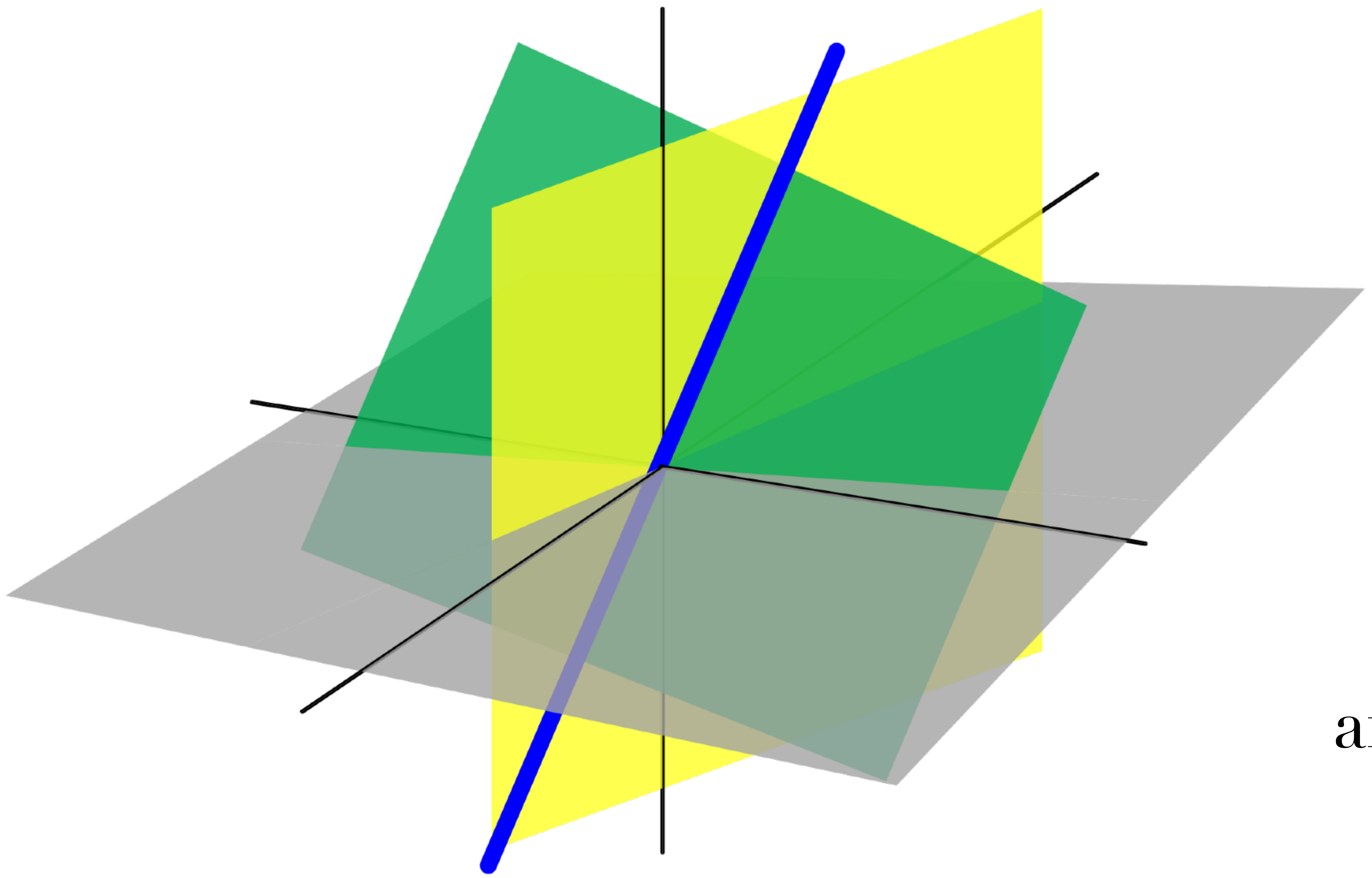
MathemAmplitudes 2019

First, I apologize that "Module Intersection"
is slightly different from the intersection theory for cohomology classes …



Module intersection
is similar to linear space intersection
but over polynomials only …
an computational algebraic geometry problem

“Module intersection” is to apply intersection of polynomial modules for finding particularly simple IBP vectors

Based on

Larsen, **YZ** 2015

YZ 2016

1805.01873, JHEP 1809 (2018) 024
Boehm, Schoenemann, Georgoudis
Larsen, **YZ**

1908.04301
Bendle, Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, **YZ**

IBP is a fascinating problem for scattering amplitudes

If one wants to reduce IBP relations by linear algebra,
then one need to make the linear system as small as possible …

Truncate IBP linear systems

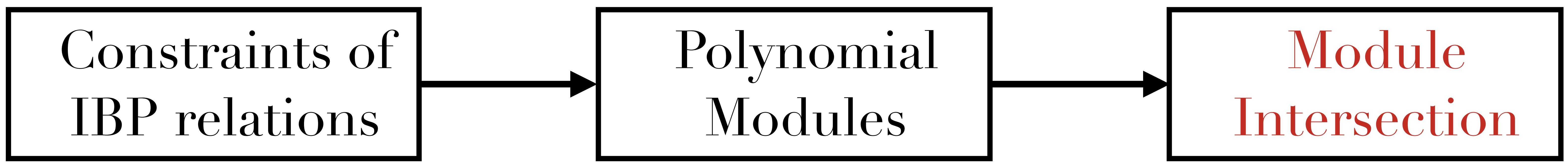
- (Gluza, Kajda, Kosower, 2010) no double propagator
- (Lee 2014) scalar integral only

all based on syzygies (linear constraints with polynomial coefficients) computation
but direct syzygy computation may be impractical …

Module Intersection

Larsen YZ 2015 YZ 2016

IBP relations in Baikov representation



- Easily get IBPs without double propagators (or propagator-degree increase)
- Naturally adaptable with unitarity cuts
- Usually much faster than direct syzygy approaches

In parallel with the developments of numeric unitarity
Ita 2015, Abreu, Cordero, Dormans, Ita, Page, Sotnikov
JHEP 1811 (2018) 116, Phys.Rev.Lett. 122 (2019) no.8, 082002, JHEP 1905 (2019) 084

Natural way to construct integrand with IBPs without doubled propagators
very efficient for constructing multi-loop integrand

IBP in Baikov representation

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} \quad \propto \int_\Omega dz_1 \ldots dz_k \frac{F^{\frac{D-L-E-1}{2}}}{z_1^{\alpha_1} \ldots z_k^{\alpha_k}}$$

Baikov

$$\boxed{0 = \int_\Omega dz_1 \ldots dz_k \sum_{i=1}^{k} \frac{\partial}{\partial z_i} \left(a_i(z) \frac{F^{\frac{D-L-E-1}{2}}}{z_1^{\alpha_1} \ldots z_k^{\alpha_k}} \right)}$$

No boundary term
feel free to set some of z's to zero (unitary cut)

IBP in Baikov representation with constraints

polynomials

Require

1. no shifted exponent: $\sum_{j=1}^{k} a_j(z)\frac{\partial F}{\partial z_j}+\beta(z)F=0$ These $\big(a_1(z),\ldots a_k(z)\big)$ form a module $M_1 \subset R^k$.

2. no propagator degree increase: $a_i(z)\in\langle z_i\rangle,\quad 1\le i\le m$ These $\big(a_1(z),\ldots a_k(z)\big)$ form a module $M_2 \subset R^k$.

Larsen, YZ 2015
YZ 2016

Both M_1 and M_2 are pretty simple ...

$$M_1 \cap M_2$$

Intersection of two modules

Determine the first module

$$\sum_{j=1}^{k} a_j(z)\frac{\partial F}{\partial z_j} + \beta(z)F = 0$$

More Advanced

- syzygy for the $\{\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_k}, F\}$ Roman Lee's trick
- $\mathrm{Ann}(F^s)$, annihilator of F^s in Weyl algebra. Bitoun, Bogner, Klausen, Panzer Lett.Math.Phys. 109 (2019) no.3, 497-564

If F is a determinant matrix whose elements are free variables, this kind of syzygy module is simple.

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

equivalent to canonical IBP in momentum space

Laplace expansion

$$\sum_j a_{k,j}\frac{\partial(\det A)}{\partial a_{i,j}} - \delta_{k,i}\cdot \det A = 0$$

Get all first order annihilator, proved by Gulliksen–Negard and Jozefiak exact sequences

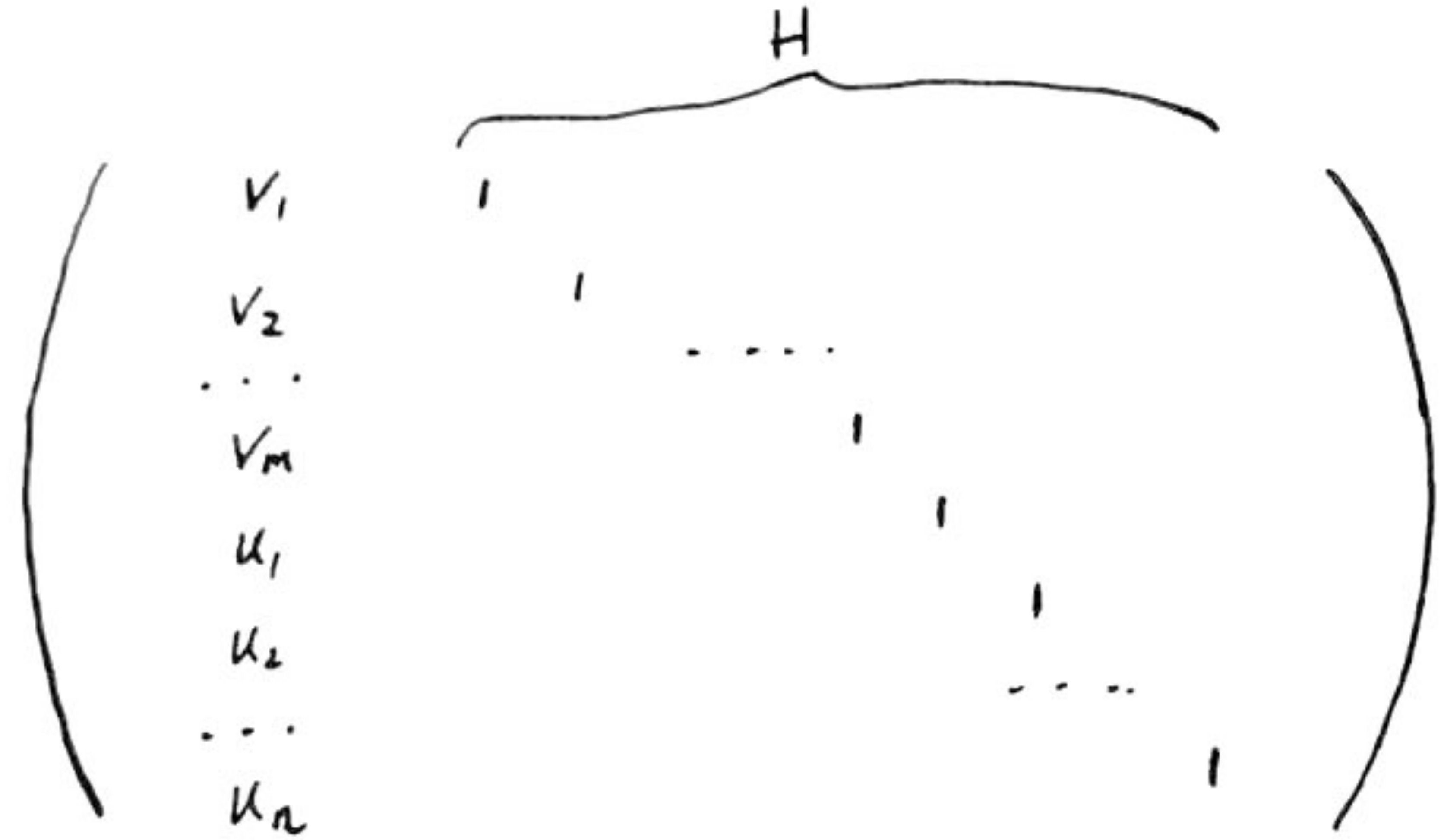
Boehm, Georgoudis, Larsen, Schulze, YZ 2017

Module Intersection

computational algebraic geometry problem

$M_1 = \langle v_1, v_2, \dots v_m \rangle$ each v t-dim row

$M_2 = \langle u_1, u_2, \dots u_n \rangle$ each u t-dim row



Compute the Gröbner basis w.r.t. rows

Position > Term

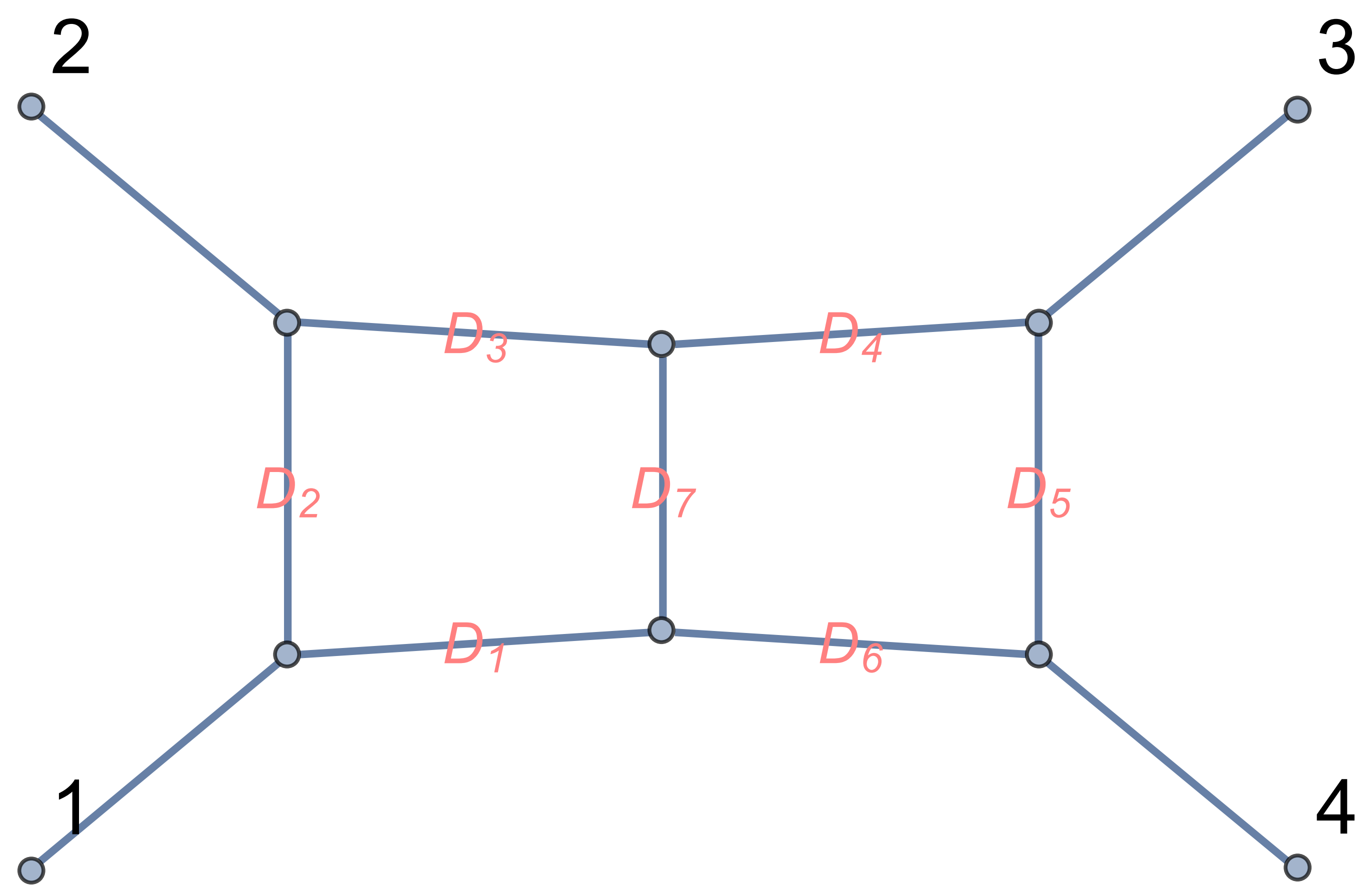
Find the Gröbner basis elements in H

controlled by degree bound

Example, massless double box

$\mathbb{Q}(s,t)[z_1,\ldots z_9]$: 2 parameters, 9 variables

(Each row is a module generator)



$$M_2 = \begin{pmatrix} z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} z_1 - z_2 & z_1 - z_2 & -s + z_1 - z_2 & 0 & 0 & 0 & z_1 - z_2 - z_6 + z_9 & t + z_1 - z_2 & 0 \\ 0 & 0 & 0 & s - z_6 + z_9 & -t - z_6 + z_9 & -z_6 + z_9 & z_1 - z_2 - z_6 + z_9 & 0 & -z_6 + z_9 \\ s + z_2 - z_3 & z_2 - z_3 & z_2 - z_3 & 0 & 0 & 0 & z_2 - z_3 + z_4 - z_9 & -t + z_2 - z_3 & 0 \\ 0 & 0 & 0 & z_4 - z_9 & t + z_4 - z_9 & -s + z_4 - z_9 & z_2 - z_3 + z_4 - z_9 & 0 & z_4 - z_9 \\ -z_1 + z_8 & -t - z_1 + z_8 & s - z_1 + z_8 & 0 & 0 & 0 & -z_1 - z_5 + z_6 + z_8 & -z_1 + z_8 & 0 \\ 0 & 0 & 0 & -s - z_5 + z_6 & -z_5 + z_6 & -z_5 + z_6 & -z_1 - z_5 + z_6 + z_8 & 0 & t - z_5 + z_6 \\ 2 z_1 & z_1 + z_2 & -s + z_1 + z_3 & 0 & 0 & 0 & z_1 - z_6 + z_7 & z_1 + z_8 & 0 \\ 0 & 0 & 0 & s - z_3 - z_6 + z_7 & -z_6 + z_7 - z_8 & -z_1 - z_6 + z_7 & z_1 - z_6 + z_7 & 0 & -z_2 - z_6 + z_7 \\ -z_1 - z_6 + z_7 & -z_1 + z_7 - z_9 & s - z_1 - z_4 + z_7 & 0 & 0 & 0 & -z_1 + z_6 + z_7 & -z_1 - z_5 + z_7 & 0 \\ 0 & 0 & 0 & -s + z_4 + z_6 & z_5 + z_6 & 2 z_6 & -z_1 + z_6 + z_7 & 0 & z_6 + z_9 \end{pmatrix}$$

$M_1 \cap M_2$ is computed within seconds, with **Singular** 4.1.2's **intersect**

Implements

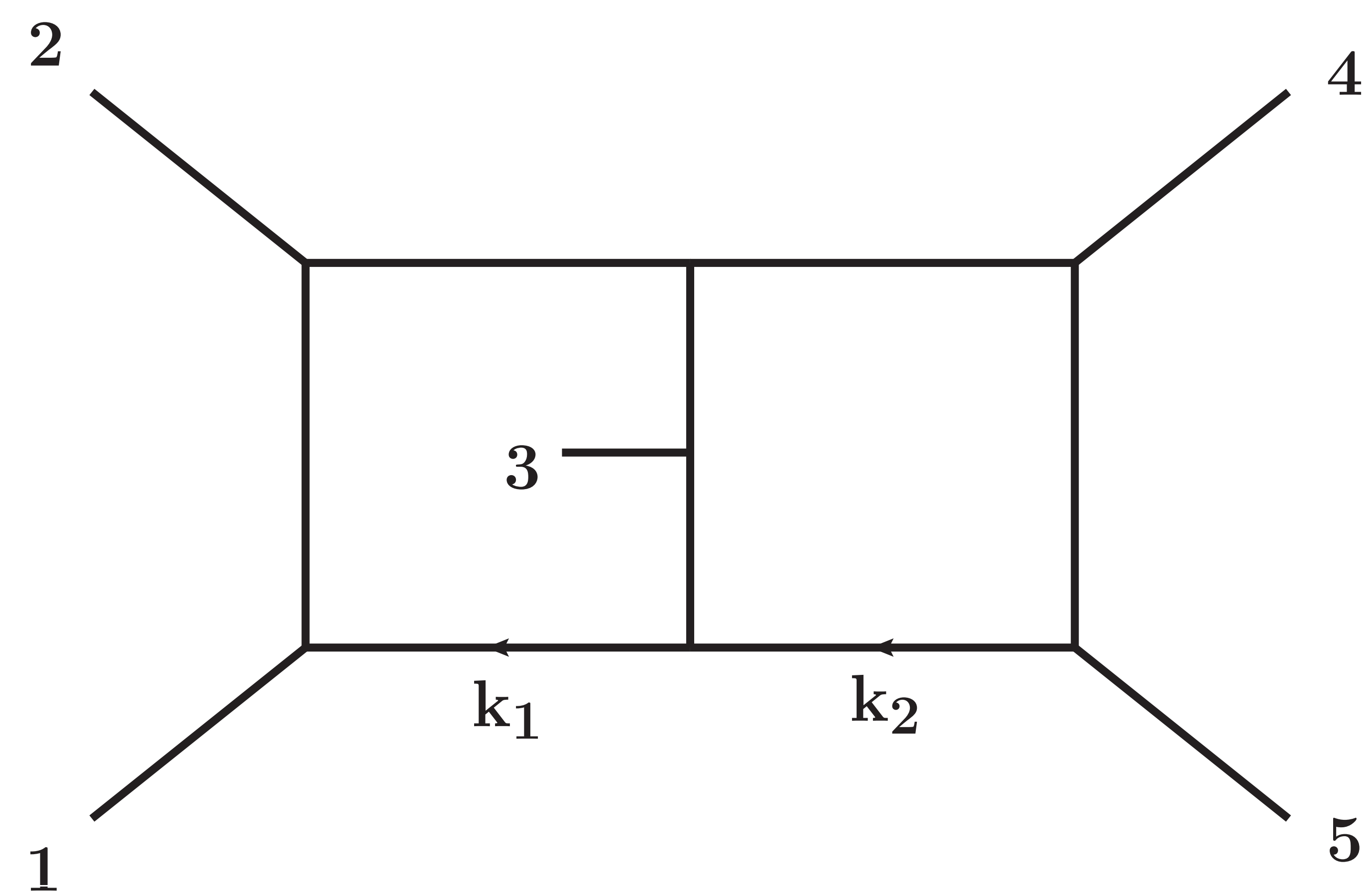
- Use unitarity cuts
- Use degree bound
- Localization trick

Treat parameters as variables, and compute in a block ordering

$$[\text{variables}] > [\text{parameters}]$$

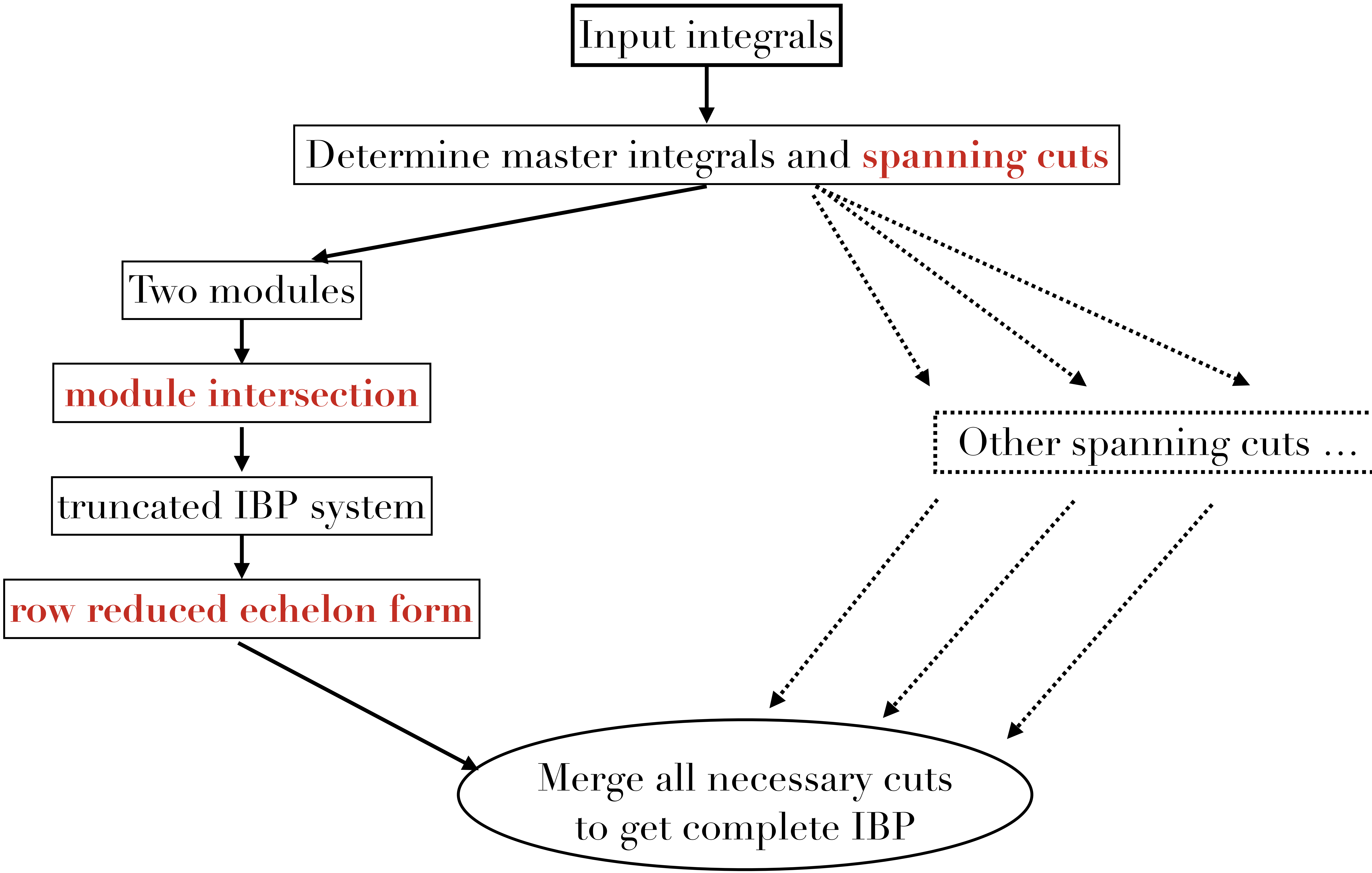
A famous trick in computational algebra

Now module intersection is really fast

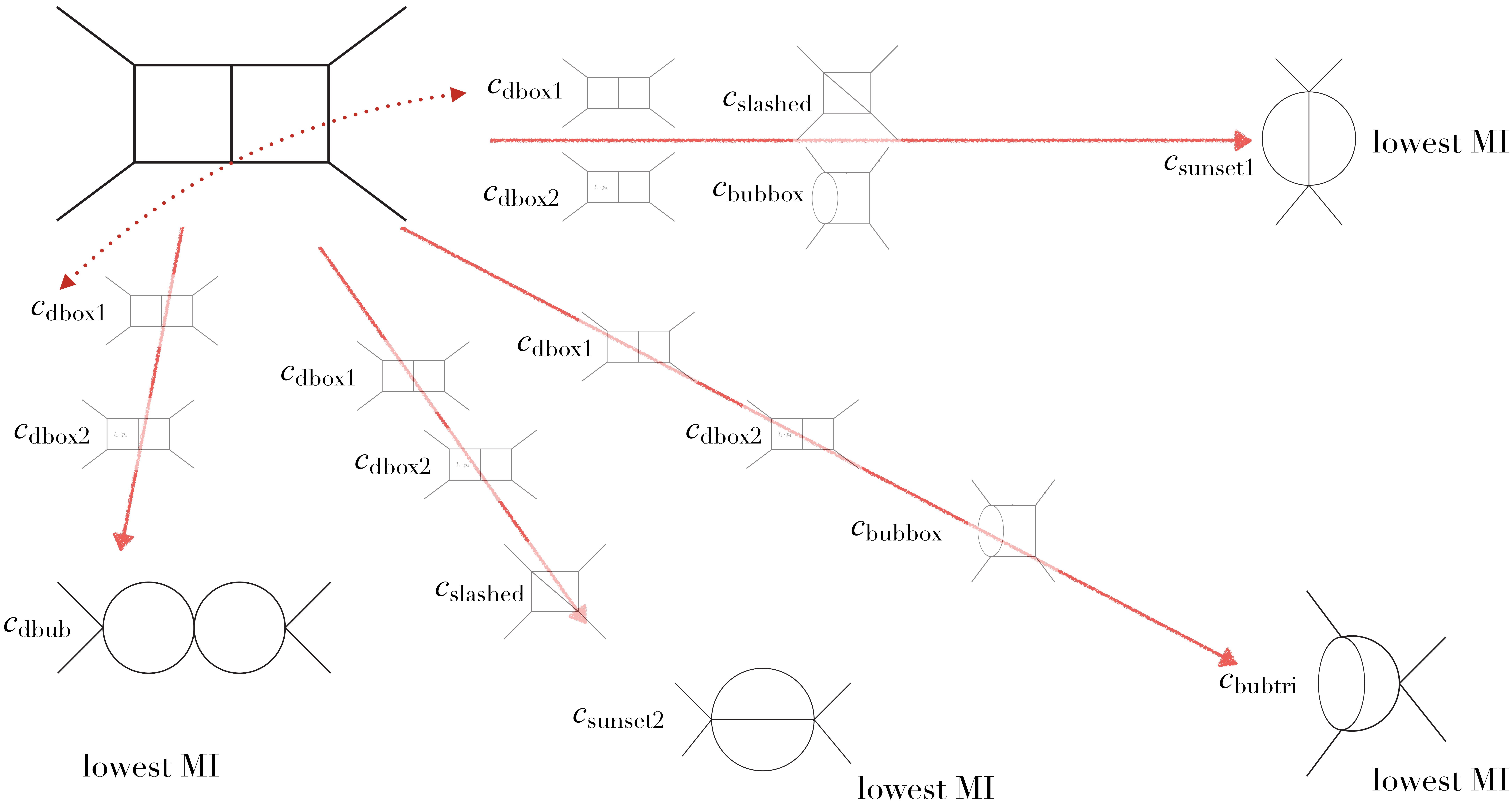


5 Mandelstam variables, with a triple cut
seconds to get the module intersection (and truncated IBPs)

After module



Example, massless double box with spanning cut



Remove cuts overlap

$$\begin{pmatrix} 1 & \boxed{12} & 0 & 0 & \boxed{-124} \\ 0 & 0 & 1 & 0 & 31 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix}$$

Set one non-pivot column (one master integral) to zero before reduction,
does NOT change other non-pivot columns after reduction

Chawdhry, Lim, Mitov Phys. Rev. D 99, 076011 (2019)
also implemented in Kira

If one master integral appears on two cuts, pick up one cut and set this integral to zero.

Can also be used for double propagator integrals

$$0 = \int_\Omega dz_1 \ldots dz_4 \sum_{i=1}^{4} \frac{\partial}{\partial z_i} \left(a_i(z) \frac{F^{\frac{D-L-E-1}{2}}}{z_1 z_2 z_3 z_4^2} \right)$$

the fourth index must be less or equal 2

$$(-5+d)G[1,0,1,2] + (5-d)G[1,1,1,1] - tG[1,1,1,2] = 0$$

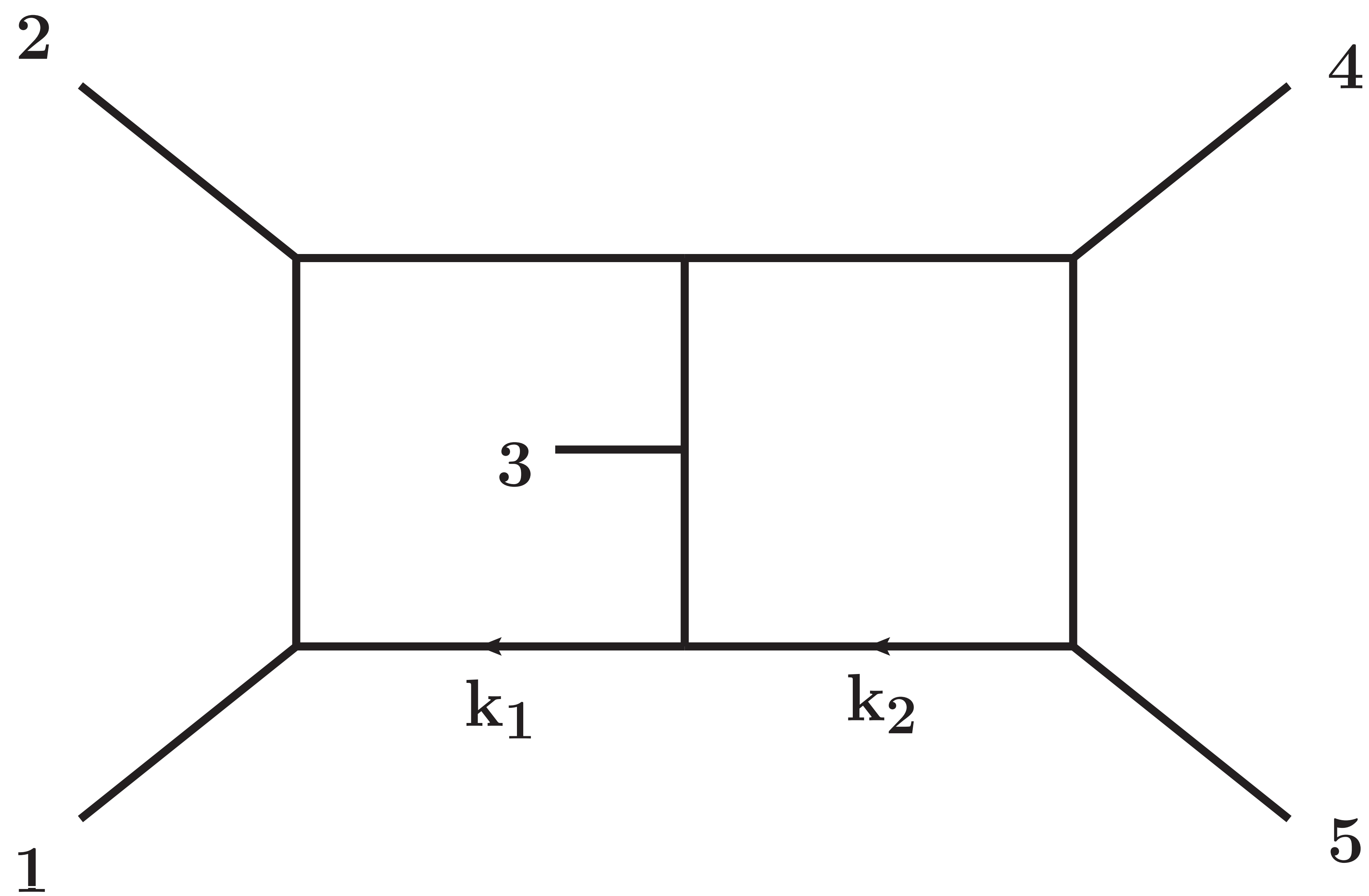
targeted reduction for integrals with doubled (fourth-)propagator

Non-trivial example

Towards an industry-level row-reduction program

- Row Reduction code written in **Singular**
- With numeric fitting, powered by the large-scale parallelization framework **GPI-space**

Janko Boehm's talk

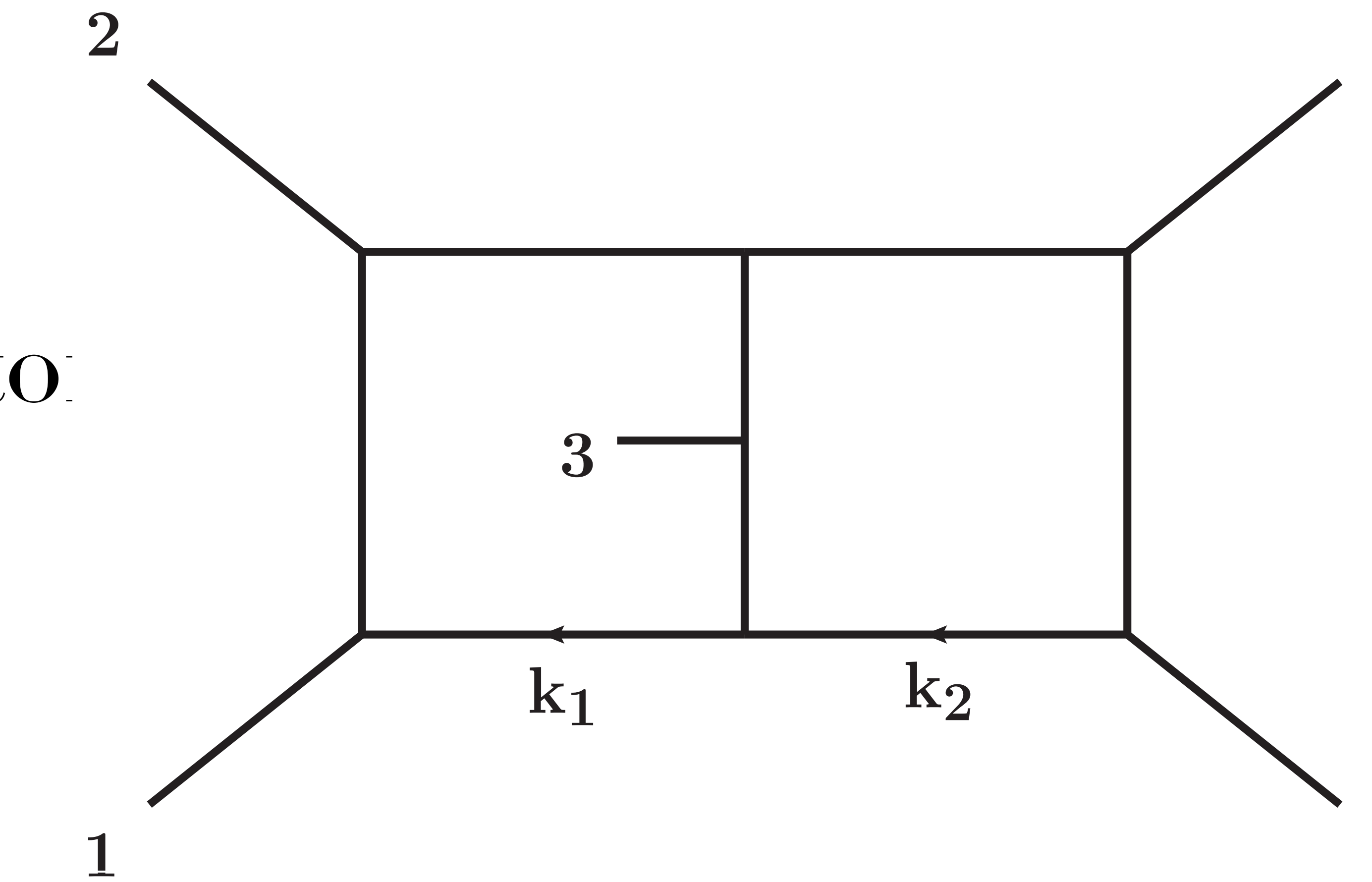


Analytic (symbol) Abreu, Dixon, Herrman, Page, Zeng
"*The two-loop five-point amplitude in N=4 sYM theory* ", Phys.Rev.Lett. 122 (2019), no. 12 121603

Analytic (function) Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia
"*All master integrals for three-jet production at NNLO*", Phys.Rev.Lett. 123 (2019), no. 4 041603

Row Reduction

reduce all rank-4 numerato
to master integrals



Cut	# relations	# integrals	size
{1,5,7}	1134	1182	0.77 MB
{1,5,8}	1141	1192	0.85 MB
{1,6,8}	1203	1205	1.1 MB
{2,4,8}	1245	1247	1.1 MB
{2,5,7}	1164	1211	0.84 MB
{2,6,7}	1147	1206	0.62 MB
{2,6,8}	1126	1177	0.83 MB
{3,4,7}	1172	1221	0.78 MB
{3,4,8}	1180	1226	1.0 MB
{3,6,7}	1115	1165	0.82 MB
{1,3,4,5}	721	762	0.43MB

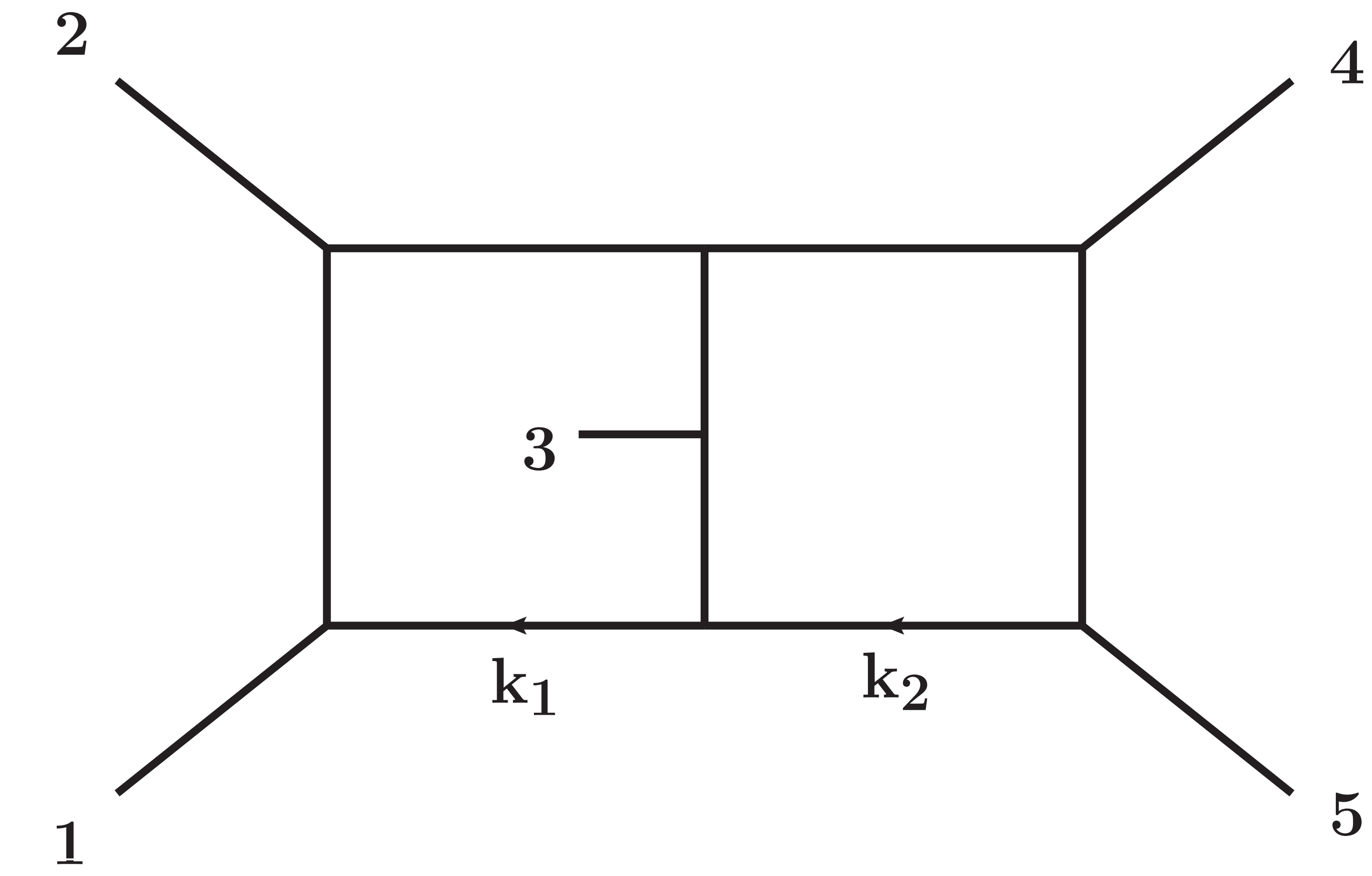
The analytical IBP reduction is actually NOT needed.

Just use these truncated, sparse IBP systems to generate numeric IBP for amplitude, and then interpolate!

Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, YZ 1908.04301

Row Reduction

reduce all rank-4 numerators analytically
to master integrals



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{1,3,4,5}	721	762	0.43MB

Really want the analytic IBP reduction?

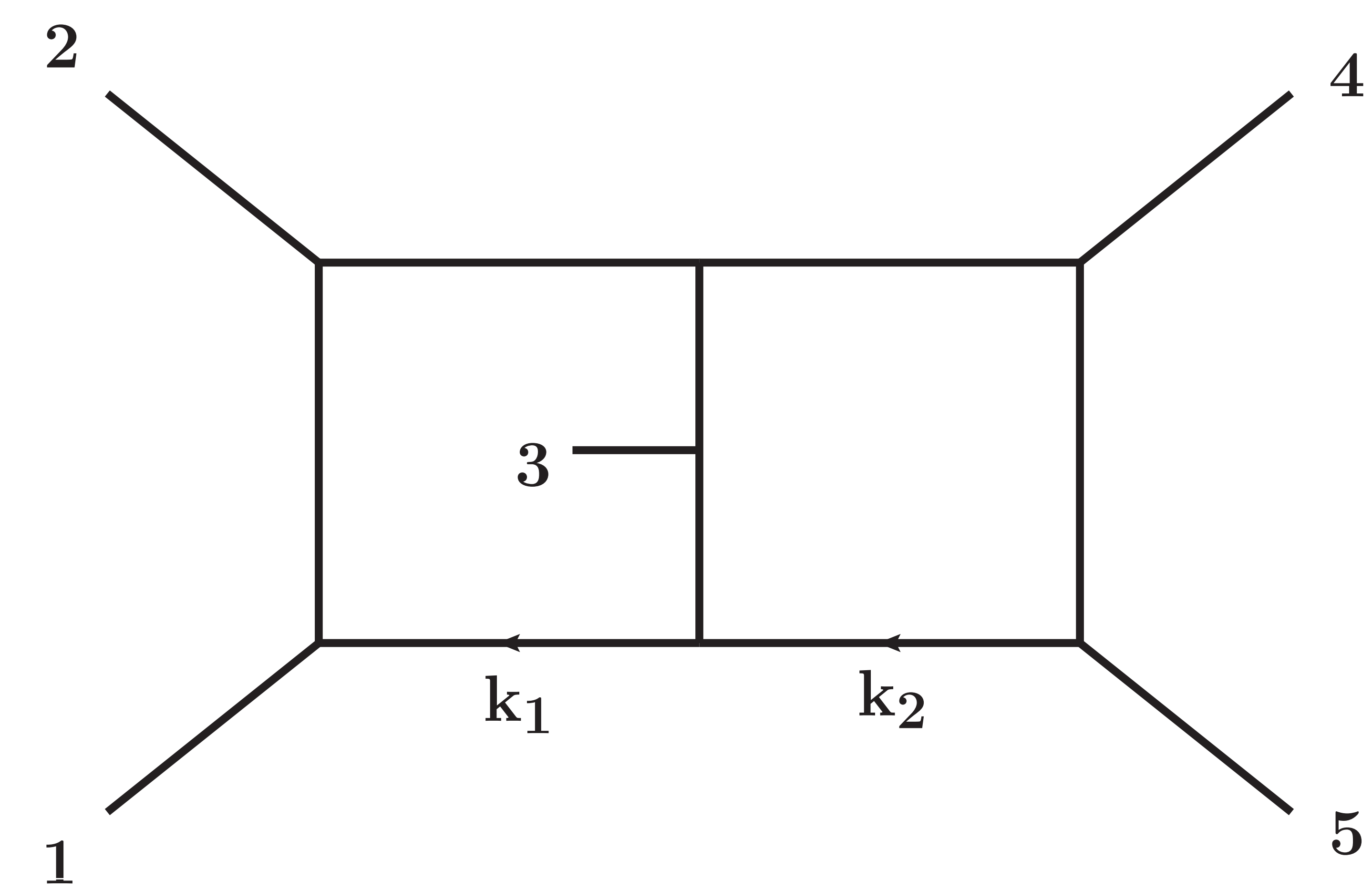
Hardest cut: done within 12 hours, 384 cores
GPI-space

Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, YZ 1908.04301

Use UT integrals to simplify coefficients

UT integrals (Johannes Henn's talk)

	IBP total size	Highest power in s
Normal Basis	~2.0 GB	35
UT Basis	0.48 GB	20



shorter coefficients

fewer numeric computations needed!

Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, YZ 1908.04301

You may ask about the rank-5 reduction …

It is on the way …
Sorry I am busy teaching this semester …

Summary

- Module intersection + Large-scale parallelzation with GPI-space
- a powerful IBP algorithm
- Since we used Baikov cut form everywhere, some relation to Intersection Theory?
- efficient Lee-Pomeransky IBPs?

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If you have interesting IBP problems,

you may send them to yzhphy@ustc.edu.cn. Thanks!