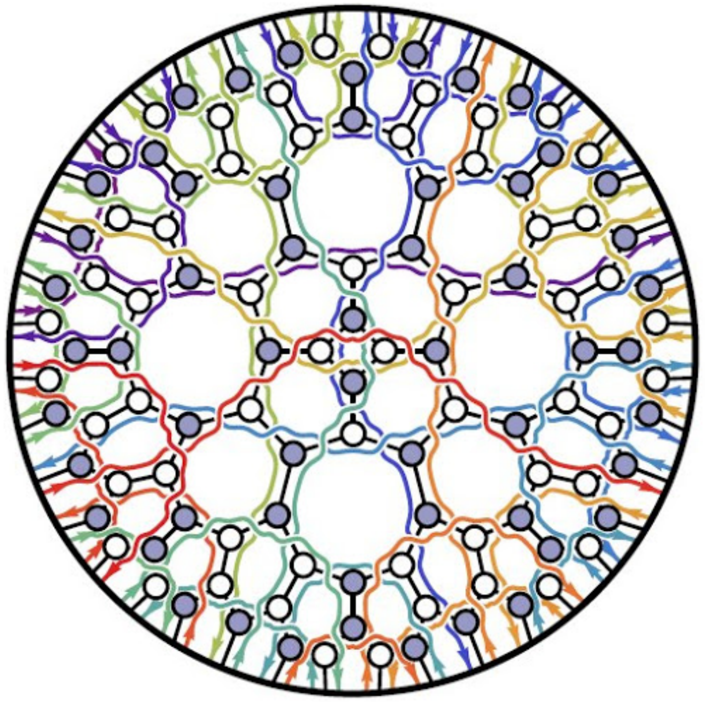
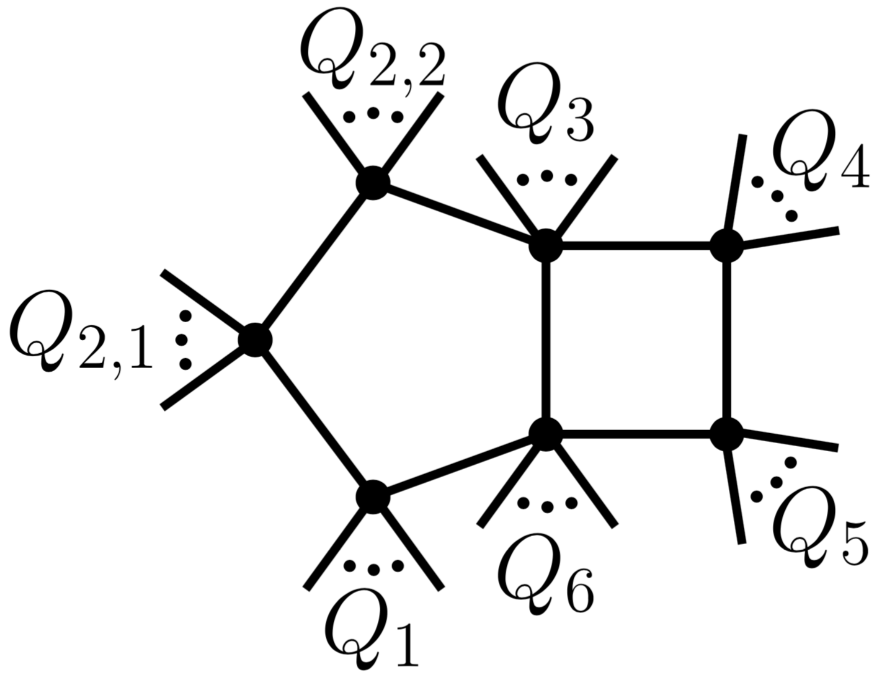




Enrico Herrmann (SLAC)

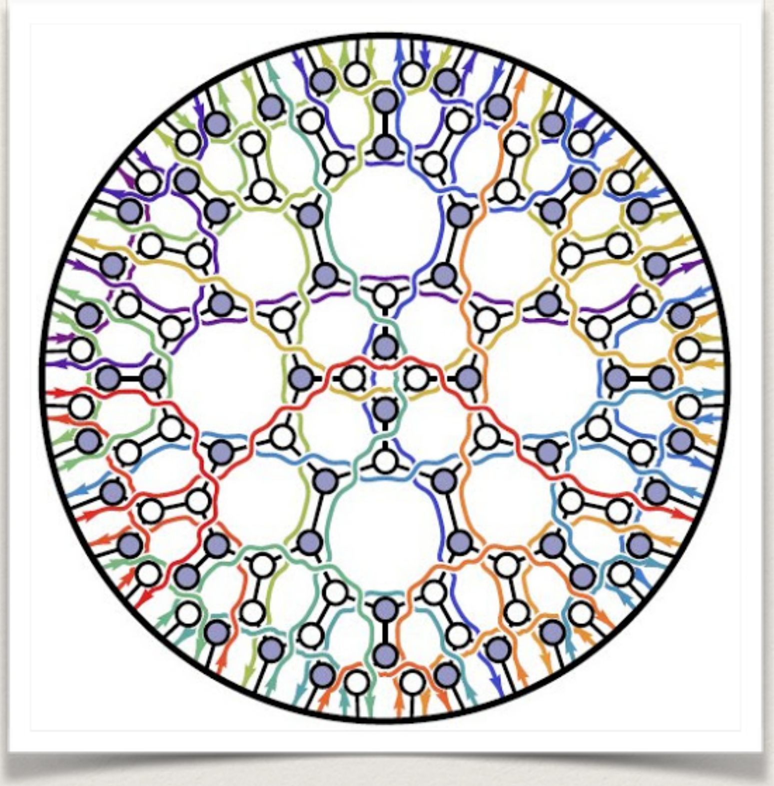
background: Arkani-Hamed,Cachazo,Goncharov,Postnikov,Trnka: 1212.5605
Arkani-Hamed,Trnka: 1312.2007,1312.7878
Bern,EH,Litsey,Stankowicz,Trnka: 1512.08591
Arkani-Hamed,Bai,Lam: 1703.04541
based on: EH, Parra-Martinez: 1909.04777 incredible thanks to Simon Caron-Huot for support

Logarithmic forms and differential equations for Feynman integrals

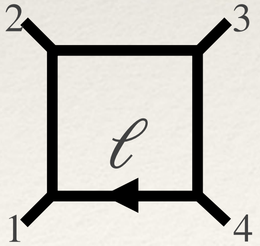
MathemAmplitudes 2019 -
Padova
12/20/2019

(0) Motivation

- scattering amplitudes: playground for fascinating interplay between physics & geometry, maths
- (before integration) novel geometric structures:
    - Grassmannian [space of k-planes in n-dim]
      [Arkani-Hamed,Cachazo,Goncharov,Postnikov,Trnka: 1212.5605]
    - Amplituhedron
      [Arkani-Hamed,Trnka: 1312.2007]
- common theme: associated to geometries are canonical dlog-forms
- here: new dlog-representation for
  Feynman integrals



[see J. Henn talk on Wednesday]



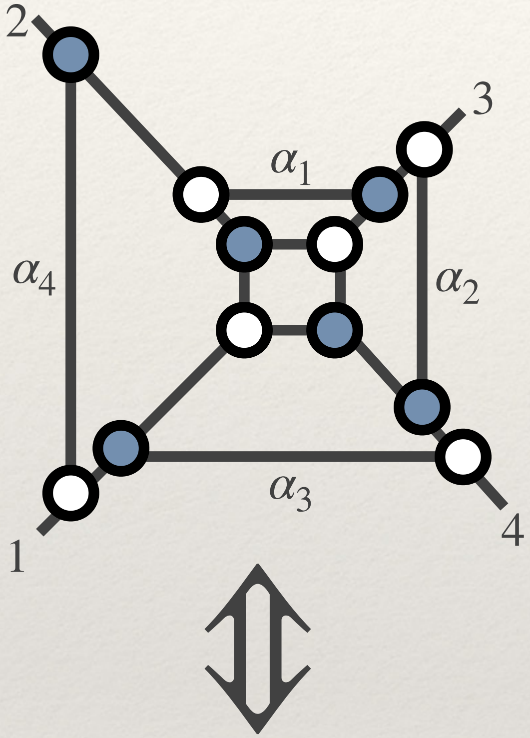
$$\Omega = d\log\frac{\ell^2}{(\ell-\ell^*)^2}d\log\frac{(\ell-p_1)^2}{(\ell-\ell^*)^2}d\log\frac{(\ell-p_1-p_2)^2}{(\ell-\ell^*)^2}d\log\frac{(\ell+p_4)^2}{(\ell-\ell^*)^2}$$

(1) Outline

- i) dlog-forms of Feynman integrals (FIs)
    - theoretical physics interest: do more general QFTs admit dual geometric description?
    - practical interest: *dlog* integrals → simplified diff. eqs.

- ii) spacetime geometry of Feynman integrals
    - *dlogs* are almost primitives → beg to be integrated

(3) Feynman integrals in dlog-form

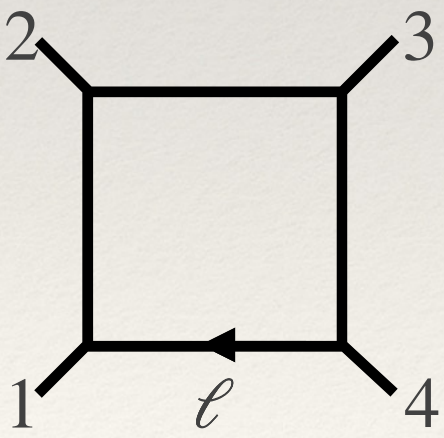


$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4} \times \delta(C \cdot \mathscr{Z})$$

- logarithmic form, dictated by the Grassmannian

- can identify and solve for Feynman loop variables ℓ^μ

[Arkani-Hamed,Cachazo,Goncharov,Postnikov,Trnka: 1212.5605]

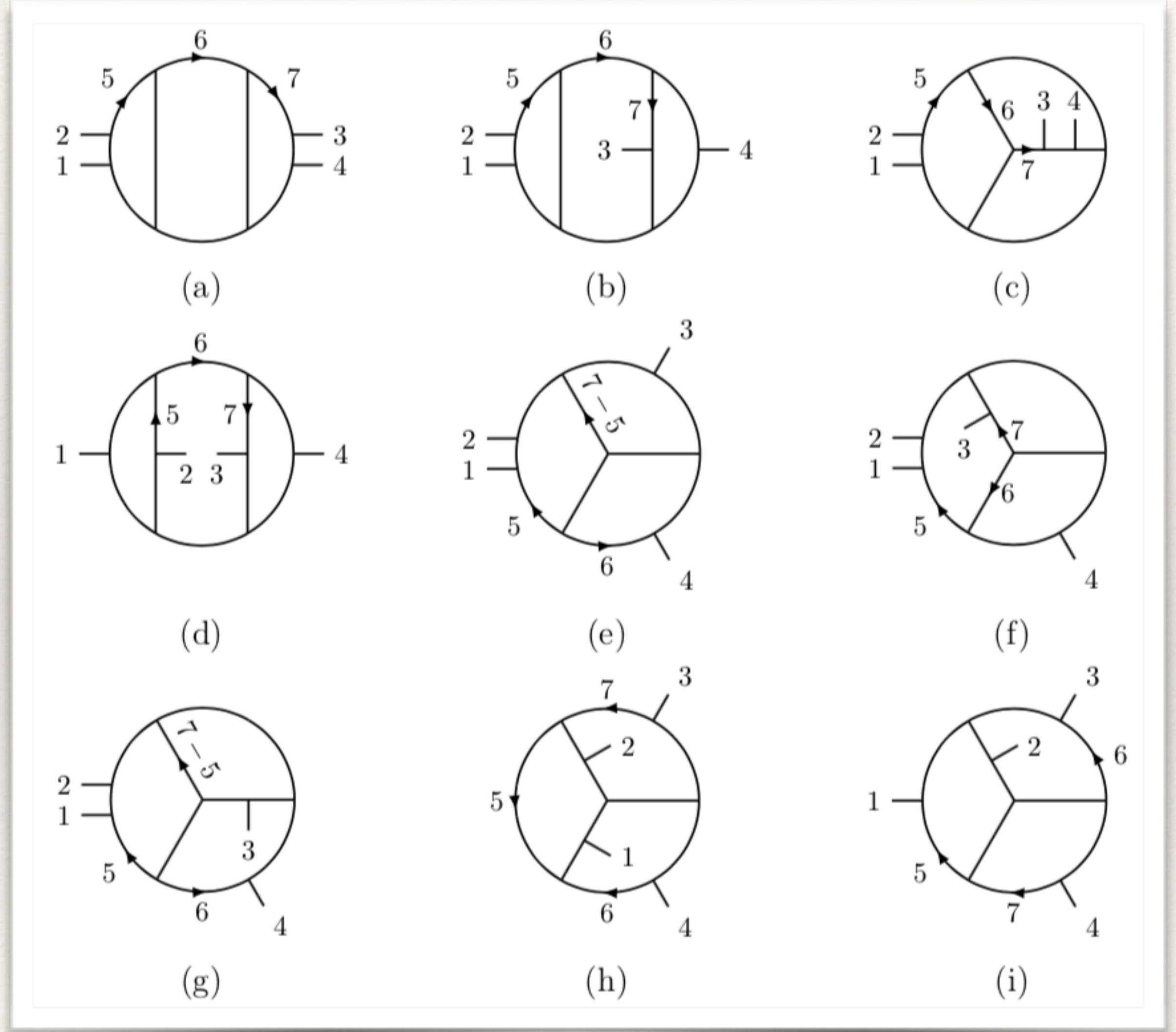


$$\Omega = d\log\frac{\ell^2}{(\ell-\ell^*)^2}d\log\frac{(\ell-p_1)^2}{(\ell-\ell^*)^2}d\log\frac{(\ell-p_1-p_2)^2}{(\ell-\ell^*)^2}d\log\frac{(\ell+p_4)^2}{(\ell-\ell^*)^2}$$

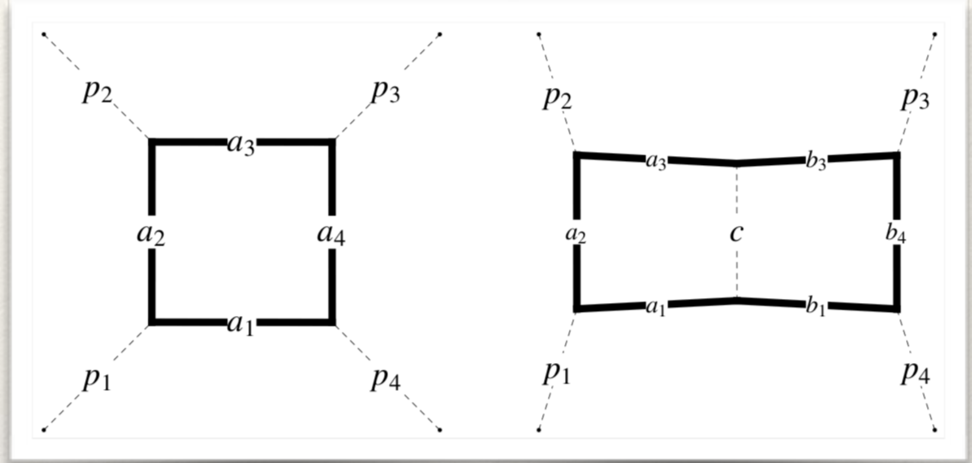
new representation of Feynman integrals

dlog-representation exists for more general FI

[Bern,EH,Litsey,Stankowicz,Trnka: 1412.8584, 1512.08591]
[Henn,Mistlberger: 1608.00850]



[EH,Parra-Martinez: 1909.04777]
[Caron-Huot,Henn: 1404.2922]



dlog forms exist for special integrals

- related to UT conjecture of $\mathcal{N} = 4$ sYM
- basis of integrals for Henn diff. eqs.
- new symmetries of nonplanar theories?
- potential geometric interpretation?

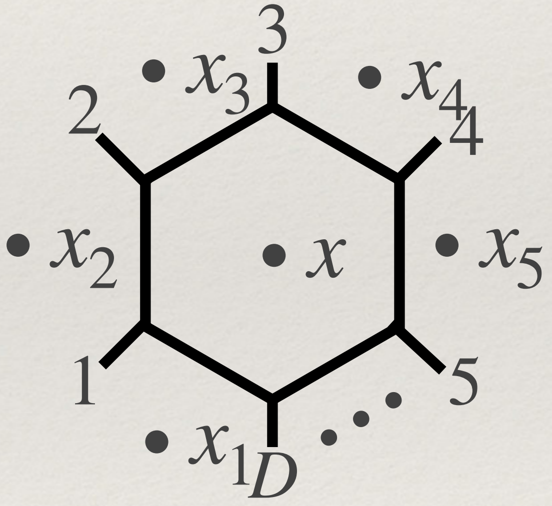
have nice integrals in *dlog* form, now what?

(3) Integration of dlog-form Feynman integrals

What to do with *dlog* forms?

- almost have *primitive*, they want to be integrated!
- why are L-loop FI of transcendental weight 2L in d=4?
- learn sth. new about these integrals

Study some concrete examples: $D-$ gons in D dim



$$I_D = \int\limits_{\Sigma_D} \frac{d^D x}{\pi^{D/2}} \frac{N[x_i, m_i^2]}{[(x-x_1)^2 + m_1^2]\cdots[(x-x_D)^2+m_D^2]}$$

- $D \in \mathbb{N}$
- $(x-x_i)^2 = \| x - x_i \|^2$ Eucl. Norm
- compact, convergent integral > 0

well defined FI, typically done in parametric space

(3.0) baby toy example

Single variable example for integrating *dlog* forms:

- finding the primitive of the integrand

$$I(a,b) = \int_0^a \frac{dx}{x+b} = \int_0^a d\log(x+b) = \log\left(\frac{a+b}{b}\right)$$

- consider the differential of $I(a,b)$ given by Leibniz rule

$$dI(a,b) = \underbrace{\frac{da}{a+b}}_{\text{variation of bdry.}} + \underbrace{\left[\frac{db}{a+b} - \frac{db}{b}\right]}_{\text{variation of the integrand}} = d\log\left(\frac{a+b}{b}\right)$$

- change variables $x = ay$: $I(a,b) = \int_0^1 d\log(ya+b)$

- $d\log(ya+b)$ is a closed form on the full space (y,a,b)

$$d^2\log(ya+b) = 0 \quad \Rightarrow \quad d_{a,b}\big(d_y\log(ya+b)\big) = -d_y\big(d_{a,b}\log(ya+b)\big)$$

differenital of I(a,b), given by *localization* to integration boundary

(3.1) kindergarten toy example

two variable example for integrating *dlog* forms:

$$I(a) = \frac{1}{2\pi i}\int_{\Sigma} \omega = \frac{1}{2\pi i}\int_{\Sigma} dz\,d\bar{z}\,\frac{\sqrt{1+4a}}{(z\bar{z}+a)[(z+1)(\bar{z}+1)+a]}$$

$\in R^+$

integration cycle, real slice in $\mathbb{C}_2$ where $\bar{z} = z^*$

in polar coordinates, can integrate easily:

$$I(a) = \log\left(\frac{1+2a+\sqrt{1+4a}}{1+2a-\sqrt{1+4a}}\right) = \log\left(\frac{\sqrt{1+4a}+1}{\sqrt{1+4a}-1}\right)^2$$

reproduce the result in an interesting way that generalizes:

$$\omega = \frac{1}{2}\, d\log\frac{z\bar{z}+a}{(z+1)(\bar{z}+1)+a}\; d\log\frac{(z-z_+)(\bar{z}-\bar{z}_+)}{(z-z_-)(\bar{z}-\bar{z}_-)}$$

$\{z_\pm, \bar{z}_\pm\}$, solutions to quadratic equations

$$z\bar{z}+a=0\,, \quad (z+1)(\bar{z}+1)+a=0$$

$$z_\pm = \bar{z}_\mp = -\frac{1}{2}\left(1\pm\sqrt{1+4a}\right)$$

Equivalence of integrand forms is easy to check, harder to derive

(3.1) kindergarten toy example

$$I(a) = \frac{1}{2\pi i}\int_{\Sigma} \omega$$

$$\omega = \frac{1}{2}\, d\log\frac{z\overline{z}+a}{(z+1)(\overline{z}+1)+a}\, d\log\frac{(z-z_+)(\overline{z}-\overline{z}_+)}{(z-z_-)(\overline{z}-\overline{z}_-)}$$

consider ω on the full space $\{z, \bar{z}, a\}$ with differential $d = d_i + d_a$

internal/integration variables

external parameters

$$\omega = \omega^{(2,0)} + \omega^{(1,1)}$$

on the full space, ω is a closed form, i.e. $d\omega = 0$

$$d\omega = 0 \quad \leftrightarrow \quad d_a\omega^{(2,0)} = -d_i\omega^{(1,1)}$$

total derivative in integration variables

simple relation has far-reaching consequences for differential eqs.

(3.1) kindergarten toy example

$$d\omega = 0 \quad \leftrightarrow \quad d_a\omega^{(2,0)} = -d_i\omega^{(1,1)}$$

total derivative in integration variables

How is I(a) nonzero, if its differential is related to a total derivative?

$$\omega = \frac{1}{2}\, d\log\frac{z\bar{z}+a}{(z+1)(\bar{z}+1)+a}\, d\log\frac{(z-z_+)(\bar{z}-\bar{z}_+)}{(z-z_-)(\bar{z}-\bar{z}_-)}$$

contains additional poles

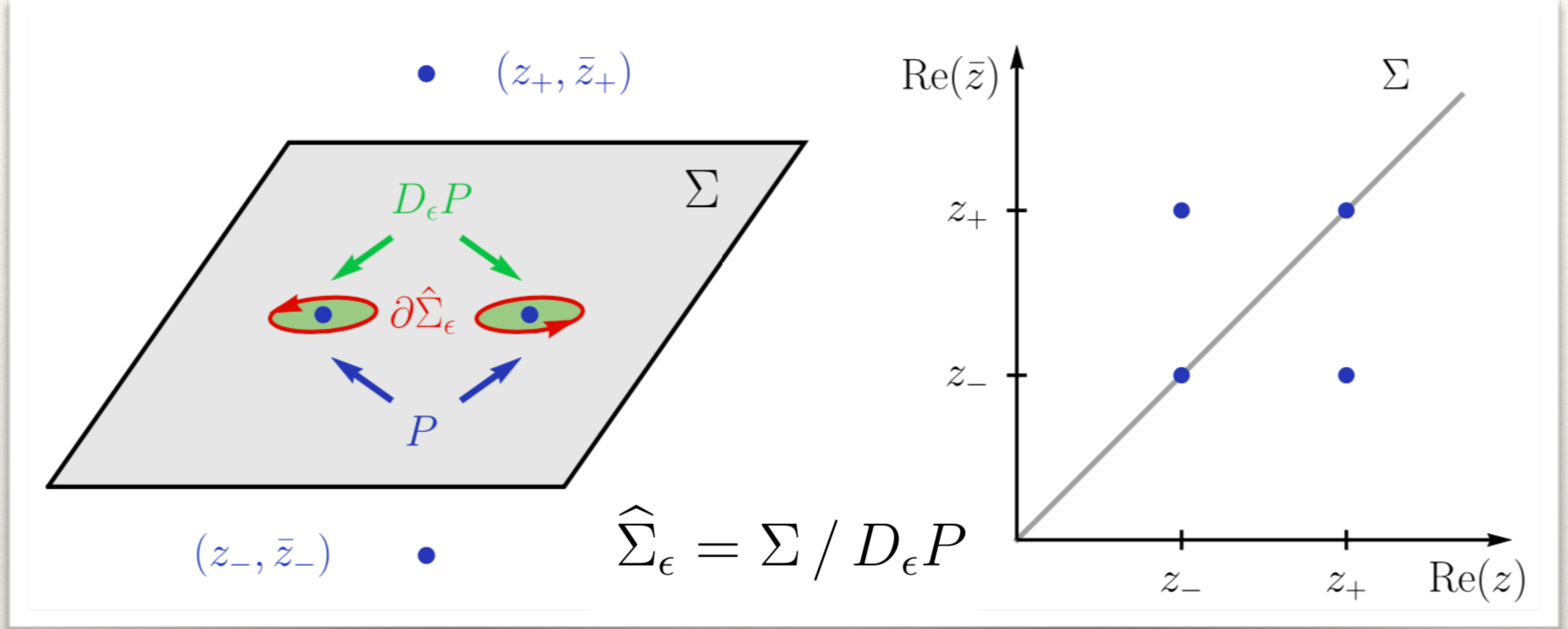
$$\omega^{(1,1)} = \frac{1+2a(2+z+3\bar{z})+2\bar{z}-(z-\bar{z})(2z\bar{z}+z+\bar{z})}{2\sqrt{1+4a}\,[z-z_+][z-z_-][z\bar{z}-a][(z+1)(\bar{z}+1)+a]}\,dz\,da + (z\leftrightarrow\bar{z})$$

singular points intersect integration cycle

$$P \equiv \{(z,\bar{z}) = (z_+,\bar{z}_-) \cup (z,\bar{z}) = (z_-,\bar{z}_+)\}$$

(3.1) kindergarten toy example

Excising singularities from the integration domain



$$\widehat{\Sigma}_\epsilon = \Sigma \,/\, D_\epsilon P$$

total derivative *localizes* by Stoke's theorem to boundary

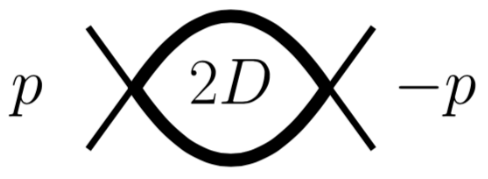
$$dI = -\frac{1}{2\pi i}\int_{\widehat{\Sigma}_\epsilon} d_i\omega^{(1,1)} = -\frac{1}{2\pi i}\int_{\partial\widehat{\Sigma}_\epsilon} \omega^{(1,1)}$$

Residue at the singular points

$$dI = -\mathrm{Res}_P[\omega^{(1,1)}] = d\log\frac{[z_+\bar{z}_- + a][(z_- + 1)(\bar{z}_+ + 1) + a]}{[z_-\bar{z}_+ + a][(z_+ + 1)(\bar{z}_- + 1) + a]}$$

(3.1) bubble integral in D=2

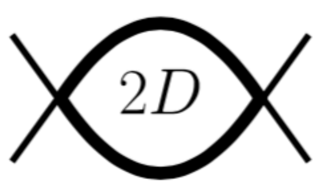
secretly, I described the bubble in D=2



$$= \int \frac{d^2\ell}{i\pi} \frac{\sqrt{p^2(p^2-4m^2)}}{[\ell^2-m^2][(\ell+p)^2-m^2]}$$

$$\omega = \frac{1}{2}\, d\log \frac{\ell^2-m^2}{(\ell+p)^2-m^2}\, d\log \frac{(\ell-\ell_+)^2}{(\ell-\ell_-)^2}$$

- light-cone variables: $\ell^0-\ell^1 = z\,(p^0-p^1)\,, \quad \ell^0+\ell^1 = \bar{z}\,(p^0+p^1)$



$$d\left[\quad\right] = d\log \frac{[\ell_\bullet^2-m^2][(\ell_\circ+p)^2-m^2]}{[\ell_\circ^2-m^2][(\ell_\bullet+p)^2-m^2]}$$

solutions to $(\ell-\ell_+)^2=0$
$(\ell-\ell_-)^2=0$

- new about these integrals: 3-step integration process

  1) find the primitive, i.e. *dlog*-form

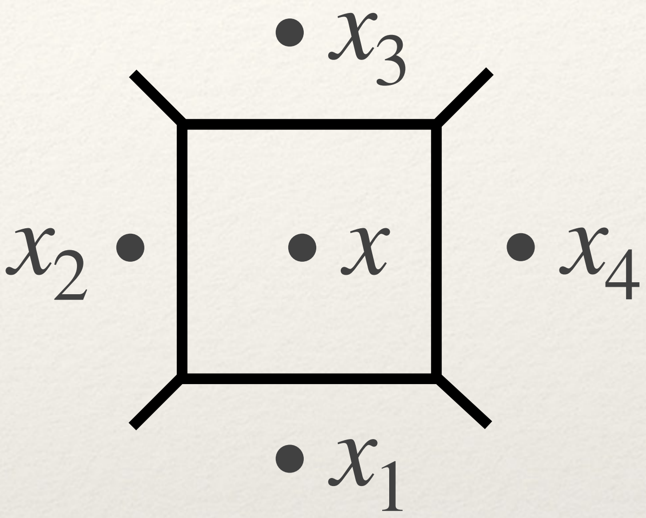
  2) localization step ["integrate"]

  3) partial fraction (later)

⇒ localization is key!
drops $D \to D-2$

(3.2) D=4 box integral

$$dI_4 = d\int \Omega_4 = d\frac{1}{2\pi^2}\int_{\Sigma_4} d\log\frac{D_1}{D_2}d\log\frac{D_2}{D_3}d\log\frac{D_3}{D_4}d\log\frac{D_+}{D_-}$$



$$D_+ = (x - x_+)^2, \qquad D_- = (x - x_-)^2$$

$$x_\pm \leftrightarrow D_1 = D_2 = D_3 = D_4 = 0$$

$$x_{ij}^2 \neq 0, \quad x_{ij} = x_i - x_j$$

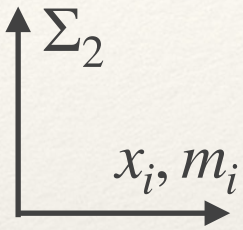
– same trick as before, IBP to localize on Σ_2 where $d\log\frac{D_+}{D_-}$ is 0 in int. region

$$dI_4 = \int_{\Sigma_2} d\log\frac{D_1}{D_2}d\log\frac{D_2}{D_3}d\log\frac{D_3}{D_4}$$

– localization reduces dim by 2 ⇒ exposes recursive structure!

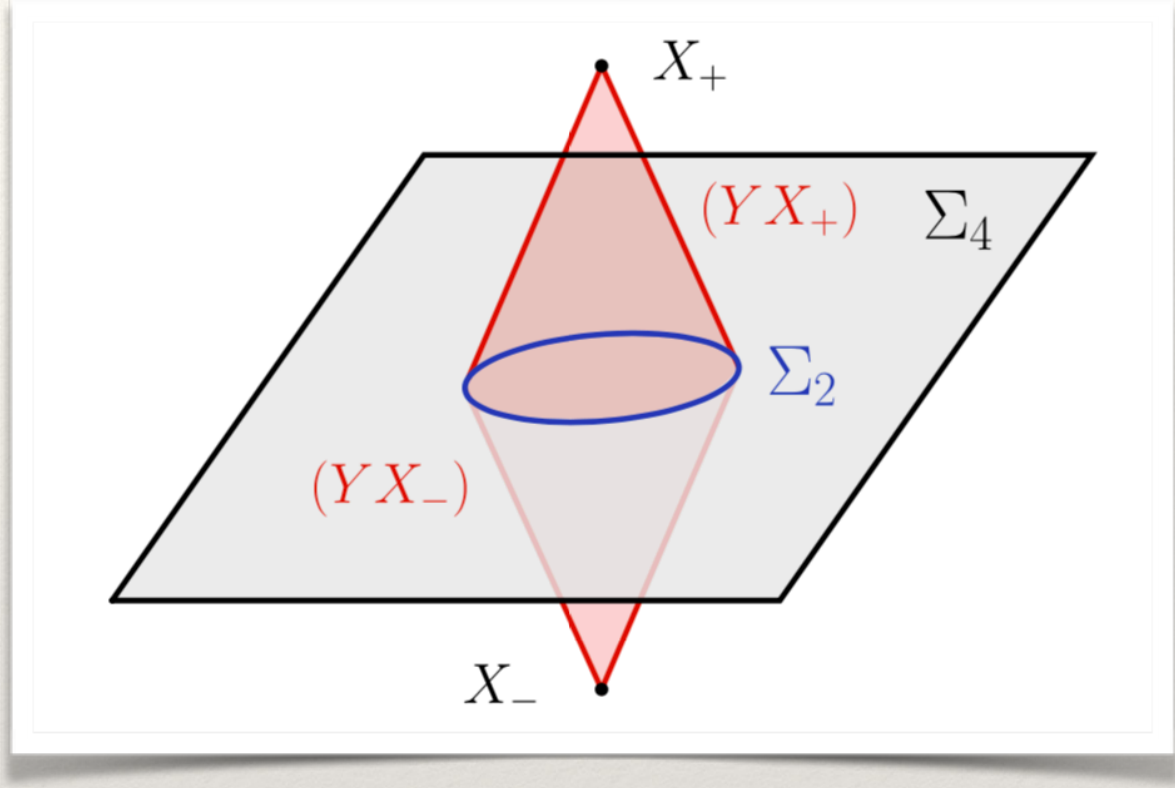
(3.3) partial fractioning - taking d's

$$dI_4 = \int_{\Sigma_2} \underbrace{d\log\frac{D_1}{D_2}\, d\log\frac{D_2}{D_3}\, d\log\frac{D_3}{D_4}}$$



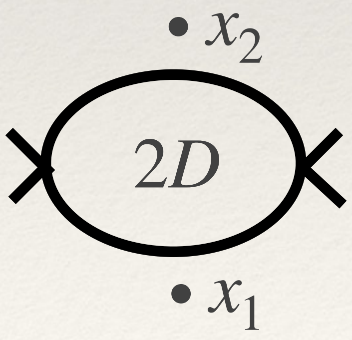
rational form ⇒ partial fraction

a.k.a. generalized unitarity



$-$expand in terms of $\frac{1}{D_i D_j}$ on the Σ_2 ⇒ sum of six 2D bubbles:

$$dI_4 = d\log\frac{D_3(x^*_{34})}{D_4(x^*_{34})}\frac{D_4(\bar{x}^*_{34})}{D_3(\bar{x}^*_{34})} \times$$



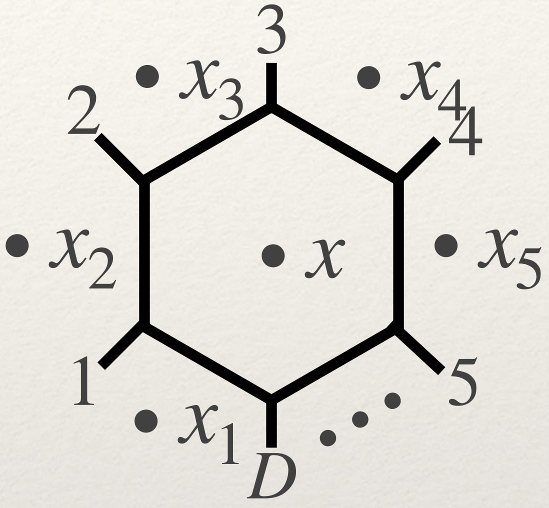
$$+ \cdots$$

$x^*_{34}, \bar{x}^*_{34} \leftrightarrow D_+ = D_- = D_1 = D_2 = 0$

(3.3) recursive formula

D-gons in D spacetime dimensions:

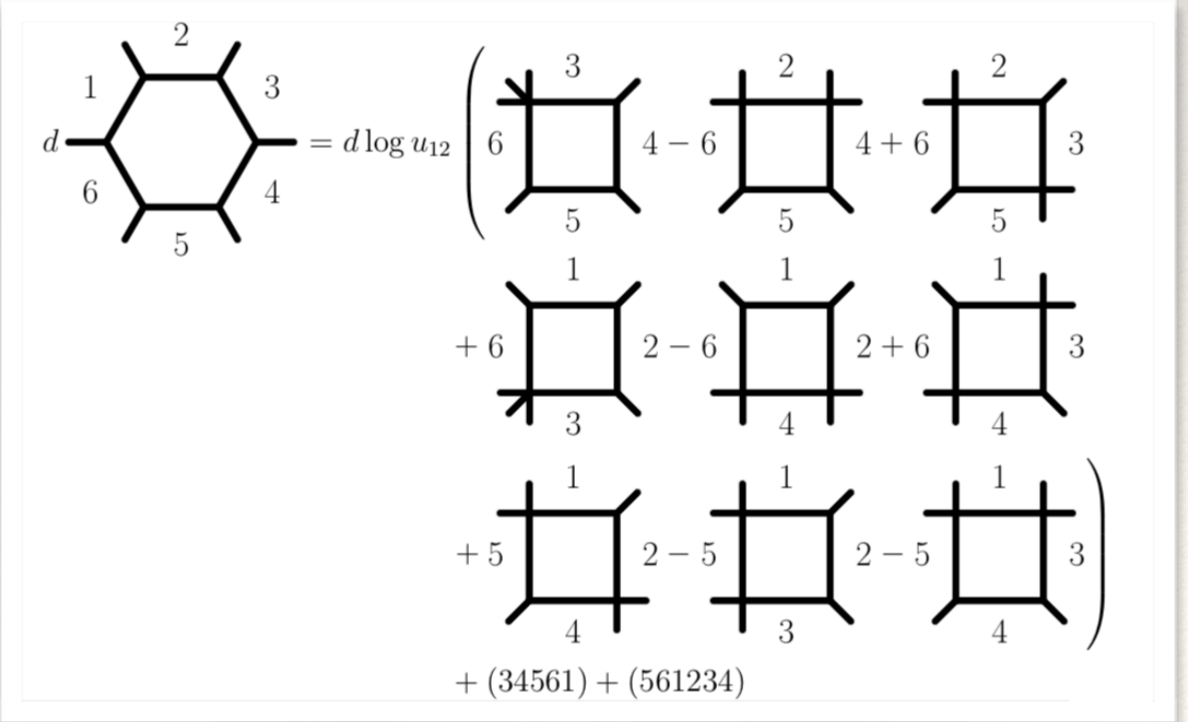
$$I_D = \frac{1}{2}\int_{\Sigma_D} d\log\frac{D_1}{D_2}\, d\log\frac{D_2}{D_3}\,\cdots\, d\log\frac{D_{D-1}}{D_D}\, d\log\frac{D_+}{D_-}$$



$$\boxed{dI_D = \sum_{i,j} d\log\frac{D_i(x^*_{ij})}{D_j(x^*_{ij})}\frac{D_j(\bar{x}^*_{ij})}{D_i(\bar{x}^*_{ij})} \times I_{D-2}(x_1\ldots\hat{x}_i\ldots\hat{x}_j\ldots x_D)}$$

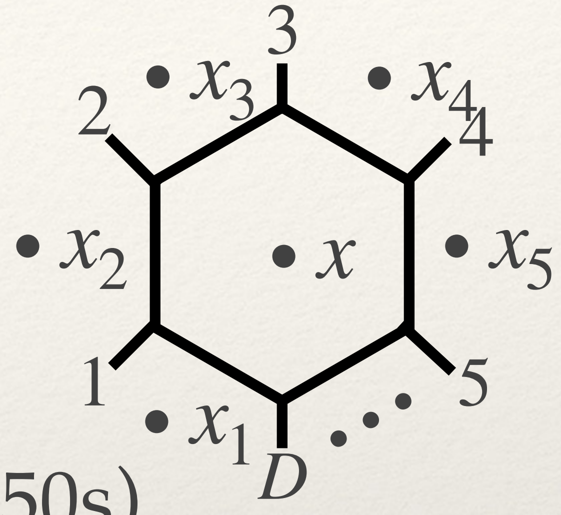
massless hexagon in D=6:

use *residue theorems* to bring differential eqs. into *minimal form*

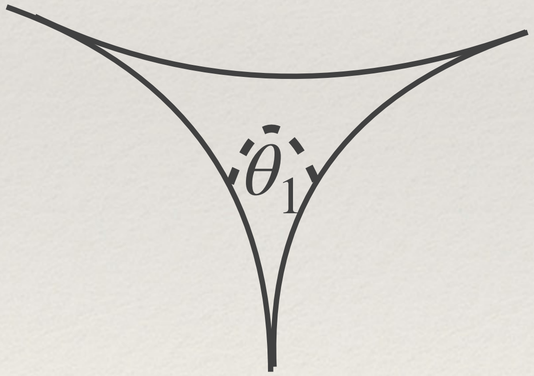


(3.3) recursive formula

$$dI_D = \sum_{i,j} d\log \frac{D_i(x^*_{ij})}{D_j(x^*_{ij})} \frac{D_j(\bar{x}^*_{ij})}{D_i(\bar{x}^*_{ij})} \times I_{D-2}(x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_D)$$



–Schläfli formula for volumes of hyperbolic simplices (1850s)
[Aomoto; Davydychev,Delbourgo; Nandan,Paulos,Spradlin,Volovich]



$$H^2 : \quad A = \pi - \theta_1 - \theta_2 - \theta_3$$

$$dA = -(d\theta_1 + d\theta_2 + d\theta_3)$$

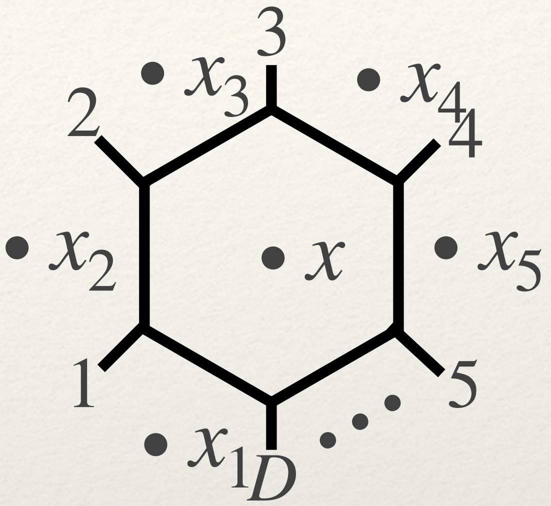
higher D generalization: $dV_n = \sum_{j=1}^{n} d\theta_j \times V^{(j)}_{n-2}$

–related to Goncharov's motivic formula; [Spradlin, Volovich 1105.2024]

–see also: [Arkani-Hamed, Yuan 1712.09991]

Comments on dlog integration

$$dI_D = \sum_{i,j} d\log \frac{D_i(x^*_{ij})}{D_j(x^*_{ij})} \frac{D_j(\bar{x}^*_{ij})}{D_i(\bar{x}^*_{ij})} \times I_{D-2}(x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_D)$$

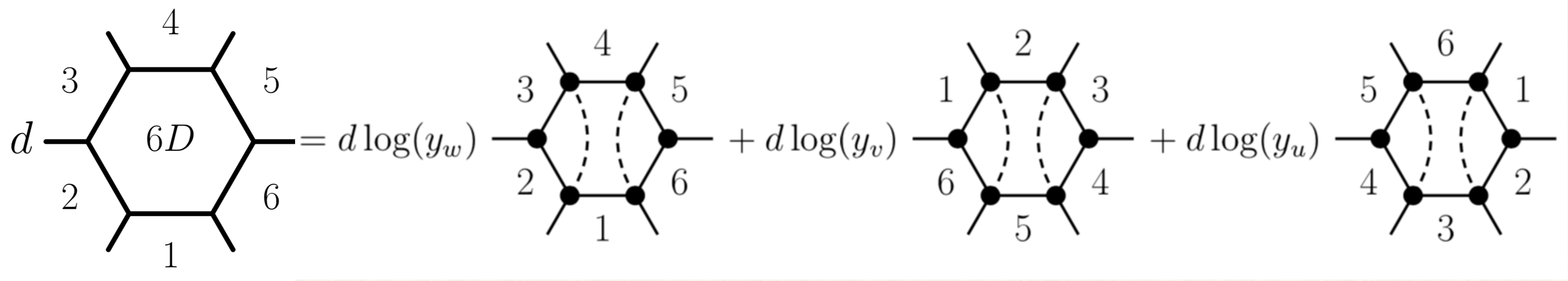


– trivializes differential equations

– relation to recently proposed "diagrammatic coaction"

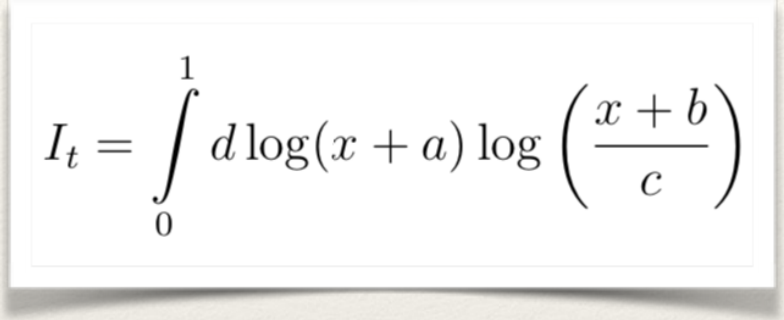
[Britto talk] [Abreu,Britto,Duhr,Gardi 1704.07931]

– differential of FI is related to other FI's in d-2



higher loop generalization

let's again start with a baby toy example [Caron-Huot,He; Panzer; Henn, EH,Parra-Martinez]



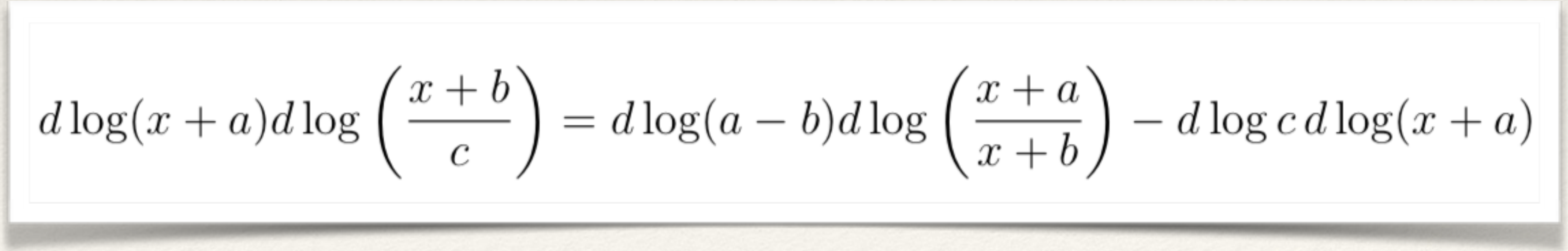
$$I_t = \int_0^1 d\log(x+a)\log\left(\frac{x+b}{c}\right)$$

(a, b, c) : external parameters

$$dI_t = I_t^{\partial} + I_t^{\text{bulk}}$$

$$I_t^{\partial} = d\log(x+a)\,\log\left(\frac{x+b}{c}\right)\Bigg|_0^1 \qquad I_t^{\text{bulk}} = \int_0^1 d\log(x+a)\,d\log\left(\frac{x+b}{c}\right)$$

partial fractioning I_t^{bulk} :

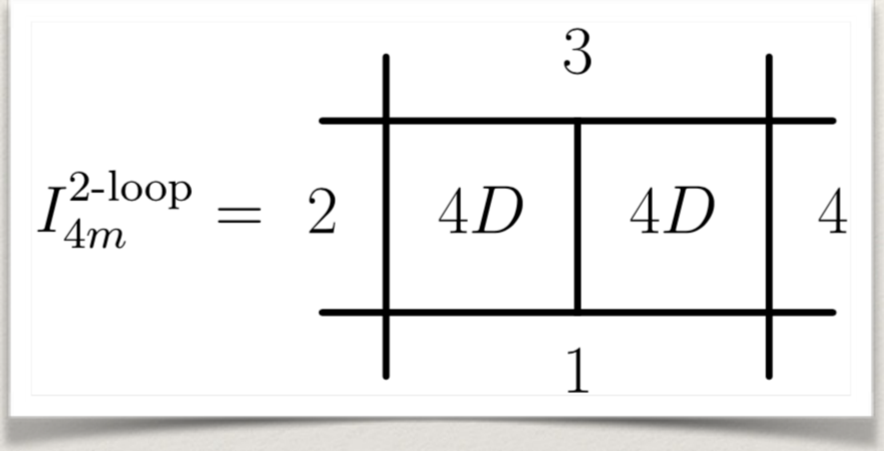


$$d\log(x+a)d\log\left(\frac{x+b}{c}\right) = d\log(a-b)d\log\left(\frac{x+a}{x+b}\right) - d\log c\, d\log(x+a)$$

extra ingredient, I^bulk, plays a role for higher-loop integrals

higher loop generalization

$$I^{2-\text{loop}}_{4m} = \int_{\Sigma^L_4 \cup \Sigma^R_4} \frac{d^4x_L d^4x_R\, x^2_{13}\sqrt{-\det x^2_{ij}}}{x^2_{L1}\, x^2_{L2}\, x^2_{L3}\, x^2_{LR}\, x^2_{R3}\, x^2_{R4}\, x^2_{R1}} \qquad u = \frac{x^2_{12}\, x^2_{34}}{x^2_{13}x^2_{24}} = z\bar{z}\,, \quad v = \frac{x^2_{23}\, x^2_{14}}{x^2_{13}x^2_{24}} = (1-z)(1-\bar{z})$$



integrate one loop at a time: F_R known funct.

$$I^{2-\text{loop}}_{4m} = \int_{\Sigma^L_4} \omega_L F_R$$

ω_L is a little different from previous 1-loop pieces

$$\omega_L = \frac{d^4x_L\, x^2_{13}\sqrt{-\det \mathrm{x}^2_{\mathrm{ij}}}}{x^2_{L1}\, x^2_{L2}\, x^2_{L3}\sqrt{-\det(x_1, x_L, x_3, x_4)}} = \frac{1}{2} d\log\frac{x^2_{L1}}{x^2_{L2}}\, d\log\frac{x^2_{L2}}{x^2_{L3}}\, d\log\frac{(x_1 - x^{13}_{+,R})^2(x_3 - x^{13}_{-,R})^2}{(x_1 - x^{13}_{-,R})^2(x_3 - x^{13}_{+,R})^2}\, d\log\frac{(x_L - x^L_+)^2}{(x_L - x^L_-)^2}$$

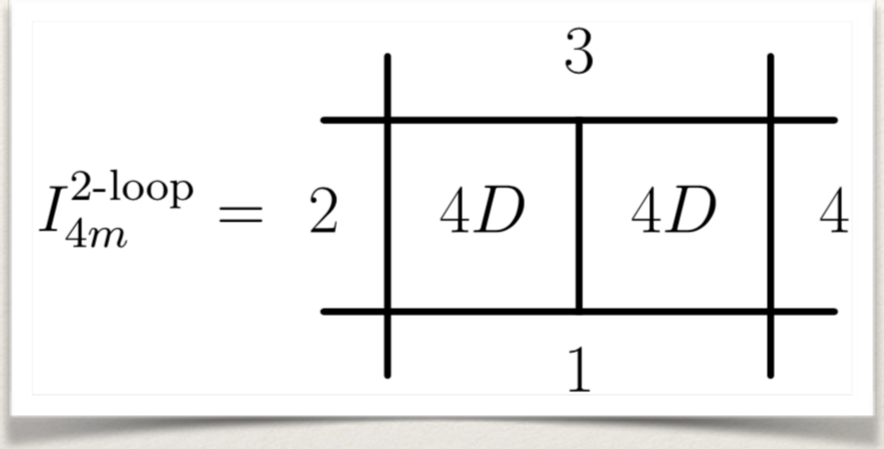
$x^L_\pm$: sols. to: $x^2_{L1} = x^2_{L2} = x^2_{L3} = x^2_{L4} = 0$

$x^R_\pm$: sols. to: $x^2_{R1} = x^2_{RL} = x^2_{R3} = x^2_{R4} = 0$

$x^{13}_{\pm,R}$: sols. to: $(x_R - x^R_+)^2 = (x_R - x^R_-)^2 = x^2_{RL} = x^2_{R4} = 0$

higher loop generalization

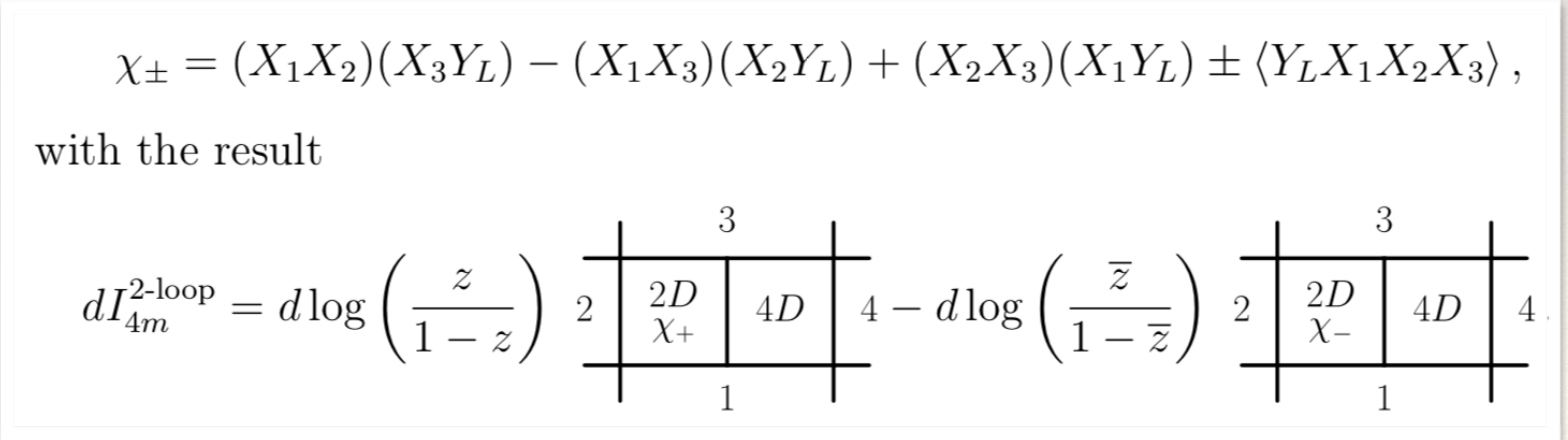
differential of the integral simplifies:



$I_{4m}^{\text{2-loop}} =$

$$dI_{4m}^{2-\text{loop}} = \int_{\Sigma_4^L} F_R \, d\omega_L + \int_{\Sigma_4^L} \omega_L dF_R$$

can explicitly partial fraction the forms:



$$\chi_\pm = (X_1X_2)(X_3Y_L) - (X_1X_3)(X_2Y_L) + (X_2X_3)(X_1Y_L) \pm \langle Y_L X_1 X_2 X_3 \rangle \, ,$$

with the result

$dI_{4m}^{\text{2-loop}} = d\log\left(\frac{z}{1-z}\right)$ … $- \, d\log\left(\frac{\bar{z}}{1-\bar{z}}\right)$ …

Conclusions

- new geometric formulations of QFT
    - Grassmannian, Amplituhedron in N=4 sYM
    - geometry ⟷ canonical differential *dlog* forms
- new dlog-representation of Feynman integrals
    - new geometric formulations for a wider class of theories?
    - give rise to new integration techniques
    - geometric structures directly in spacetime
- localization of Feynman integrals
    - clarifies the transcendentally drop of 4-fold integral to weight 2L functions
    - strategy works well for generalized polylogs, generalizations?

THANK YOU FOR THIS STIMULATING WORKSHOP!