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NLO+NLL QED corrections to electron PDFs

Based on: 1909.03886 (SF), 1911.12040 (Bertone, Cacciari, SF, Stagnitto)

3rd FCC Workshop, CERN, 14/1/2020
**Goal:** increase the accuracy in the computations of $e^+e^-$ cross sections

**Framework:** a factorisation formula

- aka structure-function approach: best to *not* use this terminology

**By means of:** more accurate PDFs

- PDFs aka structure functions: best to *not* use this terminology
- improve the LL+LO accuracy, $(\alpha \log(E/m))^k$, by including NLL+NLO terms, $(\alpha \log(E/m))^k + \alpha (\alpha \log(E/m))^{k-1}$, in the PDFs
- the corresponding increased accuracy of short-distance cross sections is widely available, and is understood here
Current $z$-space LO+LL PDFs $(\alpha \log(E/m))^k$: 

- $0 \leq k \leq \infty$ for $z \simeq 1$ (Gribov, Lipatov)
- $0 \leq k \leq 3$ for $z < 1$ (Skrzypek, Jadach; Cacciari, Deandrea, Montagna, Nicrosini; Skrzypek)
- matching between these two regimes
Current $z$-space LO+LL PDFs \((\alpha \log(E/m))^k\):

- \(0 \leq k \leq \infty\) for \(z \simeq 1\) (Gribov, Lipatov)
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Sought $z$-space NLO+NLL PDFs \((\alpha \log(E/m))^k + \alpha (\alpha \log(E/m))^{k-1}\):

- \(0 \leq k \leq \infty\) for \(z \simeq 1\)
- \(0 \leq k \leq \{3, 2\}\) for \(z < 1\) \(\iff\) \(O(\alpha^3)\)
- matching between these two regimes
- for \(e^+, e^-,\) and \(\gamma\)
- both numerical and analytical

Main tool: the solution of PDFs evolution equations
Consider the production of a system $X$ at an $e^+e^-$ collider:

$$e^+(P_{e^+}) + e^-(P_{e^-}) \rightarrow X$$

Its cross section is written as follows:

$$d\Sigma_{e^+e^-}(P_{e^+}, P_{e^-}) = \sum_{kl} \int dy_+ dy_- \mathcal{B}_{kl}(y_+, y_-) d\sigma_{kl}(y_+P_{e^+}, y_-P_{e^-})$$

To be definite, let’s stipulate that:

$$k \in \{e^+, \gamma\}, \quad l \in \{e^-, \gamma\}$$

which is immediate to generalise, if need be. Then:

♦ $d\Sigma_{e^+e^-}$: the collider-level cross section

♦ $d\sigma_{kl}$: the particle-level cross section

♦ $\mathcal{B}_{kl}(y_+, y_-)$: describes beam dynamics

♦ $e^+, e^-$ on the lhs: the beams

♦ $e^+, e^-, \gamma$ on the rhs: the particles
I’ll only talk about particles and particle-level cross sections.

The parametrisation of beam dynamics is supposed to be given.

I sum over polarisations.

Write any particle cross section by means of a factorisation formula, quite similar to its QCD counterpart.
\[ d\bar{\sigma}_{kl}(p_k, p_l) = \sum_{ij=e^+, e^-, \gamma} \int d\gamma_+ d\gamma_- \Gamma_i/k(z_+, \mu^2, m^2) \Gamma_j/l(z_-, \mu^2, m^2) \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2) + \Delta \]

with:

\[ d\bar{\sigma}_{kl} = d\sigma_{kl} + \mathcal{O}\left(\left(\frac{m^2}{s}\right)^p\right), \quad s = (p_k + p_l)^2, \quad p \geq 1 \]

♦ \(d\bar{\sigma}_{kl}\): the particle-level cross section, with power-suppressed terms discarded

♦ \(d\hat{\sigma}_{ij}\): the subtracted parton-level cross section. Independent of \(m\)

♦ \(e^+, e^-, \gamma\) on the lhs: the particles

♦ \(e^+, e^-, \gamma\) on the rhs: the partons

♦ \(\Gamma_i/k\): the PDF of parton \(i\) inside particle \(k\). It can be computed perturbatively

♦ \(\mu\): the hard scale, \(m^2 \ll \mu^2 \sim s\)
Differences wrt QCD:

♦ PDFs and power-suppressed terms can be computed perturbatively

♦ An object (e.g. $e^-$) may play the role of both particle and parton

As in QCD, a particle is a physical object, a parton is not
\[ d\tilde{\sigma}_{kl}(p_k, p_l) = \sum_{ij=e^+, e^-, \gamma} \int dz_+ dz_- \Gamma_{i/k}(z_+, \mu^2, m^2) \Gamma_{j/l}(z_-, \mu^2, m^2) \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2) + \Delta \]

This formula can be used in several ways:

A: to solve for the PDFs, given the particle and parton cross sections

B: for the computation of the particle cross section, given the parton cross section and the PDFs

C: for cross checks, given both cross sections and the PDFs
BTW: my cross sections are either fully inclusive on final state objects, or such that final-state objects are defined through fragmentation functions.

Note that e.g. bare-electron or bare-photon cross sections are:

\- a) a very bad idea;
\- b) unphysical quantities as soon as one considers EW corrections.
\[ d\tilde{\sigma}_{kl}(p_k, p_l) = \sum_{ij=e^+, e^-, \gamma} \int dz_+ dz_- \Gamma_{i/k}(z_+, \mu^2, m^2) \Gamma_{j/l}(z_-, \mu^2, m^2) \times d\tilde{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2) + \Delta \]

This formula can be used in several ways:

A: to solve for the PDFs, given the particle and parton cross sections

- Compute the cross sections to the desired perturbative order
- Formally expand the PDFs to the same order
- Set \( \Delta = 0 \) and solve for the PDFs

This is one of the possible procedures to compute the initial conditions for PDF evolution – see 1909.03886 for NLO results
\[ d\tilde{\sigma}_{kl}(p_k, p_l) = \sum_{ij=e^+, e^-, \gamma} \int dz_+ dz_- \Gamma_{i/k}(z_+, \mu^2, m^2) \Gamma_{j/l}(z_-, \mu^2, m^2) \times d\tilde{\sigma}_{ij}(z_+p_k, z_-p_l, \mu^2) + \Delta \]

This formula can be used in several ways:

**B:** for the computation of the particle cross section, given the parton cross section and the PDFs

- This is the standard usage: set \( \Delta = 0 \) and use the available parton cross section and PDFs
- PDFs are understood to be evolved – if so, the lhs does not contain large logs of the mass
- If PDFs are not evolved, but expanded perturbatively, the lhs does contain large logs of the mass: no phenomenological interest
\[ d\tilde{\sigma}_{kl}(p_k, p_l) = \sum_{ij=e^+, e^-, \gamma} \int dz_+ dz_- \Gamma_{i/k}(z_+, \mu^2, m^2) \Gamma_{j/l}(z_-, \mu^2, m^2) \times d\tilde{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2) + \Delta \]

This formula can be used in several ways:

**C:** for cross checks, given both cross sections and the PDFs

- If all quantities are computed at the same perturbative order: must obtain \( \Delta = 0 \)
- If the PDFs are evolved: \( \Delta \) features large logs of the mass

Of no interest phenomenologically
Henceforth, I consider the dominant production mechanism at an $e^+e^-$ collider, namely that associated with partons inside an electron.

Simplified notation:

$$\Gamma_i(z, \mu^2) \equiv \Gamma_{i/e^-(z, \mu^2)}$$

*The case of the positron is identical, at least in QED, and will be understood*
NLO initial conditions (1909.03886)

Conventions for the perturbative coefficients:

\[ \Gamma_i = \Gamma_i^{[0]} + \frac{\alpha}{2\pi} \Gamma_i^{[1]} + \mathcal{O}(\alpha^2) \]

Results:

\[
\begin{align*}
\Gamma_i^{[0]}(z, \mu_0^2) &= \delta_{i\epsilon} \delta(1 - z) \\
\Gamma_e^{-}(z, \mu_0^2) &= \left[ \frac{1 + z^2}{1 - z} \left( \log \frac{\mu_0^2}{m^2} - 2 \log(1 - z) - 1 \right) \right]_+ + K_{ee}(z) \\
\Gamma_\gamma^{[1]}(z, \mu_0^2) &= \frac{1 + (1 - z)^2}{z} \left( \log \frac{\mu_0^2}{m^2} - 2 \log z - 1 \right) + K_{\gamma e}(z) \\
\Gamma_{e^+}^{[1]}(z, \mu_0^2) &= 0
\end{align*}
\]

Note:

- Meaningful only if \( \mu_0 \sim m \)
- In \( \overline{\text{MS}} \), \( K_{ij}(z) = 0 \); in general, these functions define an IR scheme
NLL evolution (1911.12040)

General idea: solve the evolution equations starting from the initial conditions computed previously

\[
\frac{\partial \Gamma_i(z, \mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} [P_{ij} \otimes \Gamma_j](z, \mu^2) \quad \iff \quad \frac{\partial \Gamma(z, \mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} [\mathbb{P} \otimes \Gamma](z, \mu^2),
\]

Done conveniently in terms of non-singlet, singlet, and photon

Two ways:

♦ Mellin space: suited to both numerical solution and all-order, large-\(z\) analytical solution (called *asymptotic solution*)

♦ Directly in \(z\) space in an integrated form: suited to fixed-order, all-\(z\) analytical solution (called *recursive solution*)
A technicality: owing to the running of $\alpha$, it is best to evolve in $t$ rather than in $\mu$, with: ($\sim$ Furmanski, Petronzio)

\[
t = \frac{1}{2\pi b_0} \log \frac{\alpha(\mu)}{\alpha(\mu_0)} = \frac{\alpha(\mu)}{2\pi} L - \frac{\alpha^2(\mu)}{4\pi} \left( b_0 L^2 - \frac{2b_1}{b_0} L \right) + \mathcal{O}(\alpha^3), \quad L = \log \frac{\mu^2}{\mu_0^2}.
\]

Note:

- $t \longleftrightarrow \mu$; notation-wise, the dependence on $t$ is equivalent to the dependence on $\mu$
- $t = 0 \iff \mu = \mu_0$
- $L$ is my “large log”
- Tricky: fixed-$\alpha$ expressions are obtained with $t = \alpha L/(2\pi)$ (and not $t = 0$)
Mellin space

Introduce the evolution operator $\mathbb{E}_N$

$$\Gamma_N(\mu^2) = \mathbb{E}_N(t) \Gamma_{0,N}, \quad \mathbb{E}_N(0) = I, \quad \Gamma_{0,N} \equiv \Gamma_N(\mu_0^2)$$

The PDFs evolution equations are then re-expressed by means of an evolution equation for the evolution operator:

$$\frac{\partial \mathbb{E}_N(t)}{\partial t} = \frac{b_0 \alpha^2(\mu)}{\beta(\alpha(\mu))} \sum_{k=0}^{\infty} \left( \frac{\alpha(\mu)}{2\pi} \right)^k \mathbb{P}_N^{[k]} \mathbb{E}_N(t)$$

Can be solved numerically

Can be solved analytically in a closed form under simplifying assumptions.

Chiefly: large-$z$ is equivalent to large-$N$

I’l show results for the non-singlet $\equiv$ singlet. The photon is feasible as well (see 1911.12040), but technically very involved
Show first that this formalism allows one to quickly re-obtain the known LL result:

$$\Gamma^{[0]}_{0,N} = 1 \implies \Gamma_{\text{LL}}(z, \mu^2) = M^{-1}\left[ \exp \left( \log E_N \right) \right]$$

From the explicit expression of the AP $f f$ kernel:

$$\log E_N = \frac{\alpha}{2\pi} P_N^{[0]} L \xrightarrow{N \to \infty} -\eta_0 \left( \log \tilde{N} - \lambda_0 \right)$$

$$\eta_0 = \frac{\alpha}{\pi} L, \quad \tilde{N} = N e^{\gamma_E}, \quad \lambda_0 = \frac{3}{4}$$

The computation of the inverse Mellin transform is trivial:

$$\Gamma_{\text{LL}}(z, \mu^2) = \frac{e^{-\gamma_E \eta_0} e^{\lambda_0 \eta_0}}{\Gamma(1 + \eta_0)} \eta_0 (1 - z)^{-1+\eta_0}$$

The usual form, bar for the “−1” of soft origin (we’re resumming collinear logs here)
The NLL case is only slightly more complicated; we use:

\[ \Gamma_{NLL}(z, \mu^2) = M^{-1} \left[ \exp \left( \log E_N \right) \right] \otimes \Gamma_{NLO}(z, \mu_0^2) \]

which is convenient because the form of the evolution operator is functionally the same as at the LL:

\[ \log E_N \xrightarrow{N \to \infty} -\xi_1 \log \bar{N} + \hat{\xi}_1 \]

with:

\[ \xi_1 = 2t - \frac{\alpha(\mu)}{4\pi^2b_0} \left( 1 - e^{-2\pi b_0 t} \right) \left( \frac{20}{9} n_F + \frac{4\pi b_1}{b_0} \right) = 2t + \mathcal{O}(\alpha t) = \eta_0 + \ldots \]

\[ \hat{\xi}_1 = \frac{3}{2} t + \frac{\alpha(\mu)}{4\pi^2b_0} \left( 1 - e^{-2\pi b_0 t} \right) \left( \lambda_1 - \frac{3\pi b_1}{b_0} \right) = \frac{3}{2} t + \mathcal{O}(\alpha t) = \lambda_0 \eta_0 + \ldots \]

\[ \lambda_1 = \frac{3}{8} - \frac{\pi^2}{2} + 6 \zeta_3 - \frac{n_F}{18} (3 + 4\pi^2) \]
Thence:

\[
\Gamma_{\text{NLL}}(z, \mu^2) = \frac{e^{-\gamma_E \xi_1} e^{\xi_1}}{\Gamma(1 + \xi_1)} \xi_1 (1 - z)^{-1 + \xi_1}
\]

\[
\times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[ \left( \log \frac{\mu_0^2}{m^2} - 1 \right) \left( A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} 
\right.
\right.
\]

\[
+ \left. \left( \log \frac{\mu_0^2}{m^2} - 1 - 2A(\xi_1) \right) \log(1 - z) - \frac{1}{2} \log^2(1 - z) \right] \right\}
\]

where:

\[
A(\kappa) = -\gamma_E - \psi_0(\kappa)
\]

\[
B(\kappa) = \frac{1}{2} \gamma_E^2 + \frac{\pi^2}{12} + \gamma_E \psi_0(\kappa) + \frac{1}{2} \psi_0(\kappa)^2 - \frac{1}{2} \psi_1(\kappa)
\]
In $z$ space

Use integrated PDFs (so as to simplify the treatment of endpoints)

$$ \mathcal{F}(z, t) = \int_{0}^{1} dy \, \Theta(y - z) \Gamma(y, \mu^2) \quad \implies \quad \Gamma(z, \mu^2) = -\frac{\partial}{\partial z} \mathcal{F}(z, t) $$

in terms of which the formal solution of the evolution equation is:

$$ \mathcal{F}(z, t) = \mathcal{F}(z, 0) + \int_{0}^{t} du \, \frac{b_0 \alpha^2(u)}{\beta(\alpha(u))} \left[ \mathbb{P} \otimes \mathcal{F} \right](z, u) $$

By inserting the representation:

$$ \mathcal{F}(z, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( \mathcal{J}_{k}^{LL}(z) + \frac{\alpha(t)}{2\pi} \mathcal{J}_{k}^{NLL}(z) \right) $$

on both sides of the solution, one obtains recursive equations, whereby a $\mathcal{J}_k$ is determined by all $\mathcal{J}_p$ with $p < k$. The recursion starts from $\mathcal{J}_0$, which are the integrated initial conditions.
For the record, the recursive equations are:

\[ \mathcal{J}_{k}^{LL} = \mathbb{P}^{[0]} \otimes \mathcal{J}_{k-1}^{LL} \]

\[ \mathcal{J}_{k}^{NLL} = (-)^{k}(2\pi b_{0})^{k} \mathcal{F}^{[1]}(\mu_{0}^{2}) \]

\[ + \sum_{p=0}^{k-1} (-)^{p}(2\pi b_{0})^{p} \left( \mathbb{P}^{[0]} \otimes \mathcal{J}_{k-1-p}^{NLL} + \mathbb{P}^{[1]} \otimes \mathcal{J}_{k-1-p}^{LL} \right) \]

\[ - \frac{2\pi b_{1}}{b_{0}} \mathbb{P}^{[0]} \otimes \mathcal{J}_{k-1-p}^{LL} \]

We have computed these for \( k \leq 3 \) (\( \mathcal{J}^{LL} \)) and \( k \leq 2 \) (\( \mathcal{J}^{NLL} \)), ie to \( \mathcal{O}(\alpha^{3}) \)

Results in 1911.12040 and its ancillary files
A remarkable result

Our asymptotic solutions, expanded in $\alpha$, feature \emph{all} of the terms:

\[
\begin{align*}
\frac{\log^q(1 - z)}{1 - z} & \quad \text{singlet, non–singlet} \\
\log^q(1 - z) & \quad \text{photon}
\end{align*}
\]

of our recursive solutions

Non-trivial; stems from keeping subleading terms \((at \ z \to 1)\) in the AP kernels
Illustrative results for PDFs

♦ Analytical results obtained by means of an additive matching between the recursive and the asymptotic solutions

♦ All are in $\overline{\text{MS}}$

♦ Bear in mind that PDFs are unphysical quantities
\(e^-\) vs \(\gamma\) vs \(e^+\). Note that \(e^-\) in the right-hand panel is strongly damped.
Numerical vs analytical, non-singlet
NLL vs LL, non-singlet. The insets show the double ratio, i.e., numerical vs analytical.
In order to understand the large-$z$ bit of the previous plots:

\[
\Gamma_{\text{LL}}(z, \mu^2) = \frac{e^{-\gamma_E \eta_0} e^{\lambda_0 \eta_0}}{\Gamma(1 + \eta_0)} \eta_0 (1 - z)^{-1 + \eta_0}
\]

\[
\Gamma_{\text{NLL}}(z, \mu^2) = \frac{e^{-\gamma_E \xi_1} e^{\hat{\xi}_1}}{\Gamma(1 + \xi_1)} \xi_1 (1 - z)^{-1 + \xi_1}
\]

\[
\times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[ \left( \log \frac{\mu_0^2}{m^2} - 1 \right) \left( A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} \\
\quad + \left( \log \frac{\mu_0^2}{m^2} - 1 - 2A(\xi_1) \right) \log(1 - z) - \log^2(1 - z) \right] \right\}
\]

with:

\[
\xi_1 \simeq \eta_0, \quad \hat{\xi}_1 \simeq \lambda_0 \eta_0
\]

\[
A(\kappa) = \frac{1}{\kappa} + \mathcal{O}(\kappa) \implies \log(1 - z) \text{ dominates}
\]

\[
B(\kappa) = -\frac{\pi^2}{6} + 2\zeta_3 \kappa + \mathcal{O}(\kappa^2)
\]
Conclusions

♦ We have computed all NLO initial conditions for PDFs and FFs (1909.03886), unpolarised

♦ We have NLL-evolved those relevant to the electron PDFs (1911.12040), both analytically and numerically

♦ These can be obtained at:
  
  https://github.com/gstagnit/ePDF

Many results are based on establishing a “dictionary” QCD $\longrightarrow$ QED, which works at any order in $\alpha_s$ and $\alpha$
To be done

♦ Assess the impact of PDFs NLL effects on physical cross sections

♦ The inclusion of these results in MG5_aMC@NLO v3.X is the only missing ingredient in the latter for the computation of NLO QED corrections in $e^+e^-$ collisions ($hh$ collisions already OK)

♦ $\gamma$ PDFs; soft effects; alternative IR schemes; FFs

♦ Polarisations?