Precise determination of the strong coupling from jet rates

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based on A. Verbytskyi, A. Banfi, A. Kardos, P. F. Monni, S. Kluth, GS, Z. Szőr, Z. Trócsányi, Z. Tulipánt, G. Zanderighi, JHEP **1908** (2019) 129 [arXiv:1902.08158 [hep-ph]]

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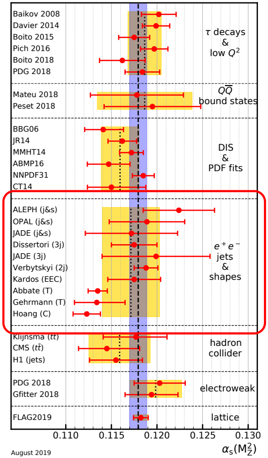
Part I: α_s from jet rates in e^+e^-

Why α_s from jet rates in e^+e^-?



[PDG, The Review of Particle Physics, 2019 update]

Why α_s in e^+e^-?

- $\alpha_s(M_Z)$ is known with $\sim 0.8\%$ precision
- however, the e^+e^- jets & shapes sub-field alone gives $\sim 2.6\%$ uncertainty
- large spread between measurements, note in particular the low values of $\alpha_s(M_Z)$ obtained from event shapes at NNLO+N^{3}LL using analytic hadronization models

Why jet rates?

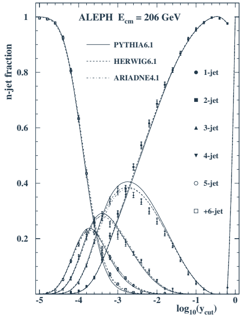
- high perturbative accuracy, especially for the two-jet rate
- compared to event shapes, jet rates are known to be less sensitive to hadronization corrections

Measurements with new approaches/data are important!

Durham jet rates

Durham jet algorithm: sequential recombination algorithm with distance measure $y_{ij} = 2\frac{\min(E_i^2, E_j^2)}{E_{\mathrm{vis}}^2}(1 - \cos\theta_{ij})$ where E_i is particle energy and θ_{ij} is the angle between tree-momenta of particles i and j; momenta recombined using the E-scheme

Jet rates: R_n is the fraction of n-jet events for given y: $R_n(y) = \frac{\sigma_{n-\mathrm{jet}}(y)}{\sigma_{\mathrm{tot}}}$



[ALEPH Coll., Eur. Phys. J. **C35**, 457 (2004)]

- R_3 was used multiple times in the past to extract $\alpha_s(M_Z)$
- Fixed-order perturbative predictions for R_3/R_2 at NNLO/N^{3}LO
  [Gehrmann-De Ridder et al., Phys. Rev. Lett. **100** (2008) 172001, Weinzierl, Phys. Rev. Lett. **101** (2008) 162001]
- Resummed predictions for R_2 at NNLL accuracy became available in 2016
  [Banfi et al., Phys. Rev. Lett. **117** (2016) 172001]
- This analysis naturally combines R_2 and R_3 for the first time

Analysis components

- Data from LEP and PETRA + new OPAL measurements used to build correlation model for older measurements.
- Fixed-order perturbative predictions + some b-mass corrections
- Resummation + matching
- Non-perturbative corrections from state-of-the-art MC event generators + Lund and cluster hadronization models

Data

Combined analysis using 20+ datasets from 4 collaborations

The data covers a **wide range of cms energies**: $\sqrt{s} = 35 - 207$ GeV

Experiment	Data $\sqrt{s}$, (average), GeV	MC $\sqrt{s}$, GeV	Events
OPAL	91.2(91.2)	91.2	1508031
OPAL	189.0(189.0)	189	3300
OPAL	183.0(183.0)	183	1082
OPAL	172.0(172.0)	172	224
OPAL	161.0(161.0)	161	281
OPAL	130.0 — 136.0(133.0)	133	630
L3	201.5 — 209.1(206.2)	206	4146
L3	199.2 — 203.8(200.2)	200	2456
L3	191.4 — 196.0(194.4)	194	2403
L3	188.4 — 189.9(188.6)	189	4479
L3	180.8 — 184.2(182.8)	183	1500
L3	161.2 — 164.7(161.3)	161	424
L3	135.9 — 140.1(136.1)	136	414
L3	129.9 — 130.4(130.1)	130	556
JADE	43.4 — 44.3(43.7)	44	4110
JADE	34.5 — 35.5(34.9)	35	29514
ALEPH	91.2(91.2)	91.2	3600000
ALEPH	206.0(206.0)	206	3578
ALEPH	189.0(189.0)	189	3578
ALEPH	183.0(183.0)	183	1319
ALEPH	172.0(172.0)	172	257
ALEPH	161.0(161.0)	161	319
ALEPH	133.0(133.0)	133	806

Data selection:

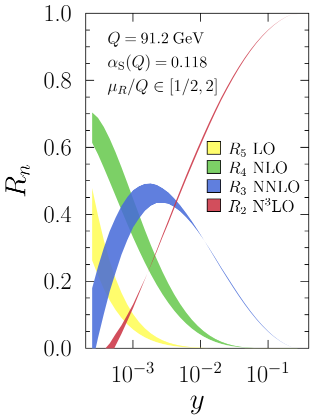
- measurements with both charged and neutral final state particles
- corrected for detector effects
- corrected for QED ISR
- no overlap with other samples
- sufficient precision
- sufficient information on dataset available

Predictions: fixed-order

Fixed-order predictions up to and including α_s^3 corrections known for some time

[Gehrmann-De Ridder et al., Phys. Rev. Lett. **100** (2008) 172001, Weinzierl, Phys. Rev. Lett. **101** (2008) 162001]

$$R_n(y) = \delta_{2,n} + \frac{\alpha_s(Q)}{2\pi} A_n(y) + \left(\frac{\alpha_s(Q)}{2\pi}\right)^2 B_n(y) + \left(\frac{\alpha_s(Q)}{2\pi}\right)^3 C_n(y) + \mathcal{O}(\alpha_s^4)$$



- R_3 computed at **NNLO** accuracy using CoLoRFulNNLO $\Rightarrow$ obtain R_2 at **N^{3}LO**

  [Del Duca et al., Phys. Rev. **D94** (2016) no.7, 074019]
- very good numerical precision and stability
- b-mass corrections from `Zbb4`: note only **NLO** for R_3 $\Rightarrow$ **NNLO** for R_2

  [Nason, Oleari, Phys. Lett. **B407**, 57 (1997)]
- mass effects included at distribution level, e.g.

$$R_2(y) = (1 - r_b) R_2^{\mathrm{N^3LO}}(y)_{m_b=0} + r_b R_2^{\mathrm{NNLO}}(y)_{m_b\neq 0}$$

where r_b is the fraction of b-quark events

$$r_b = \frac{\sigma_{m_b\neq 0}(e^+e^- \to b\bar{b})}{\sigma_{m_b\neq 0}(e^+e^- \to \text{hardons})}$$

Predictions: resummation

Resummed predictions for R_2 at **NNLL** accuracy have been computed more recently

[Banfi et al., Phys. Rev. Lett. **117** (2016) 172001]

$$R_2(y) = e^{-R_{\mathrm{NNLL}}(y)} \left[\left(1 + \frac{\alpha_\mathrm{s}(Q)}{2\pi} H^{(1)} + \frac{\alpha_\mathrm{s}(Q\sqrt{y})}{2\pi} C_{\mathrm{hc}}^{(1)} \right) \mathcal{F}_{\mathrm{NLL}}(y) + \frac{\alpha_\mathrm{s}(Q)}{2\pi} \delta\mathcal{F}_{\mathrm{NNLL}}(y) \right]$$

- resummation performed with the `ARES` program
- matching to fixed-order: $\log R$ scheme
- counting of logs (NNLL) here refers to logs in $\ln R_2$

In contrast, resummed predictions for R_3 have a much lower logarithmic accuracy

- more colored emitters
- state-of-the-art resummation includes only $\mathcal{O}(\alpha_\mathrm{s}^n L^{2n})$ and $\mathcal{O}(\alpha_\mathrm{s}^n L^{2n-1})$ terms in R_3 (note different logarithmic counting)
- in this analysis, no resummation for R_3 is performed

$\Downarrow$

Main focus on N^{3}LO+NNLL for R_2, but also simultaneous analysis with NNLO for R_3

Hadronization corrections: setups

Effects associated with the parton-to-hadron transition cannot be computed in perturbation theory and must be estimated by other means

Obtained using state-of-the-art MC event generators: $e^+e^- \to jjjjj$ merged samples with massive b-quarks

- **Default setup "H^L": `Herwig7.1.4` for $e^+e^- \to 2, 3, 4, 5$ jets, 2 and 3 jets at NLO using `MadGraph5` and `OpenLoops` + Lund fragmentation model**
- Setup for hadronization systematics "H^C": `Herwig7.1.4` for $e^+e^- \to 2, 3, 4, 5$ jets, 2 and 3 jets at NLO using `MadGraph5` and `OpenLoops` + cluster fragmentation model
- Setup for cross-checks "S^C": `Sherpa2.2.6` for $e^+e^- \to 2, 3, 4, 5$ jets, 2 jets at NLO using `AMEGIC`, `COMIX` and `OpenLoops` + cluster fragmentation model

Fit procedure

To find the optimal value of α_s, MINUIT2 is used to minimize

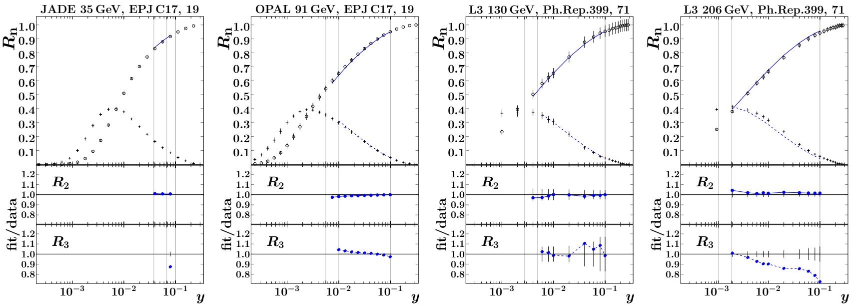
$$\chi^2(\alpha_s) = \sum_{\text{data set}} \chi^2(\alpha_s)_{\text{data set}}$$

where $\chi^2(\alpha_s)$ are computed separately for each data set

$$\chi^2(\alpha_s) = \vec{r}\,V^{-1}\vec{r}^{\,T}\,, \qquad \vec{r} = (\vec{D} - \vec{P}(\alpha_s))$$

- $\vec{D}$: vector of data points
- $\vec{P}(\alpha_s)$: vector of theoretical predictions
- V: covariance matrix for $\vec{D}$ (statistical correlations estimated from MC generated samples, systematic correlations modeled to mimic patters observed in OPAL data)

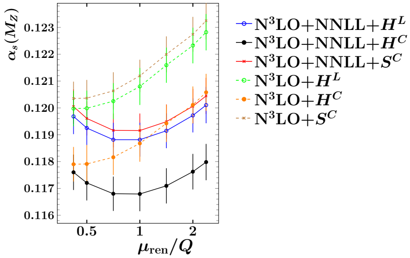
Fits: distributions

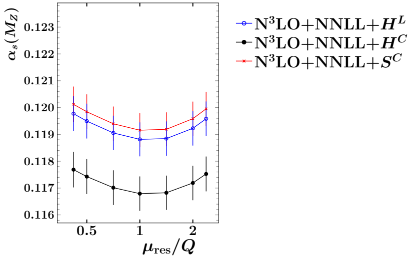


Central result and fit range selection

- avoid regions where theoretical predictions or hadronization model are unreliable
- Q^2-dependent fit range: $[-2.25 + \mathcal{L}, -1]$ for R_2 and $[-2 + \mathcal{L}, -1]$ for R_3 (if used), where $\mathcal{L} = \ln \frac{M_Z^2}{Q^2}$
- note separate fit ranges for R_2 and R_3 (if used)
- smallest $\chi^2/ndof$, low sensitivity to fit range

Fits: systematics and uncertainties





Estimate the uncertainty by

- varying the renormalization scale $\mu_{\rm ren} \in [Q/2, 2Q]$: (*ren.*)
- varying the resummation scale $\mu_{\rm res} \in [Q/2, 2Q]$: (*res.*)
- varying the hadronization model H^L vs. H^C: (*hadr.*)
- fit uncertainty is obtained from the $\chi^2 + 1$ criterion as implemented in `MINUIT2`: (*exp.*)

Notice much reduced renormalization scale uncertainty when NNLL resummation for R_2 is included

Results: R_2

Extraction of $\alpha_s(M_Z)$ from the two-jet rate R_2 measured over a wide range of cms energies in e^+e^- collisions has been performed at N^{3}LO+NNLL accuracy for the first time:

$$\alpha_s(M_Z) = 0.11881 \pm 0.00063(exp.) \pm 0.00101(hadr.) \pm 0.00045(ren.) \pm 0.00034(res.)$$
$$\alpha_s(M_Z) = 0.11881 \pm 0.00131(comb.)$$

- main source of uncertainty: hadronization modeling
- uncertainty from scale variation is considerably smaller than from hadronization
- experimental uncertainty comparable to perturbative one

Inclusion of **NNLL resummation crucial** for reducing perturbative uncertainty

Results: $R_2 + R_3$

Combined fit of R_2 at N^{3}LO+NNLL and R_3 at NNLO, taking into account for the first time the correlation between the observables gives:

$$\alpha_s(M_Z) = 0.11989 \pm 0.00045(exp.) \pm 0.00098(hadr.) \pm 0.00046(ren.) \pm 0.00017(res.)$$
$$\alpha_s(M_Z) = 0.11989 \pm 0.00118(comb.)$$

- result is fully compatible with R_2-only fit
- formally more precise than a fit based on R_2 alone,
- but much more sensitive to fit range selection

An accurate resummation of R_3 could potentially reduce the sensitivity to fit range selection and lead to an even more precise determination of $\alpha_s(M_Z)$

Final result

The following value of $\alpha_s(M_Z)$ was obtained in the analysis

$$\boldsymbol{\alpha_s(M_Z) = 0.11881 \pm 0.00063\ (exp.) \pm 0.00101\ (hadr.) \pm 0.00045\ (ren.) \pm 0.00034\ (res.)}$$
$$\boldsymbol{\alpha_s(M_Z) = 0.11881 \pm 0.00131\ (comb.)}$$

- The result agrees with the world average ($\alpha_s(M_Z)_{\mathrm{PDG2018}} = 0.1181 \pm 0.0011^{*}$) and has an uncertainty that is of the same size
- The presented result is the most precise in its subclass [Salam, arXiv:1712.05165v2]

Determination	Data and procedure
0.1175 ± 0.0025	ALEPH 3-jet rate (NNLO+MChad)
0.1199 ± 0.0059	JADE 3-jet rate (NNLO+NLL+MChad)
0.1224 ± 0.0039	ALEPH event shapes (NNLO+NLL+MChad)
0.1172 ± 0.0051	JADE event shapes (NNLO+NLL+MChad)
0.1189 ± 0.0041	OPAL event shapes (NNLO+NLL+MChad)
$0.1164^{+0.0028}_{-0.0026}$	Thrust (NNLO+NLL+anlhad)
$0.1134^{+0.0031}_{-0.0025}$	Thrust (NNLO+NNLL+anlhad)
0.1135 ± 0.0011	Thrust (SCET NNLO+N^3LL+anlhad)
0.1123 ± 0.0015	C-parameter (SCET NNLO+N^3LL+anlhad)

*The PDF average after the 2019 update reads $\alpha_s(M_Z) = 0.1179 \pm 0.0010$

Part II: lessons for FCC-ee

Improving perturbative predictions I

More legs, more N's

- beyond NNLO for 3-jet event shapes/rate?
- beyond 3-jet rate/event shapes at NNLO?
- improved logarithmic accuracy for R_2, R_3?

Mass effects, mixed EW×QCD corrections

- R_3 at NNLO with massive b-quarks?
- mixed EW×QCD corrections for R_2, R_3?

Two issues

- full 2- and 3-loop matrix elements that would be needed are presently **not known**, however **great progress**, so expect new results [see talks by C. Duhr, J. Usovitsch, K. Papadopoulos]
- computing **physical observables** using those matrix elements is a **separate issue** (definitely beyond NNLO), new ideas may be needed [see talks by G. Rodrigo, R. Pittau]

Improving perturbative predictions I

More legs, more N's

- beyond NNLO for 3-jet event shapes/rate? $\Rightarrow$ not top priority for this fit
- beyond 3-jet rate/event shapes at NNLO? $\Rightarrow$ not top priority for this fit
- **improved logarithmic accuracy for R_2, R_3?** $\Rightarrow$ **already within reach**

Mass effects, mixed EW×QCD corrections

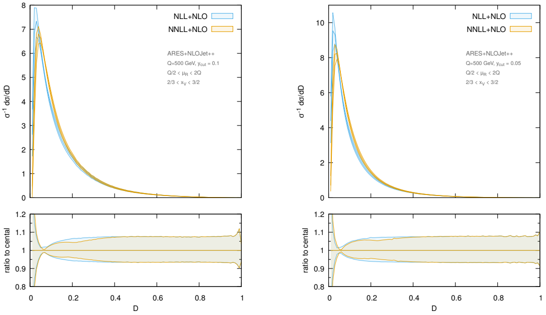
- R_3 at NNLO with massive b-quarks? $\Rightarrow$ more relevant this fit
- mixed EW×QCD corrections for R_2, R_3? $\Rightarrow$ more relevant for this fit

Two issues

- full 2- and 3-loop matrix elements that would be needed are presently **not known**, however **great progress**, so expect new results [see talks by C. Duhr, J. Usovitsch, K. Papadopoulos]
- computing **physical observables** using those matrix elements is a **separate issue** (definitely beyond NNLO), new ideas may be needed [see talks by G. Rodrigo, R. Pittau]

Improved logarithmic accuracy for R_2, R_3

- very recently the NNLL radiator for three hard emitters has been defined
- allows for NNLL resummation of event shapes in the near-to-planar limit, e.g. D-parameter at NNLL+NLO

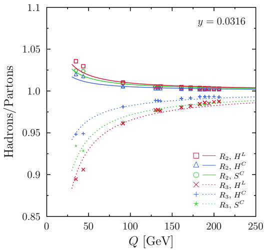


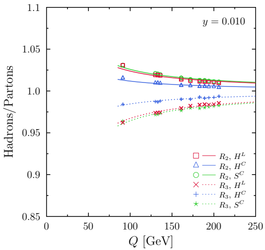
[Arpino et al. arXiv:1912.09341]

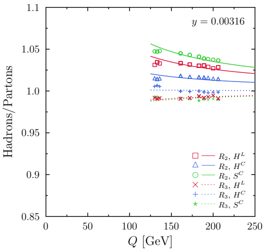
Analytic pieces in place for N^3LL and NNLL resummation for R_2 and R_3

The elephant in the room: the main source of uncertainty is due to hadronization modeling

- naively going to higher energies helps: hard. corr. $\sim 1/Q$, however. . .
- FCC-ee energy is not orders of magnitude larger than LEP
- going up in energy there is an interplay between smaller hadronization corrections but larger background and much smaller luminosity







The role of hadronization corrections

Need better MC's + hadronization models/calibration in e^+e^-

In a perfect world

- parton showers with NNLL logarithmic accuracy matched to NNLO
- hadronization models calibrated from scratch with many different observables, since current models were tuned using MC's with lower accuracy [see also talk by P. F. Monni]

Alternatively

- need a (much) more refined analytical understanding of non-perturbative corrections
- look for better observables, with smaller hadronization corrections [see talk by Z. Trócsányi]

Role of background

At LEP (and before) the signal process was $e^+e^- \to Z/\gamma \to$ hadrons, while $e^+e^- \to VV/ZH \to$ hadrons ($+$ leptons) was background to be subtracted

- introduces a lot of systematic uncertainties
- but this is what could be compared to precisely computed predictions

One way to deal with increased background at FCC-ee could be to redefine the signal process as $e^+e^- \to$ hadrons

- only background to this is from $e^+e^- \to VV/ZH \to$ hadrons $+$ leptons, e.g. $e^+e^- \to W^+W^- \to q\,\bar{q}\,l\,\bar{\nu}_l$, which can be suppressed almost completely
- however with this redefinition already the Born processes $e^+e^- \to VV/ZH \to q\bar{q}q\bar{q}$ involve four colored particles $\Rightarrow$ the precise theoretical description of all channels is a major challenge
- EW corrections to $e^+e^- \to Z/\gamma \to$ hadrons must also be addressed

Conclusions

New extraction of $\alpha_s(M_Z)$ from fit of the Durham two-jet rate R_2 in e^+e^- annihilation to $N^3LO+NNLL$ predictions + hadronization corrections extracted from state-of-the-art MC event generators:

$$\alpha_s(M_Z) = 0.11881 \pm 0.00063(exp.) \pm 0.00101(hadr.) \pm 0.00045(ren.) \pm 0.00034(res.)$$

- the result is consistent with the world average and the most precise in its subclass
- main source of uncertainty from modeling of hadronization corrections

Lessons for a similar measurement at FCC-ee [see also talk by A. Verbytskyi]

- perturbative uncertainty under control, but improvements possible
- N^4LO/N^3LO for R_2/R_3: not the priority from the point of view of this measurement
- b-quark mass corrections and EW$\times$QCD corrections seem more relevant
- $N^3LL/NNLL$ resummation for R_2/R_3 are already within reach
- better understanding of hadronization corrections crucial for improvement
- could consider a redefinition of the signal: $e^+e^- \to$ hadrons

Backup

Hadronization corrections: simultaneous corrections for R_2 and R_3

Challenge: simultaneous corrections for R_2 and R_3

- hadronization corrections derived on a bin-by-bin basis, $R_{n,\mathrm{hadron}} = R_{n,\mathrm{parton}} f_n(y)$, $n = 2, 3, 4, \ldots$ can violate physical constraints: $0 \le R_n \le 1$ and $\sum_n R_n = 1$

Solution:

- introduce ξ_1 and ξ_2 such that at parton level $R_{2,\mathrm{parton}} + R_{3,\mathrm{parton}} + R_{\ge 4,\mathrm{parton}} = 1$

$$R_{2,\mathrm{parton}} = \cos^2\xi_1\,, \qquad R_{3,\mathrm{parton}} = \sin^2\xi_1 \cos^2\xi_2\,, \qquad R_{\ge 4,\mathrm{parton}} = \sin^2\xi_1 \sin^2\xi_2\,,$$

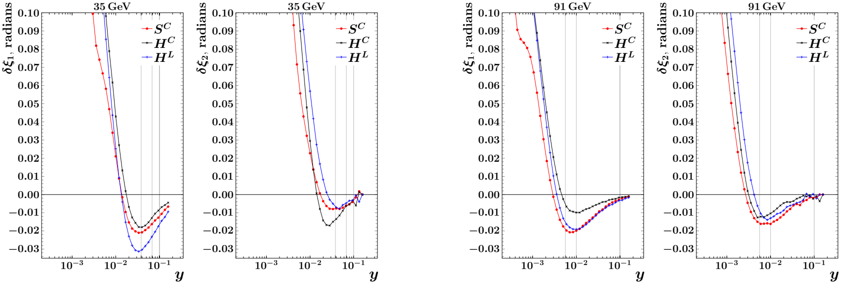
- similarly at hadron level, set

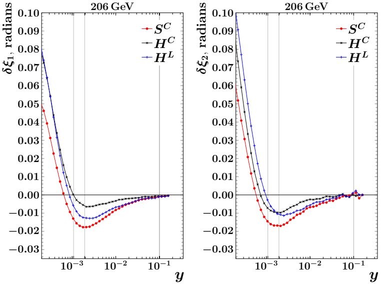
$$R_{2,\mathrm{hadron}} = \cos^2(\xi_1 + \delta\xi_1)\,, \qquad R_{3,\mathrm{hadron}} = \sin^2(\xi_1 + \delta\xi_1)\cos^2(\xi_2 + \delta\xi_2)\,,$$
$$R_{\ge 4,\mathrm{hadron}} = \sin^2(\xi_1 + \delta\xi_1)\sin^2(\xi_2 + \delta\xi_2)$$

- the functions $\delta\xi_1(y)$ and $\delta\xi_2(y)$ account for hadronization corrections and are extracted from the MC samples

This approach clearly preserves physical constraints

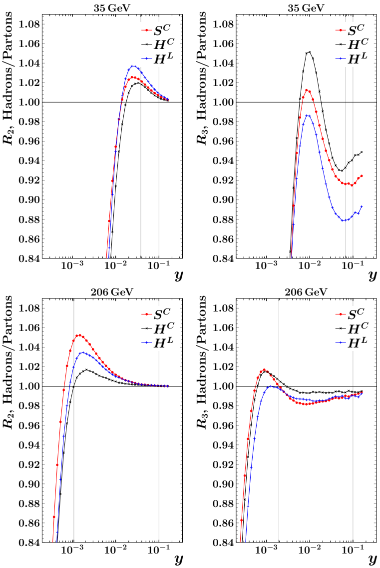
Hadronization corrections: $\delta\xi_1(y)$ and $\delta\xi_2(y)$

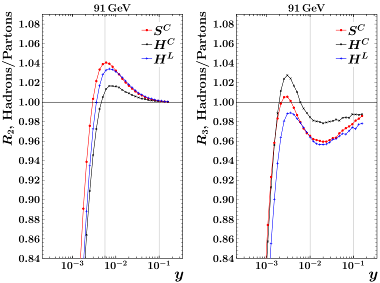




- to avoid binning effects, the hadronization corrections are parametrized with smooth functions
- vertical lines show the fit ranges for the reference fits of R_2 and R_3

Hadronization corrections: hadron to parton ratios





- to avoid binning effects, the hadronization corrections are parametrized with smooth functions
- vertical lines show the fit ranges for the reference fits of R_2 and R_3

R_2 fits

Fit of $\alpha_s(M_Z)$ from experimental data for R_2 obtained using N^{3}LO and N^{3}LO+NNLL predictions for R_2. The reported uncertainty comes from `MINUIT2`

Fit ranges, $\log y$ Hadronization	N^{3}LO $\chi^2/ndof$	N^{3}LO+NNLL $\chi^2/ndof$
$[-1.75+\mathcal{L},-1]$ S^C	0.12121 ± 0.00095 $20/86 = 0.24$	0.11849 ± 0.00092 $20/86 = 0.24$
$[-2+\mathcal{L},-1]$ S^C	0.12114 ± 0.00081 $26/100 = 0.26$	0.11864 ± 0.00075 $26/100 = 0.26$
$[-2.25+\mathcal{L},-1]$ S^C	0.12119 ± 0.00060 $44/150 = 0.29$	0.11916 ± 0.00063 $44/150 = 0.29$
$[-2.5+\mathcal{L},-1]$ S^C	0.12217 ± 0.00052 $89/180 = 0.50$	0.12075 ± 0.00055 $107/180 = 0.59$
$[-1.75+\mathcal{L},-1]$ H^C	0.11957 ± 0.00098 $22/86 = 0.26$	0.11698 ± 0.00093 $22/86 = 0.25$
$[-2+\mathcal{L},-1]$ H^C	0.11923 ± 0.00079 $29/100 = 0.29$	0.11687 ± 0.00076 $28/100 = 0.28$
$[-2.25+\mathcal{L},-1]$ H^C	0.11868 ± 0.00068 $43/150 = 0.28$	0.11679 ± 0.00064 $40/150 = 0.27$
$[-2.5+\mathcal{L},-1]$ H^C	0.11849 ± 0.00050 $58/180 = 0.32$	0.11723 ± 0.00053 $58/180 = 0.32$
$[-1.75+\mathcal{L},-1]$ H^L	0.12171 ± 0.00109 $21/86 = 0.25$	0.11897 ± 0.00092 $21/86 = 0.24$
$[-2+\mathcal{L},-1]$ H^L	0.12144 ± 0.00078 $28/100 = 0.28$	0.11893 ± 0.00075 $26/100 = 0.26$
$[-2.25+\mathcal{L},-1]$ H^L	0.12080 ± 0.00069 $43/150 = 0.28$	0.11881 ± 0.00063 $39/150 = 0.26$
$[-2.5+\mathcal{L},-1]$ H^L	0.12024 ± 0.00051 $57/180 = 0.32$	0.11897 ± 0.00053 $52/180 = 0.29$

$R_2 + R_3$ fits

Simultaneous fit of $\alpha_s(M_Z)$ from experimental data for R_2 and R_3 obtained using N^{3}LO and N^{3}LO+NNLL predictions for R_2 and NNLO predictions for R_3. The reported uncertainty comes from `MINUIT2`

Fit ranges, log y Hadronization	N^{3}LO $\chi^2/ndof$	N^{3}LO+NNLL $\chi^2/ndof$
$[-1.75+\mathcal{L},-1][-1.5+\mathcal{L},-1]$ S^C	0.12195 ± 0.00072 $120/143 = 0.84$	0.12078 ± 0.00066 $140/143 = 0.98$
$[-2+\mathcal{L},-1][-1.75+\mathcal{L},-1]$ S^C	0.12163 ± 0.00061 $153/187 = 0.82$	0.12065 ± 0.00056 $176/187 = 0.94$
$[-2.25+\mathcal{L},-1][-2+\mathcal{L},-1]$ S^C	0.12075 ± 0.00044 $208/251 = 0.83$	0.11994 ± 0.00041 $222/251 = 0.88$
$[-2.5+\mathcal{L},-1][-2.25+\mathcal{L},-1]$ S^C	0.12143 ± 0.00043 $321/331 = 0.97$	0.12089 ± 0.00044 $336/331 = 1.01$
$[-1.75+\mathcal{L},-1][-1.5+\mathcal{L},-1]$ H^C	0.12068 ± 0.00073 $126/143 = 0.88$	0.11956 ± 0.00066 $147/143 = 1.03$
$[-2+\mathcal{L},-1][-1.75+\mathcal{L},-1]$ H^C	0.12006 ± 0.00061 $163/187 = 0.87$	0.11913 ± 0.00054 $188/187 = 1.01$
$[-2.25+\mathcal{L},-1][-2+\mathcal{L},-1]$ H^C	0.11869 ± 0.00043 $221/251 = 0.88$	0.11793 ± 0.00043 $238/251 = 0.95$
$[-2.5+\mathcal{L},-1][-2.25+\mathcal{L},-1]$ H^C	0.11845 ± 0.00045 $302/331 = 0.91$	0.11799 ± 0.00047 $310/331 = 0.94$
$[-1.75+\mathcal{L},-1][-1.5+\mathcal{L},-1]$ H^L	0.12248 ± 0.00068 $121/143 = 0.85$	0.12129 ± 0.00063 $141/143 = 0.99$
$[-2+\mathcal{L},-1][-1.75+\mathcal{L},-1]$ H^L	0.12211 ± 0.00057 $155/187 = 0.83$	0.12110 ± 0.00053 $180/187 = 0.96$
$[-2.25+\mathcal{L},-1][-2+\mathcal{L},-1]$ H^L	0.12071 ± 0.00044 $209/251 = 0.83$	0.11989 ± 0.00045 $227/251 = 0.90$
$[-2.5+\mathcal{L},-1][-2.25+\mathcal{L},-1]$ H^L	0.12041 ± 0.00044 $266/331 = 0.80$	0.11990 ± 0.00044 $278/331 = 0.84$

Consistency tests

Several consistency tests performed

- simultaneous fit of $R_2 + R_3$ (see above)
- separate R_3 fit
- variation of χ^2 definition
- change of fit ranges
- multiplicative hadronization corrections
- `Sherpa` MC hadronization S^C
- stability across $\sqrt{s}$ (see below)
- exclusion of data with $\sqrt{s} < M_Z$