Numerical approach to multiloop calculations: overview

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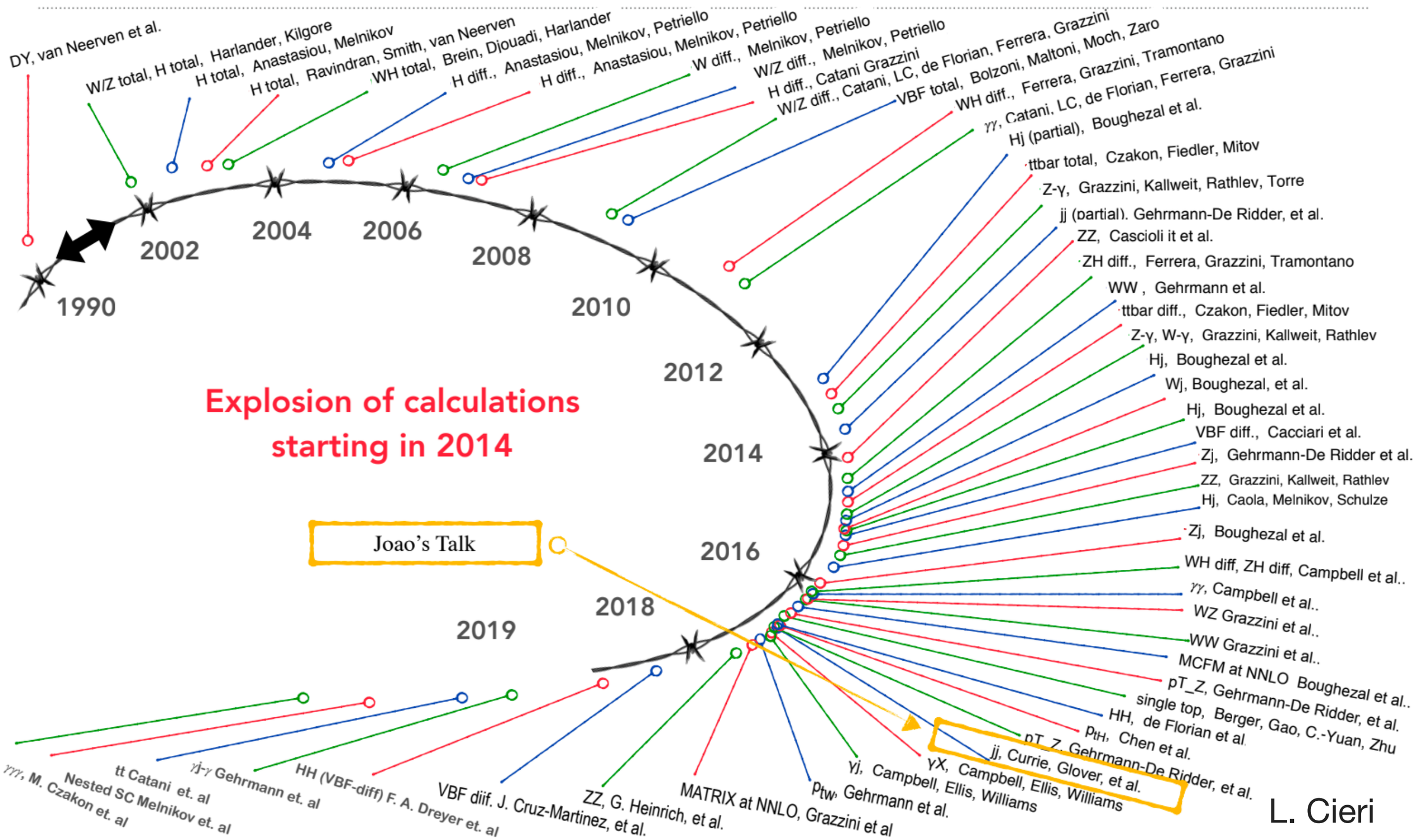
FCC hh ee he

3rd FCC Physics and Experiments Workshop
13-17 January 2020 | CERN

THE NNLO STANDARD

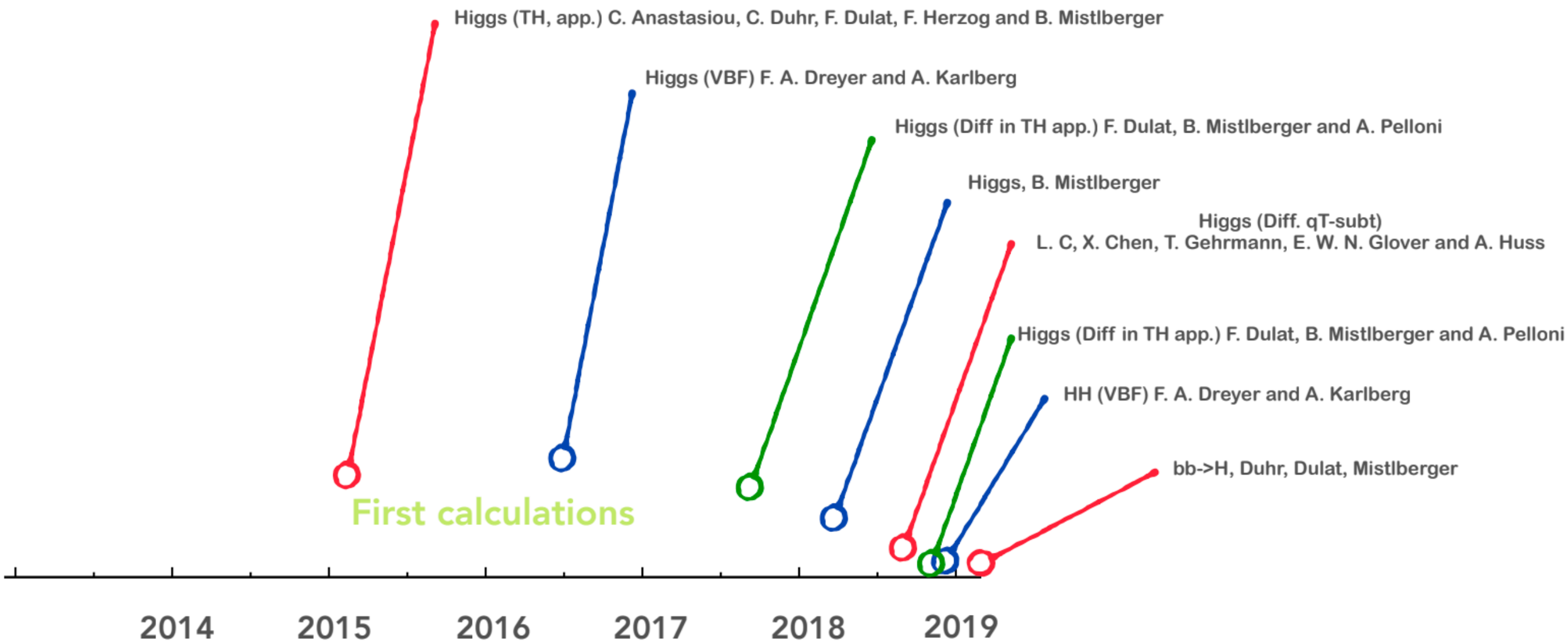
NNLO HADRON-COLLIDER CALCULATIONS VS. TIME



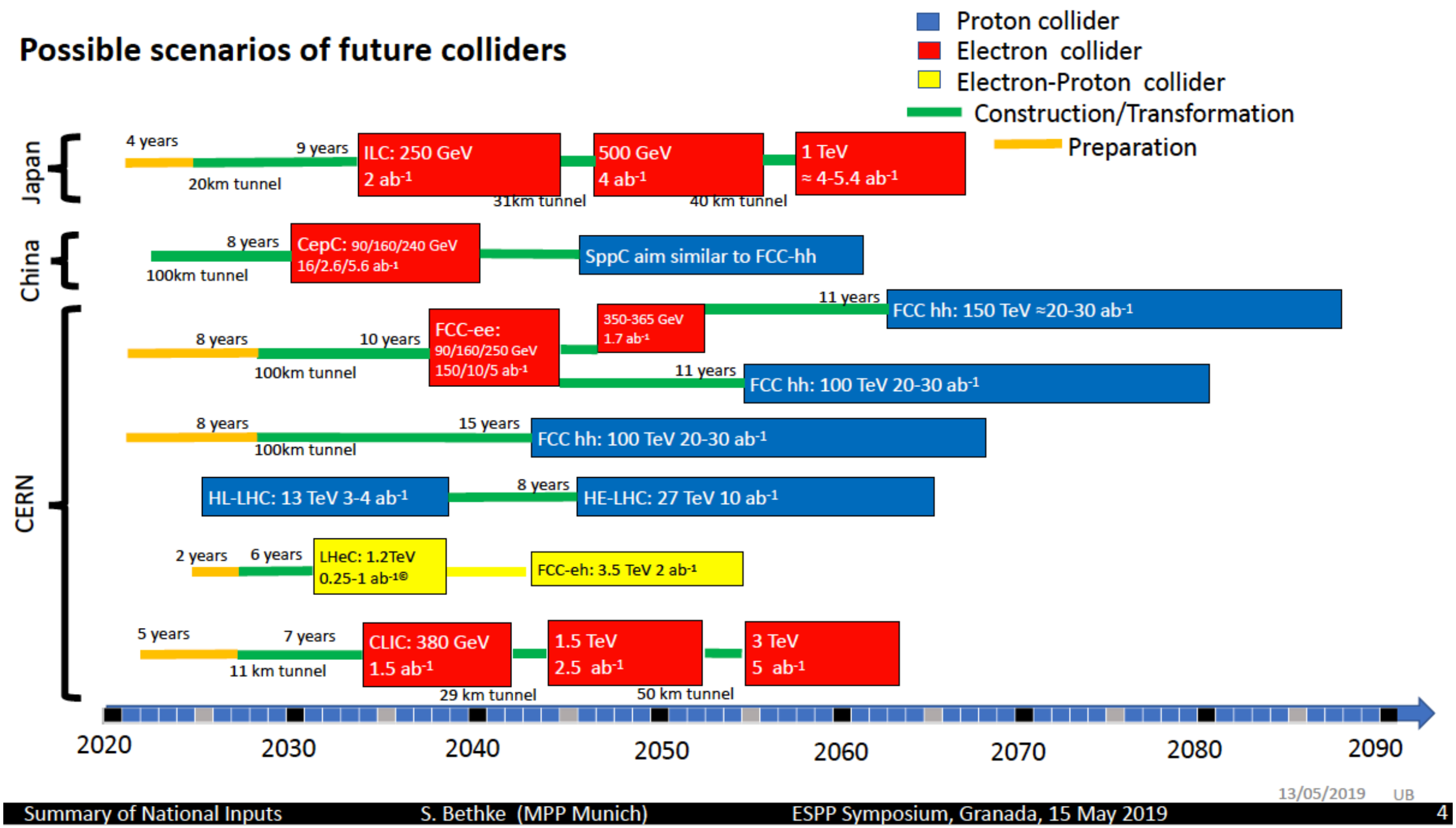
L. Cieri

THE N^{3}LO ERA

N^{3}LO HADRON-COLLIDER CALCULATIONS VS. TIME



L. Cieri



- all options aimed at **attobarn^{-1} physics**
- requires to go **far beyond NNLO for theory**
- Even conservative estimates **not reachable with current techniques**

some (semi-) numerical methods for loop integrals

[G. Heinrich]

- numerical solution of differential equations
  [Caffo, Czyz, Laporta, Remiddi '98; Czakon, Mitov '07 …]
- dispersion relations [Bauberger et al '94, Bauberger Freitas '17 …]
- use Bernstein-Sato-Tkachov theorem [Passarino, Uccirati et al '01- …]
- numerical evaluation of Mellin-Barnes representations
  [Czakon '05; … Dubovyk, Freitas, Gluza, Riemann, Usovitsch '16]
- numerical extrapolation [De Doncker, Yuasa, Kato, Fujimoto, Kurihara, Ishikawa, Olagbemi, Shimizu]
- direct numerical integration in momentum space
  [Soper '99; Gong, Soper, Nagy '09; Weinzierl, Reuschle et al. '10- …]
- loop-tree duality (4-dim) [Rodrigo, Buchta, Chachamis, Sborlini, Driencourt-Mangin et al. '08- …]
- sector decomposition [Hepp '66; Denner, Roth 96; Binoth, GH '00; …]

pro's and con's

	analytic	numerical
pole cancellation	exact	with numerical uncertainty
fast evaluation	☑ (mostly)	depends
control of integrable singularities	☑ control of analytic regions	difficult
extension to more scales	difficult	less difficult
automation	difficult	less difficult

[G. Heinrich]

MAX-PLANCK-GESELLSCHAFT

Max-Planck-Institut für Physik
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particleface

cost
EUROPEAN COOPERATION
IN SCIENCE & TECHNOLOGY

WG1 Meeting: Workstop / Thinkstart 3.0

Paving the way for alternative NNLO strategies

Galileo Galilei Institute
4-6 November 2019

To d, or not to d?

traditional dimensional schemes

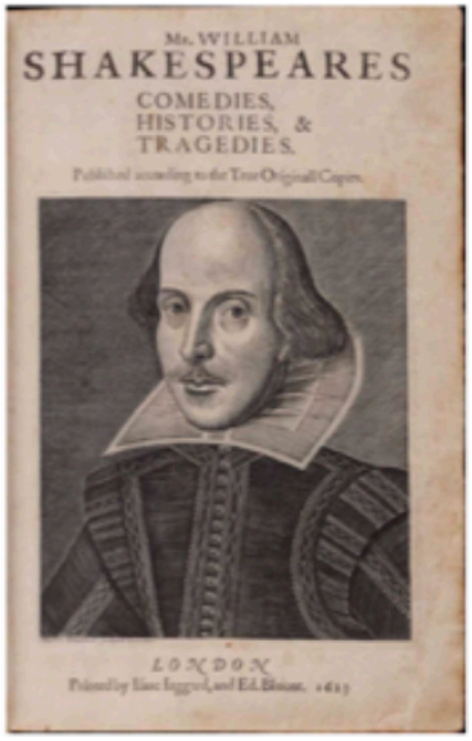
- 't Hooft / Veltman (HV) '72
- conventional dim. reg. (CDR) '73
- dim. reduction (DRED) '79
- four-dim. helicity (FDH) '92

reformulations of dimensional schemes

- six-dim. formalism (SDF) '09
- four-dim. formalism (FDF) '14

non-dimensional schemes

- implicit reg. (IREG) '98
- loop regularization (LORE) '03
- four-dim. reg. / ren. (FDR) '12
- four-dim. unsubtraction (FDU) '16



- mathematical consistency
- unitarity, causality (equivalence to $\overline{\text{MS}}$ / BPHZ)
- symmetries (gauge invariance, SUSY, ...)

- computational efficiency (analytical/numerical automation)

Overview of schemes

- DREG and variants (CDR, HV, FDH, DRED)
  standard, well developed, $QS_{[d_s]}=QS_{[d]}\oplus QS_{[n_\epsilon]}$
- FDF, SFD (4-, 6-dimensional formulation)
  =DREG/FDH, exploit properties of evanescent quantities
- Implicit regularization and
  FDR (Four-dimensional renormalization)
  stay in 4-dim! regularize by "replacement rule"
  use constraints/make choices for divergent integrals
- FDU (4-dimensional unsubtraction)
  loop-tree duality, cutting rules, combine real+virtual

Schemes and tricks to deal with the IR

Few scheme available at NLO:

- Slicing: [Giele, Glover], ...
- Subtraction: dipole[Catani, Seymour 9602277], FKS [Frixione et al. 9512328], NS [Nagy, Soper 0308127]

Many schemes available at NNLO:

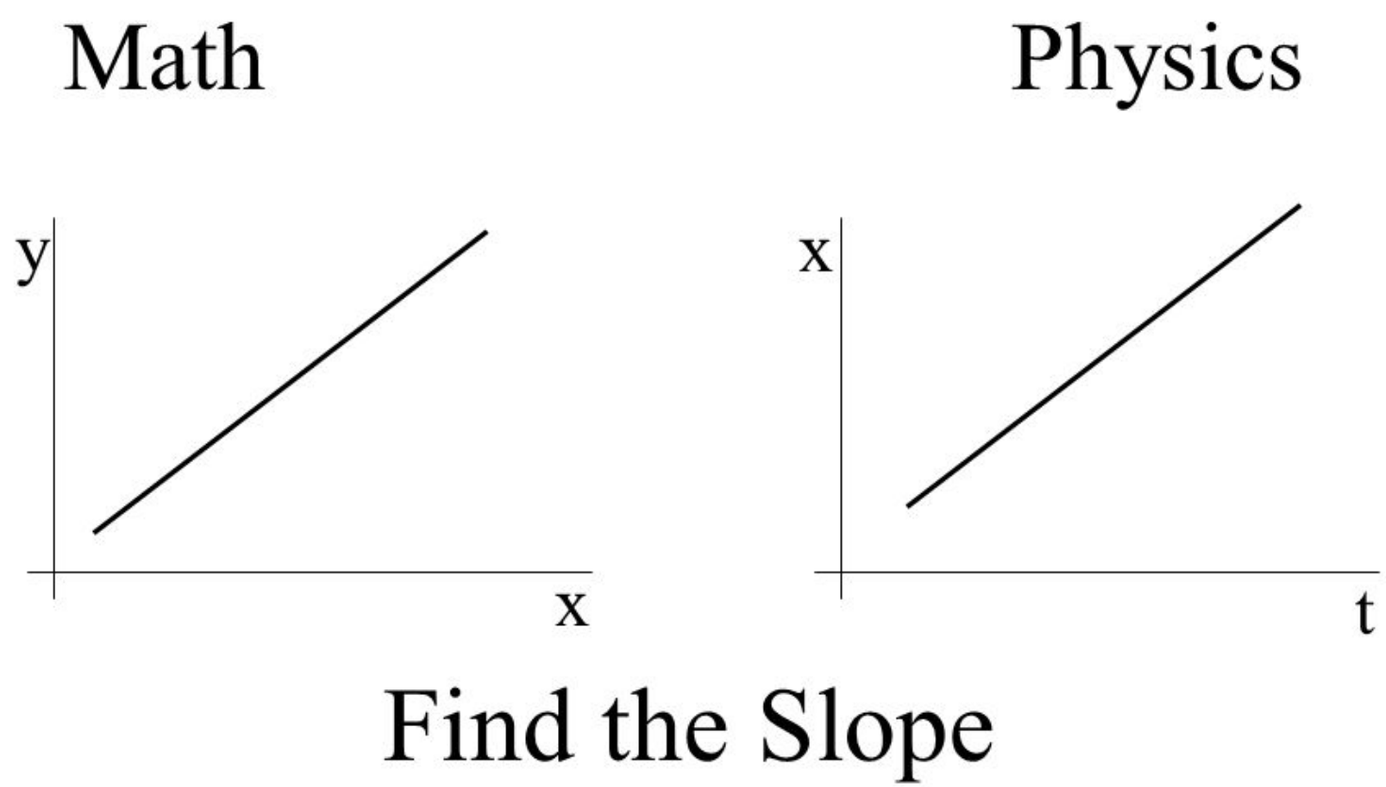
- Slicing: $q_\perp$ [Catani, Grazzini 0703012], N-Jettiness [Boughezal et al. 1505.03893, Gaunt et al. 1505.04794]
- Subtraction: Antenna [Gehrmann-DeRidder et al. 0505111], ColorfulNNLO [Del Duca et al. 1603.08927], Nested soft-collinear [Caola et al. 1702.01352], Geometric IR subtraction [Herzog 1804.07949], ϵ-prescription [Frixione, Grazzini 0411399], Sector decomposition [Bonoth et al. 0402265, Anastasiou et al. 0311311], residue subtraction [Czakon 1005.0274]
- New stategies: Unsubtraction [Sborlini et al. 1608.01584], FDR [Pittau 1208.5457]

→ Many options, but still there is room for improvement according to the *five criteria rule* [Melnikov, talk@Amplitude2019]

A good subtraction scheme should be

1. physically transparent ¿?
2. general (scaleable) ✓
3. local ✓
4. analytic numeric ?
5. efficient

The difficulties to reach higher orders arise because we have defined Quantum Field Theory not in the optimal way

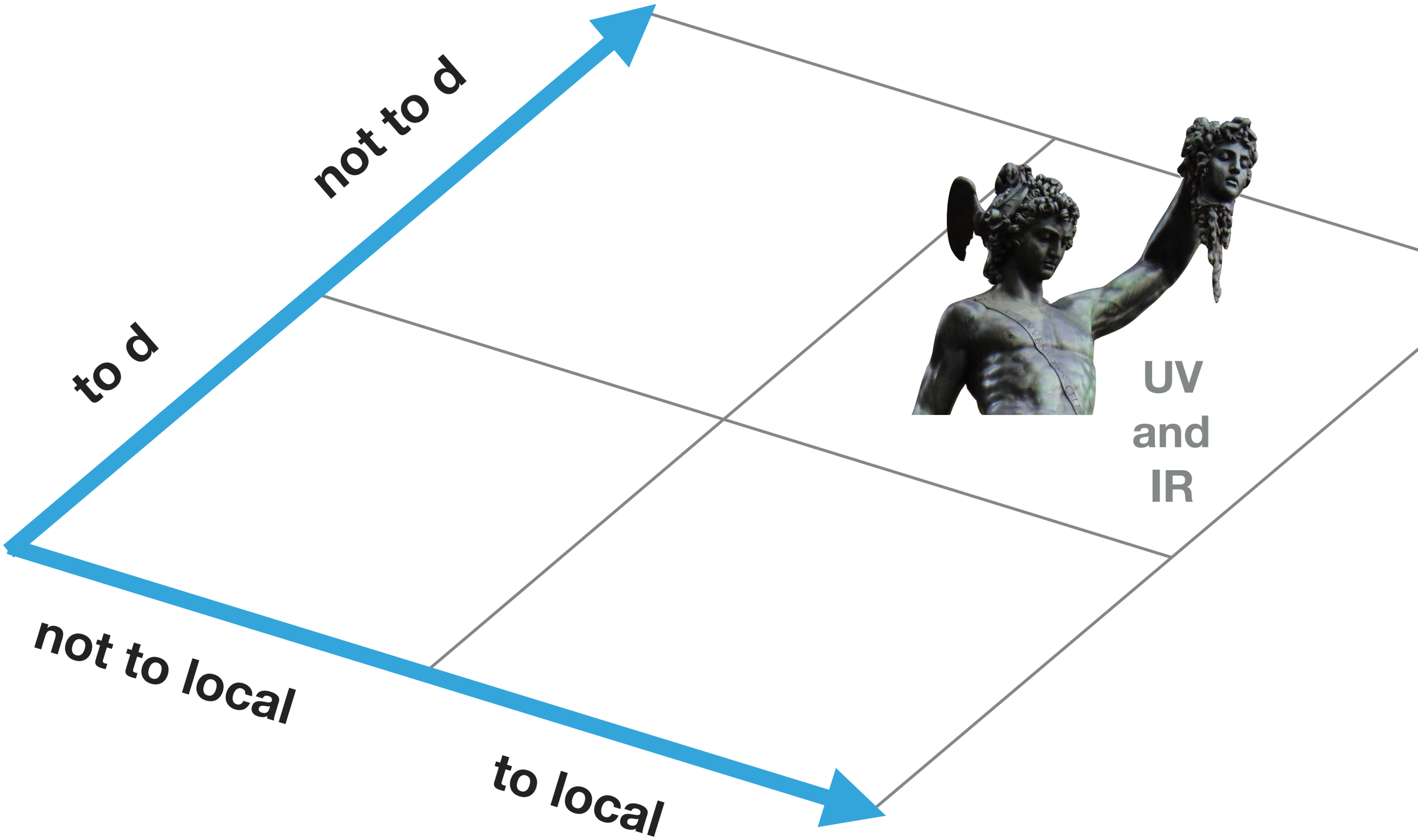


QFT = Quantum Mechanics + space-time

- Loops encode quantum fluctuations at **infinite energy** (**zero distance**): **SM/BSM** extrapolated at energies $\gg M_{\text{Plank}}$

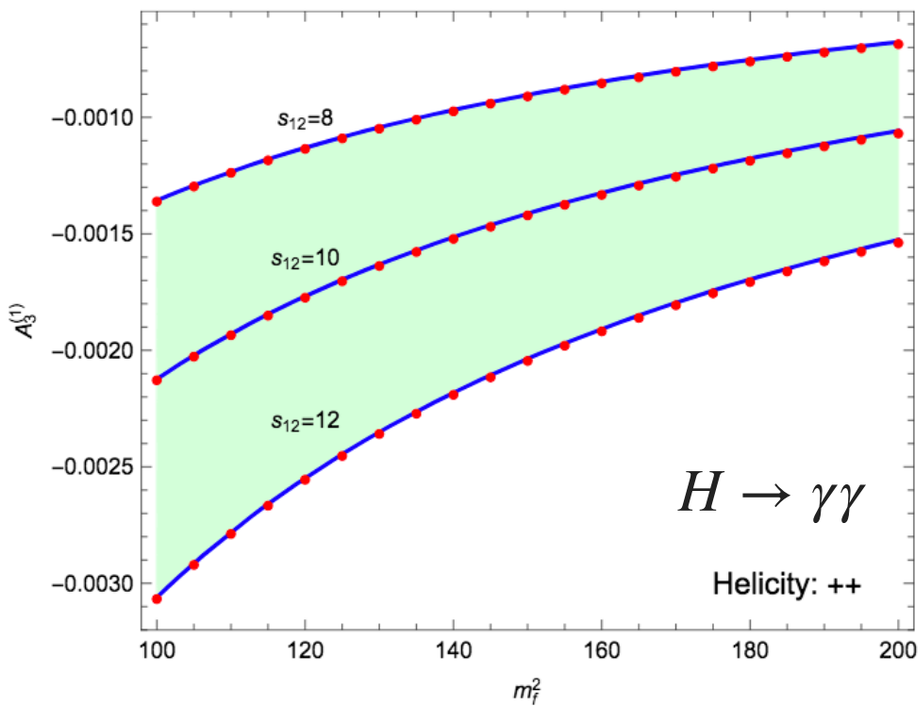
- QED/QCD massless gauge bosons/quarks: quantum state with N partons $\neq$ quantum state with **zero energy emission (infinite distance)** of extra partons

- Partons can be emitted in **exactly the same direction (zero distance)**

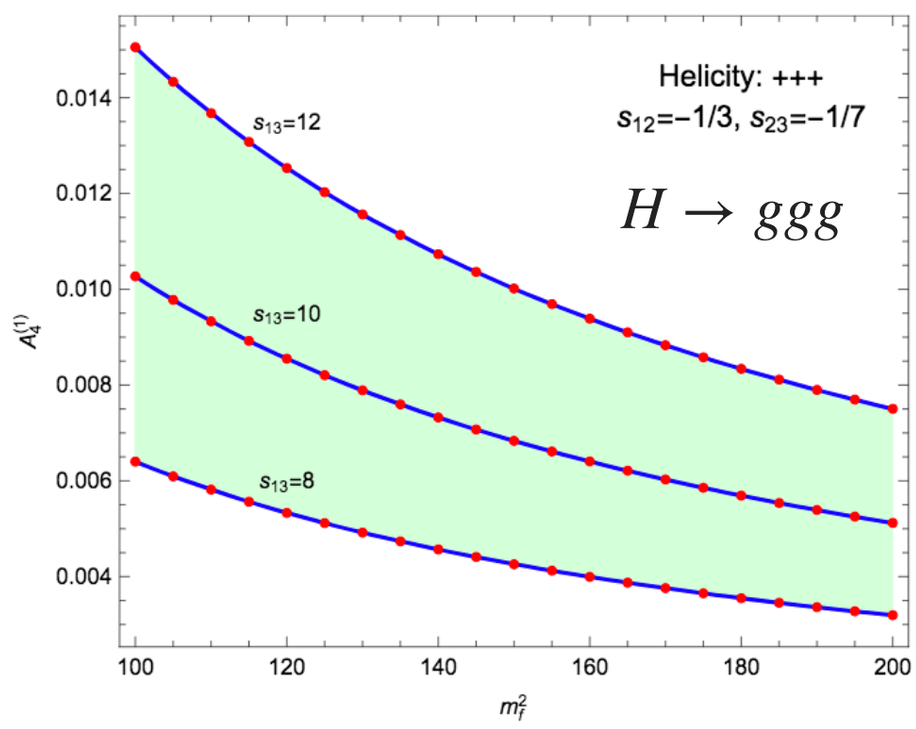

not to d
to d
not to local
to local
UV
and
IR

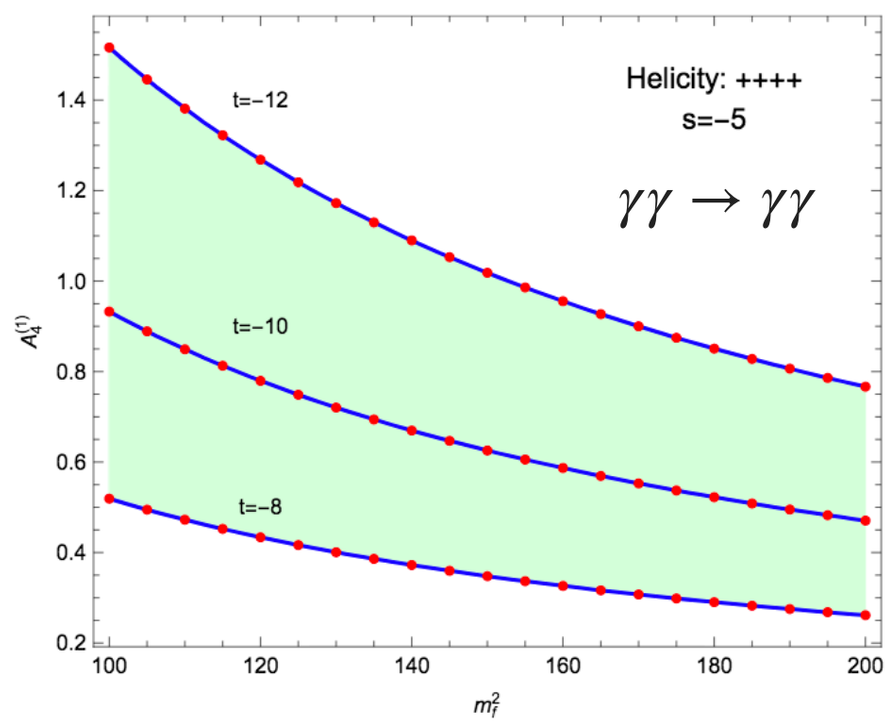
[Driencourt, GR, Sborlini, Torres, 1911.11125]

LOCAL UV RENORMALIZATION



- **UV finite** helicity amplitudes, but unintegrated amplitudes locally singular
- $\mathscr{A}_{\mathrm{R}}^{(L)} = \mathscr{A}^{(L)} - \mathscr{A}_{\mathrm{UV}}^{(L)}\Big|_{d=4}$ $\qquad \mathscr{A}_{\mathrm{UV}}^{(L)}\Big|_{d} = 0$
- Subtract not only logarithmic UV singularities, but also linear and quadratic
- Disentangle the UV from the IR behaviour in scaleless integrals (e.g. self-energies)





LOCAL UV RENORMALISATION

- Expand propagators and numerators around a UV propagator [Reuschle et al., similar to FDR in the UV - Pittau's talk]

$$G_F(q_{\rm UV}) = \frac{1}{q_{\rm UV}^2 - \mu_{\rm UV}^2 + \imath 0} \qquad \{\ell_j^2 \mid \ell_j \cdot k_i\} \to \{\lambda^2 q_{\rm UV}^2 + (1-\lambda^2)\,\mu_{\rm UV}^2 \mid \lambda\, q_{\rm UV} \cdot k_i\}$$

- and adjust **subleading** terms, $c_{\rm UV}$, to subtract only the pole ($\overline{MS}$ **scheme**), or to define any other renormalisation scheme. For the scalar two point function

$$I_{\rm UV}^{\rm cnt} = \int_\ell \frac{1}{(q_{\rm UV}^2 - \mu_{\rm UV}^2 + i0)^2}\left(1 + c_{\rm UV}\,\frac{\mu_{\rm UV}^2}{q_{\rm UV}^2 - \mu_{\rm UV}^2 + i0}\right)$$

- dual representation needs to deal with **multiple poles** [Bierenbaum et al.]

$$I_{\rm UV}^{\rm cnt} = \int_\ell \frac{\tilde{\delta}(q_{\rm UV})}{2\left(q_{\rm UV,0}^{(+)}\right)^2}\left(1 - \frac{3\,c_{\rm UV}\,\mu_{\rm UV}^2}{4\left(q_{\rm UV,0}^{(+)}\right)^2}\right) \qquad q_{\rm UV,0}^{(+)} = \sqrt{\mathbf{q}_{\rm UV}^2 + \mu_{\rm UV}^2 - i0}$$

Hernández-Pinto, Sborlini, GR, JHEP **1602,** 044

- Integration on the UV on-shell hyperboloid: loop three-momentum unconstrained, but **loop contributions suppressed** for loop energies larger than $\mu_{\rm UV}$

LOCAL UV RENORMALISATION: MULTILOOP

$$\begin{cases} |\ell_1| \to \infty \\ |\ell_2| \ \text{ fixed} \end{cases} , \qquad \begin{cases} |\ell_1| \ \text{ fixed} \\ |\ell_2| \to \infty \end{cases} , \qquad \begin{cases} |\ell_1| \to \infty \\ |\ell_2| \to \infty \end{cases} .$$

- Multiple UV limit

$$\mathrm{UV}^2 : \{\ell_j^2 \mid \ell_j \cdot \ell_k \mid \ell_j \cdot k_i\}$$
$$\to \{\lambda^2 q_{j,\mathrm{UV}}^2 + (1-\lambda^2)\,\mu_{\mathrm{UV}}^2 \mid \lambda^2 q_{j,\mathrm{UV}} \cdot q_{k,\mathrm{UV}} + (1-\lambda^2)\,\mu_{\mathrm{UV}}^2/2 \mid \lambda\, q_{j,\mathrm{UV}} \cdot k_i\}$$

- Most subtle steep the adjustment of the **subleading** terms, d_{UV2}, to be in agreement with e.g. the $\overline{MS}$ **scheme**

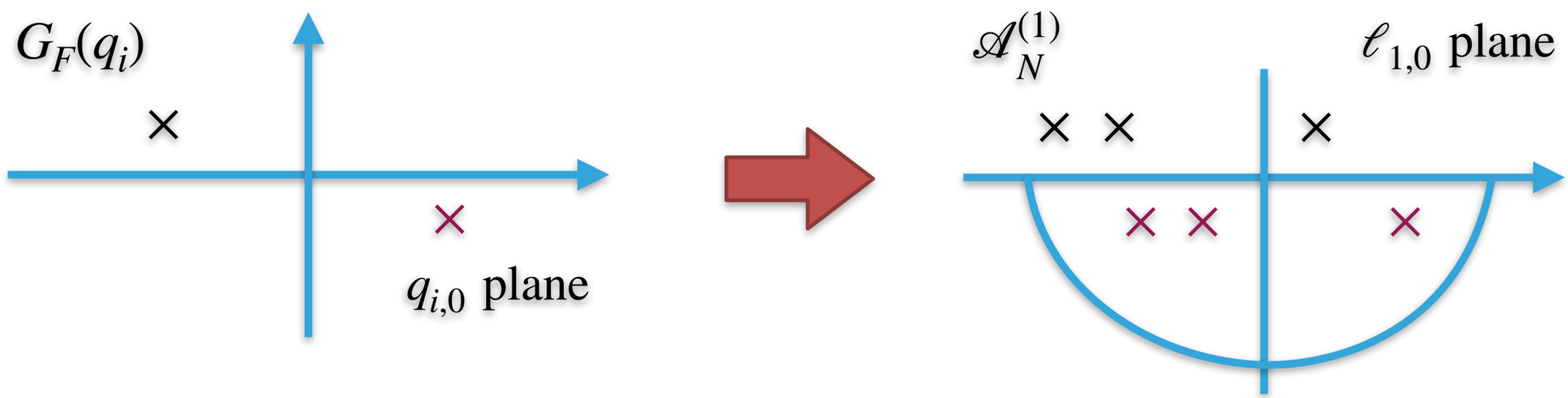
$$\left(\mathscr{A}^{(L)} - \mathscr{A}^{(L)}_{1,\mathrm{UV}} - \mathscr{A}^{(L)}_{2,\mathrm{UV}}\right)_{\mathrm{UV}^2} - d_{\mathrm{UV2}}\,\mu_{\mathrm{UV}}^4 \int_{\ell_1,\ell_2} \left(G_F(q_{1,\mathrm{UV}})\right)^3 \left(G_F(q_{2,\mathrm{UV}})\right)^3$$

[Catani et al. JHEP **0809**, 065]

THE LOOP-TREE DUALITY (LTD)

Cauchy residue theorem

in the loop energy complex plane



Feynman Propagator +i0:
positive frequencies are propagated forward in time, and negative backward

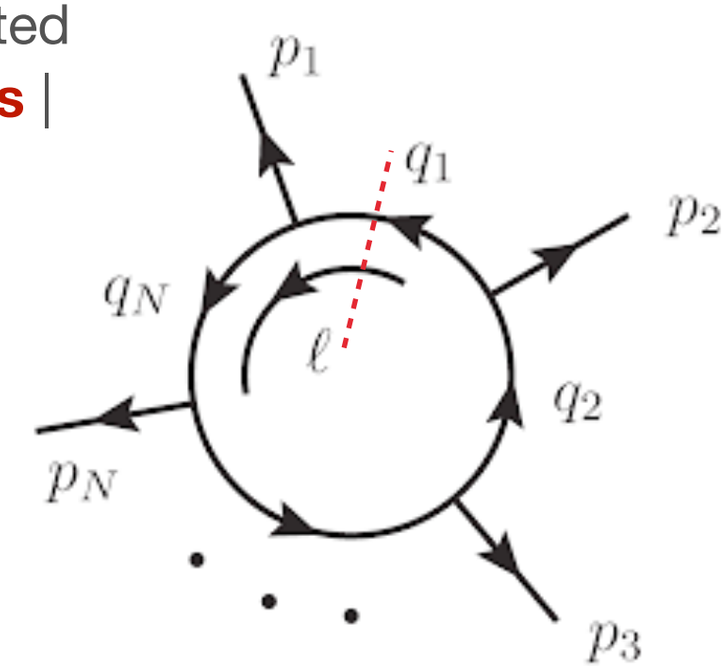
selects residues with definite **positive energy and negative imaginary part** (indeed in any other coordinate system)

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0}$$

[Catani et al. JHEP **0809**, 065]

THE LOOP-TREE DUALITY (LTD)

One-loop amplitudes in any relativistic, local and unitary QFT represented as a linear combination of *N* **single-cut phase-space/dual amplitudes** | **non-disjoint trees** (at higher orders: number of cuts equal to the number of loops)

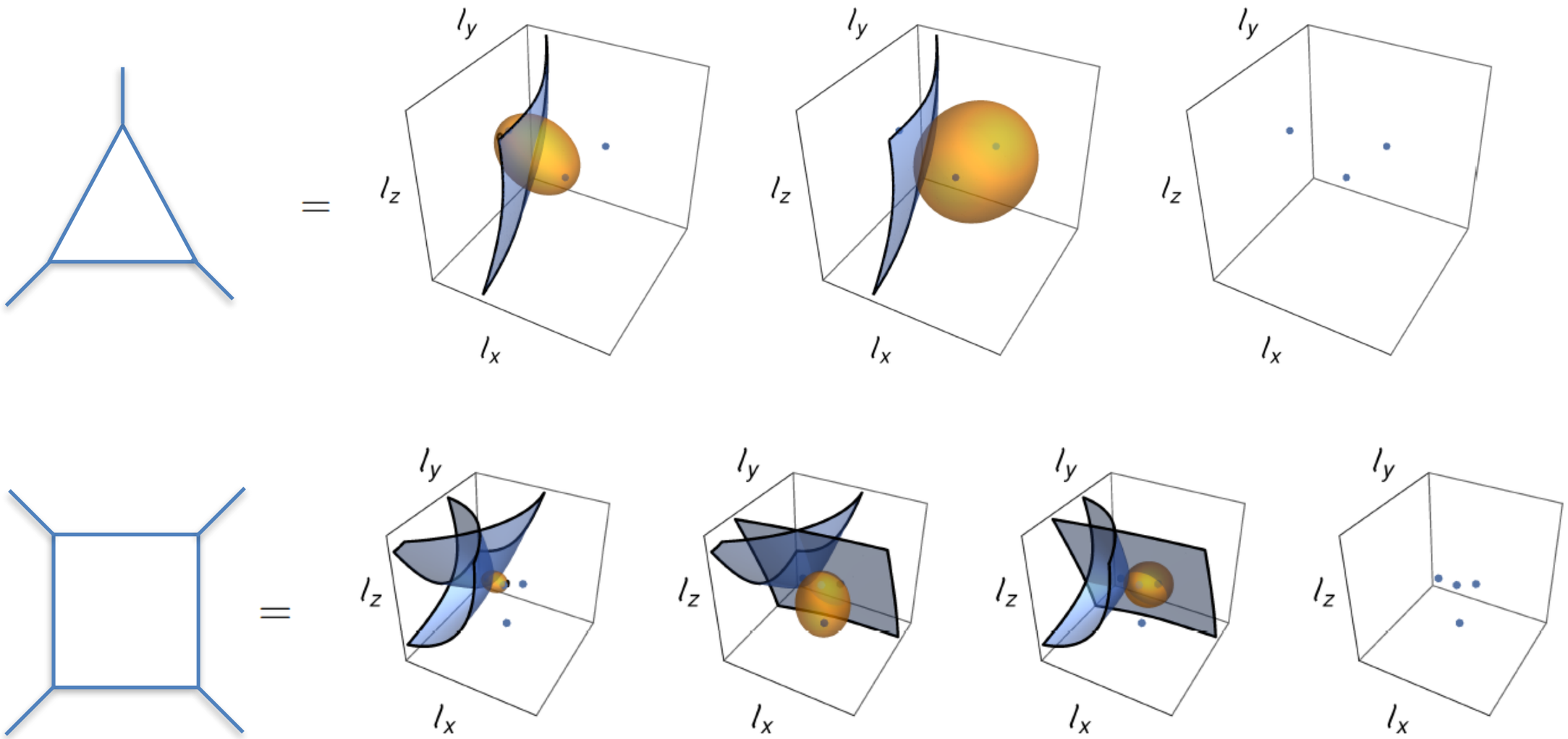


$$\int_{\ell_1} \mathcal{N}(\ell_1) \prod G_F(q_i) = -\int_{\ell_1} \mathcal{N}(\ell_1) \otimes \sum \tilde{\delta}(q_i) \prod_{i \neq j} G_D(q_i; q_j)$$

- $\tilde{\delta}(q_i) = \imath 2\pi\, \theta(q_{i,0})\, \delta(q_i^2 - m_i^2)$ sets internal line on-shell, positive energy mode
- $G_D(q_i; q_j) = \dfrac{1}{q_j^2 - m_j^2 - \imath 0\, \eta k_{ji}}$ **dual propagator** $k_{ji} = q_j - q_i$ $q_{i,0}^{(+)} = \sqrt{\mathbf{q}_i^2 + m_i^2 - \imath 0}$
- LTD realised by **modifying the customary +i0 prescription** of the Feynman propagators (only the sign matters), it encodes in a compact way the effect of **multiple-cut** contributions that appear in the **Feynman's Tree Theorem**
- **Lorentz invariant**, best choice $\eta^\mu = (1, \mathbf{0})$: energy component integrated out, remaining integration in **Euclidean space**

LTD singular scheme

[Buchta PhD2015]
Buchta, et al. EPJC **77** (2017) 274



- non-causal singularities (forward-forward in blue): undergo **dual cancellations** among dual pairs
- causal singularities (forward-backward in orange): bounded to a **compact region,** which is of the size of the **hard scale**, collapse to a finite segment for **infrared singularities (→ FDU)**
- Numerical integration in the **Euclidean** space of the loop three-momenta, CPU/GPU time do not scale significantly with the number of legs

Contour deformation: thresholds

[Buchta PhD2015]
Buchta, et al. EPJC **77** (2017) 274

- Integrable singularities / causal and anomalous thresholds through contour deformation of the loop three-momentum

$$\boldsymbol{\ell} \to \boldsymbol{\ell}' = \boldsymbol{\ell} + i\boldsymbol{\kappa}$$

- such that it matches the +i0 prescription

$$q_{i,0}^{(+)} = \sqrt{-\boldsymbol{\kappa}^2 + 2i\boldsymbol{\kappa}\cdot\mathbf{q}_i + \mathbf{q}_i^2 + m_i^2 - i0} \qquad \boldsymbol{\kappa}\cdot\mathbf{q}_i < 0$$

- e.g.

$$\boldsymbol{\kappa} = \sum_{i,j\in\text{group}} \lambda_{ij} \left(\frac{\mathbf{q}_i}{\sqrt{\mathbf{q}_i^2}} + \frac{\mathbf{q}_j}{\sqrt{\mathbf{q}_j^2}} \right) \exp\left(-\frac{G_D^{-2}(q_i;q_j)}{2\sigma_{ij}^2} \right)$$

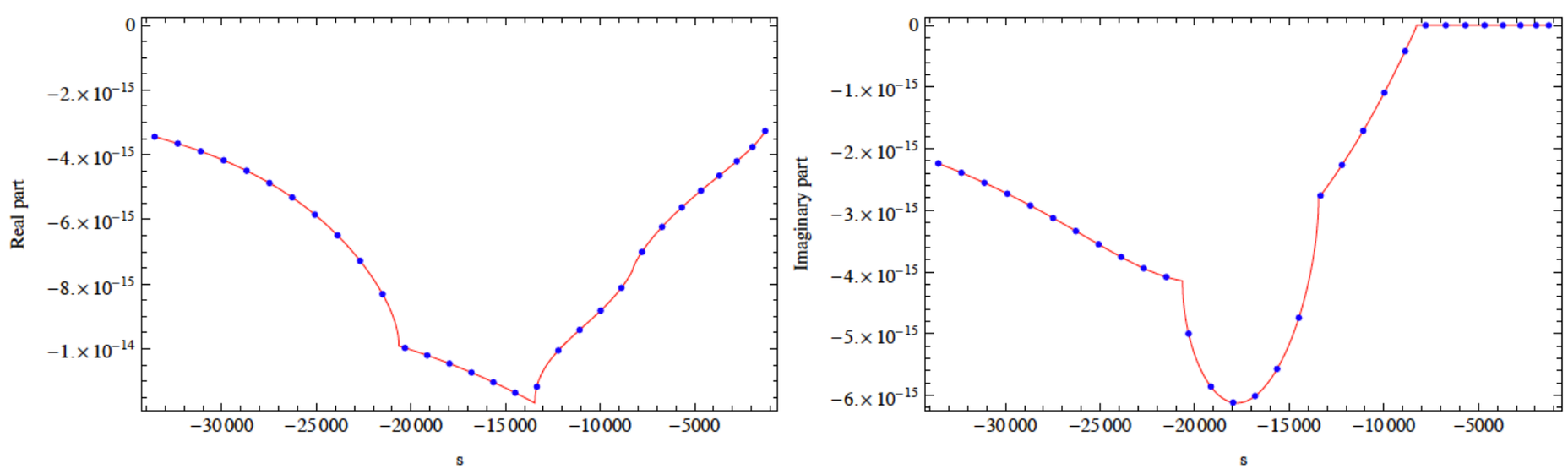


Figure 9: Energy-scan of a scalar pentagon. The red curve is done with LoopTools and the blue points are obtained with the LTD method.

Tensor / scalar

[Buchta PhD2015]
Buchta, et al. EPJC **77** (2017) 274

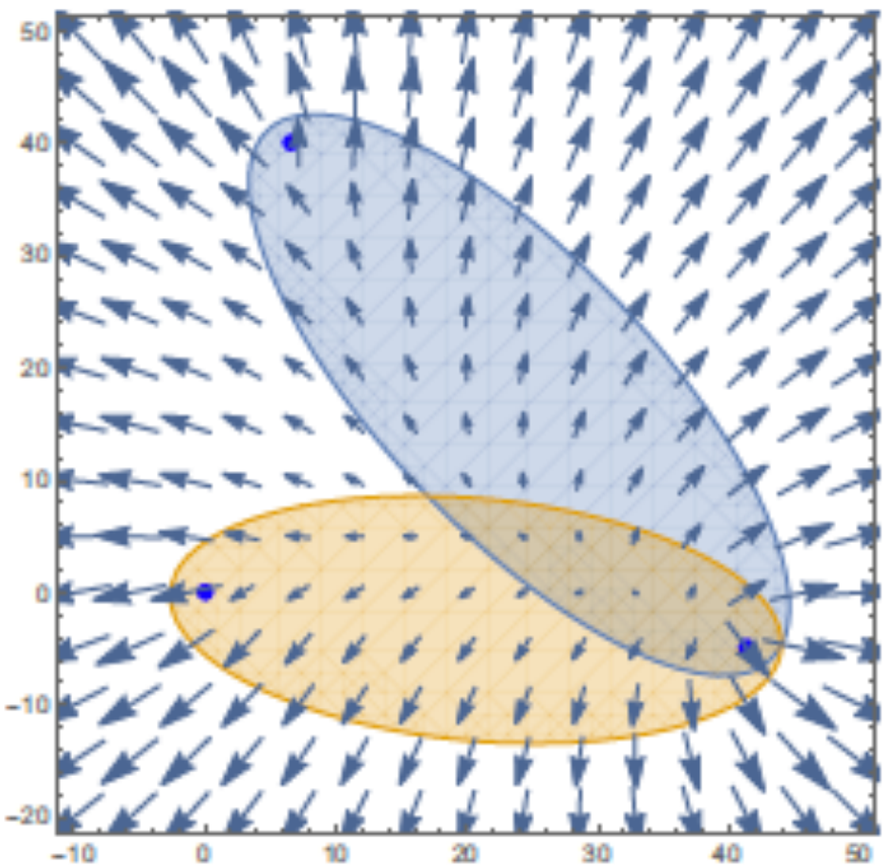
	Rank	Tensor Hexagon	Real Part	Imaginary Part	Time[s]
P20	1	SecDec	$-1.21585(12) \times 10^{-15}$		36
		LTD	$-1.21552(354) \times 10^{-15}$		6
P21	3	SecDec	$4.46117(37) \times 10^{-9}$		5498
		LTD	$4.461369(3) \times 10^{-9}$		11
P22	1	SecDec	$1.01359(23) \times 10^{-15}$	$+i\ 2.68657(26) \times 10^{-15}$	33
		LTD	$1.01345(130) \times 10^{-15}$	$+i\ 2.68633(130) \times 10^{-15}$	72
P23	2	SecDec	$2.45315(24) \times 10^{-12}$	$-i\ 2.06087(20) \times 10^{-12}$	337
		LTD	$2.45273(727) \times 10^{-12}$	$-i\ 2.06202(727) \times 10^{-12}$	75
P24	3	SecDec	$-2.07531(19) \times 10^{-6}$	$+i\ 6.97158(56) \times 10^{-7}$	14280
		LTD	$-2.07526(8) \times 10^{-6}$	$+i\ 6.97192(8) \times 10^{-7}$	85

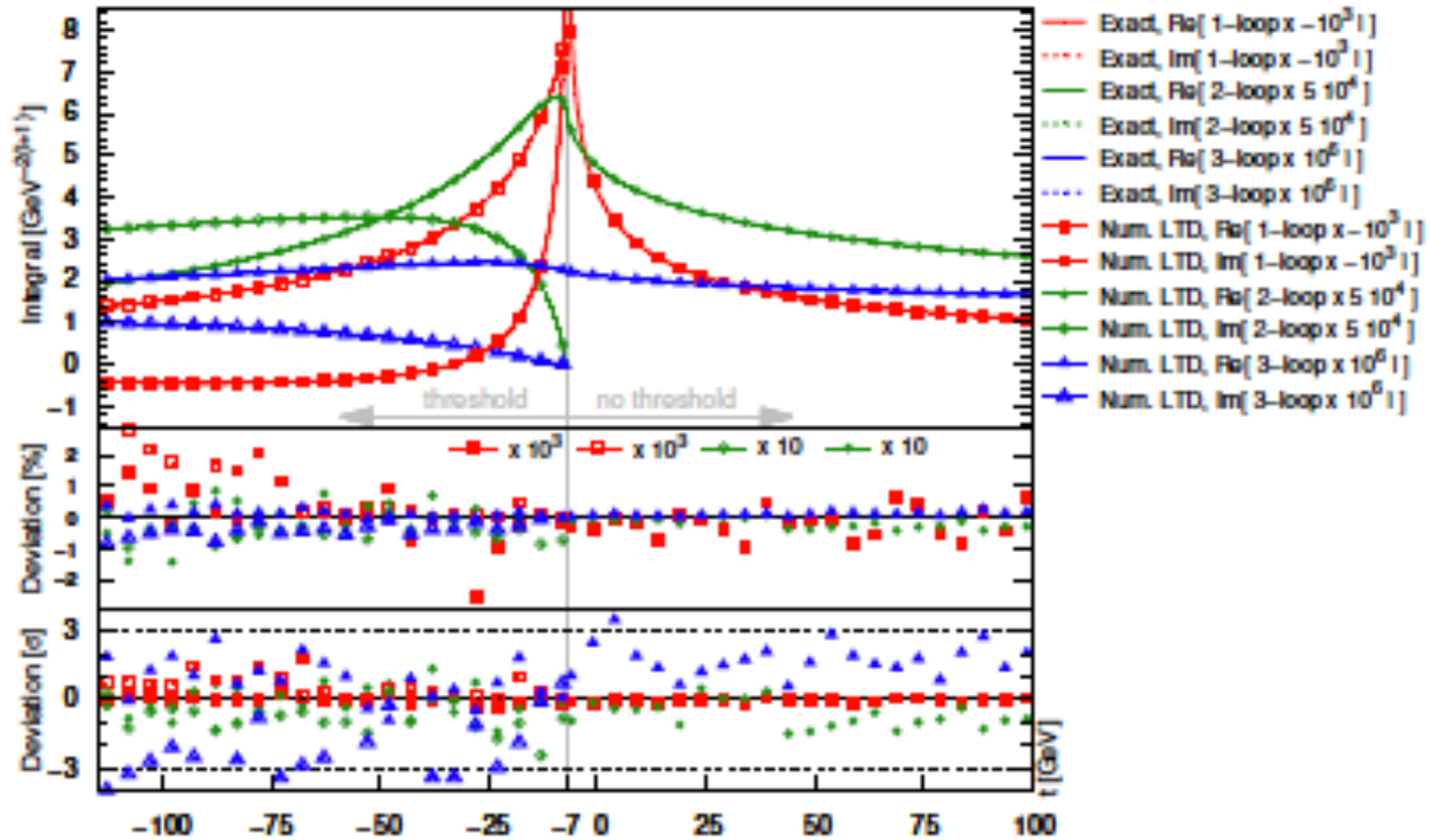
Table 5: Tensor hexagons involving numerators of rank one to three.

Propagator	Real Part	Imaginary Part
1	$2.530(4) \times 10^{-14}$	$+\,i\ 8.514(1) \times 10^{-14}$
$\ell.p_3 \times \ell.p_5$	$8.08(4) \times 10^{-15}$	$+\,i\ 6.144(5) \times 10^{-13}$

Table 1: Scalar and tensor decagon with all internal masses different.

[Capatti, Hirschi, Kermanschah, Pelloni, Ruijl, 1912.09291]





IR subtracted [Anastasiou, Sterman, JHEP **1907**, 056]

LTD BEYOND ONE-LOOP

R. P. Feynman, Closed Loop And Tree Diagrams. (talk 1972),
In *Brown, L.M. (ed.): Selected papers of Richard Feynman* 867-887

FIGURE 7.
Opening a double ring.

If there is more than one loop in the original diagram, the loops may be opened in succession. Choose any one loop; that is, integration over any one

- **After the first LTD round the position of the poles in the complex plane is momentum dependent**

(1) Use a **general identity** to transform into Feynman propagators the dual propagators that enter the successive LTD rounds [Bierenbaum et al., 2010]

- First **full two-loop** calculation ($H \to \gamma\gamma$) with local UV renormalization [Driencourt et al., 2019]
- **Analytic proof** of the dual cancellation of unphysical (non-causal) singularities, causal and anomalous thresholds as well as infrared in a compact region (→ FDU) [Aguilera et al., 2019]

(2) **Average** over all possible momentum flows [Runkel et al., 2019]: cumbersome symmetry factors

(3) Keep track of the position of the poles and close the Cauchy contour either **from above or from below** to cancel that dependence [Capatti et al., 2019]

- **Numerical test** of dual cancellations

[Aguilera, Driencourt, Hernández, Plenter, Ramírez, Rentería, GR, Sborlini, Torres, Tracz, arXiv:2001:03564]

LTD TO ALL ORDERS AND POWERS

- **Multi-loop scattering amplitude:** n sets of momenta that depend on L loop momenta or a linear combination

$$\mathcal{A}_N^{(L)}(1,\ldots,n) = \int_{\ell_1,\ldots,\ell_L} \mathcal{N}(\{\ell_i\}_L, \{p_j\}_N)\, G_F(1,\ldots,n)$$

$$= \prod_{i\in 1\cup\ldots\cup n} \left(G_F(q_i)\right)^{a_i}$$

- The **dual function** involving two sets that depend on the same loop momentum: momenta in the set t remain off-shell

$$G_D(s;t) = -2\pi\imath \sum_{i_s\in s} \mathrm{Res}\left(G_F(s,t), \mathrm{Im}(\eta q_{i_s}) < 0\right)$$

  - **Cauchy contour** always from **below** the real axis
  - valid for arbitrary powers and **Lorentz invariant** [Catani et al. JHEP **0809**, 065]
  - reverse momenta, if necessary, to keep a coherent momentum flow $t \to \bar{t} \quad (q_{i_t} \to -q_{i_t})$

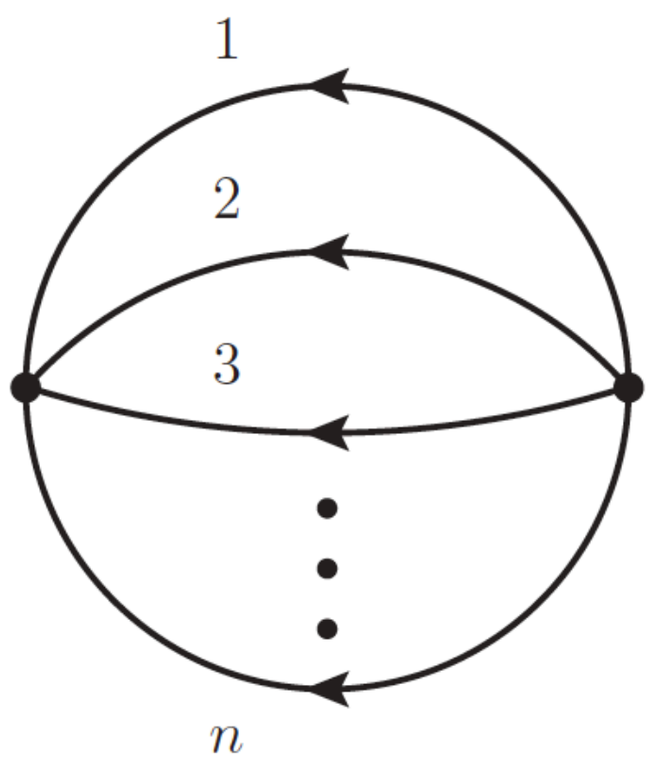
- The **nested residue** involving several sets

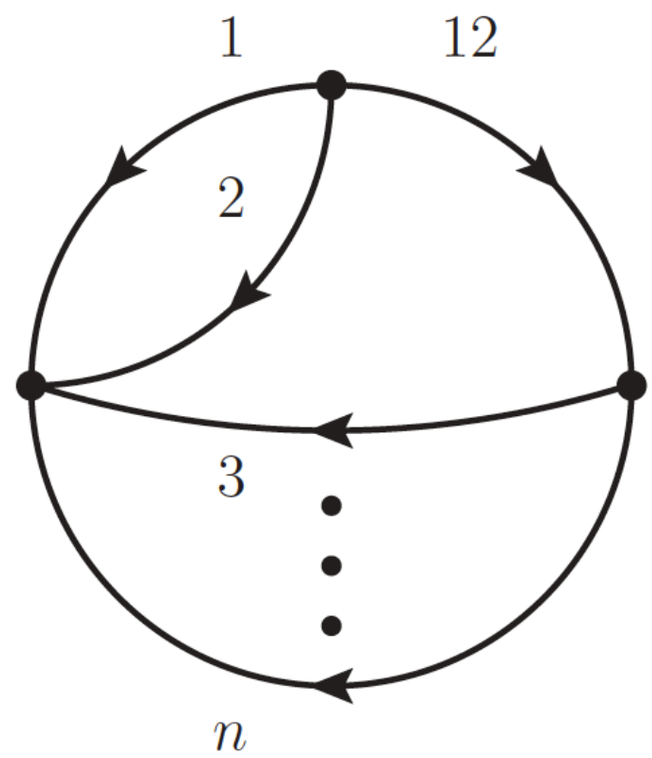
$$G_D(1,\ldots,r;n) = -2\pi\imath \sum_{i_r\in r} \mathrm{Res}\left(G_D(1,\ldots,r-1;r,n), \mathrm{Im}(\eta q_{i_r}) < 0\right)$$

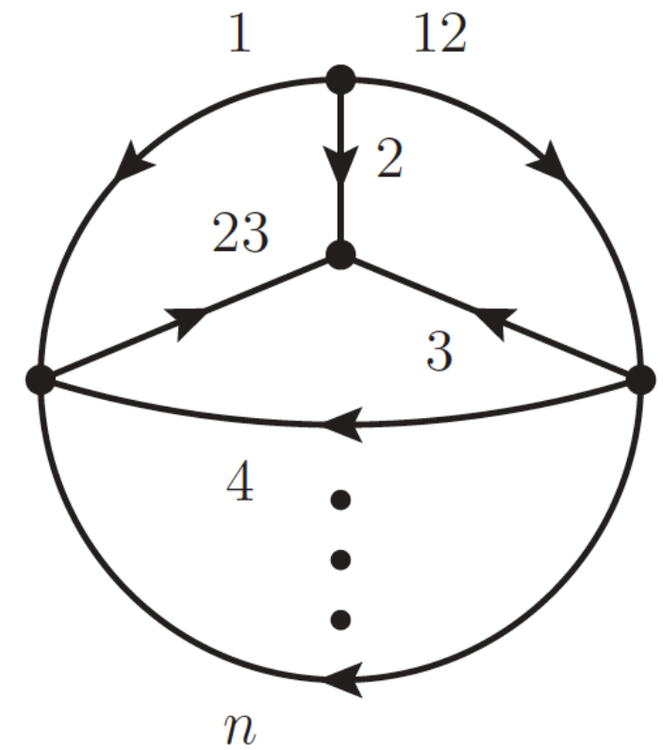
[Aguilera, Driencourt, Hernández, Plenter, Ramírez, Rentería, GR, Sborlini, Torres, Tracz, arXiv:2001:03564]

MULTILOOP TOPOLOGIES

arbitrary number of external legs attached to each line







MLT

Maximal Loop Topology

NMLT

Next-to-maximal Loop Topology

N2MLT

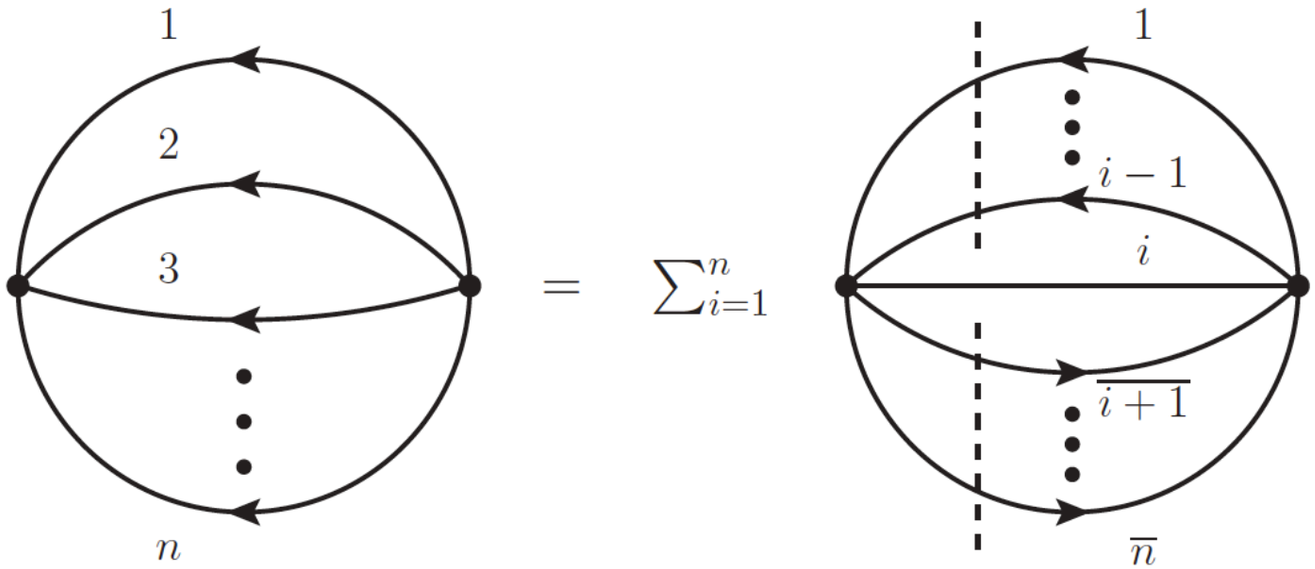
Next-to-next-to-maximal Loop Topology

single topology at two loops

starting at three loops

[Aguilera, Driencourt, Hernández, Plenter, Ramírez, Rentería, GR, Sborlini, Torres, Tracz, arXiv:2001:03564]

MAXIMAL LOOP TOPOLOGY



- **extremely simple and symmetric** LTD representation, proven by induction and directly **independent of the position of the poles** in the complex plane

$$\mathscr{A}^{(L)}_{\text{MLT}}(1,\ldots,n) = \int_{\ell_1,\ldots,\ell_L} \sum_{i=1}^{n} \mathscr{A}^{(L)}_{D}(1,\ldots,i-1,\overline{i+1},\ldots,\overline{n};i)$$

- **causal singularities** when on-shell momenta get aligned [Aguilera et al. JHEP **1912**, 163]

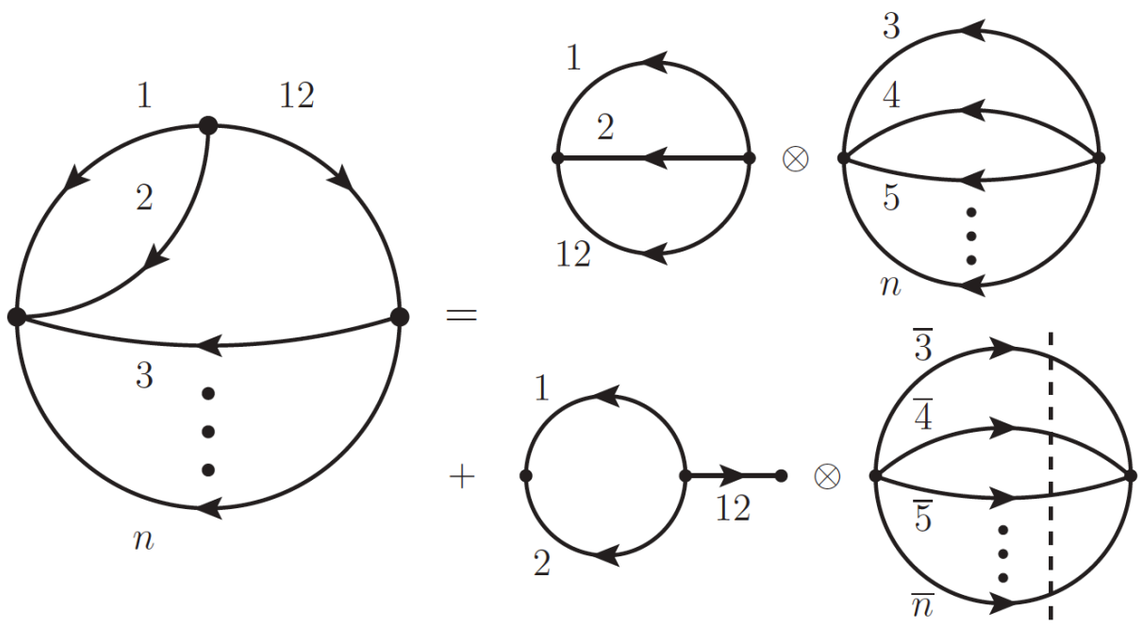
$$\mathscr{A}^{(L)}_{D}(\overline{2},\ldots,\overline{n};1) \to \mathscr{A}^{(L)}_{D}(\overline{1},\overline{2},\ldots,\overline{n})$$

- **non-causal singularities** (unphysical) entangled among dual pairs, they cancel

$$\mathscr{A}^{(L)}_{D}(\overline{2},\overline{3},\ldots,\overline{n};1) + \mathscr{A}^{(L)}_{D}(1,\overline{3},\ldots,\overline{n};2) \to \mathscr{A}^{(L)}_{D}(1,\overline{2},\overline{3},\ldots,\overline{n}) - \mathscr{A}^{(L)}_{D}(1,\overline{2},\overline{3},\ldots,\overline{n})$$

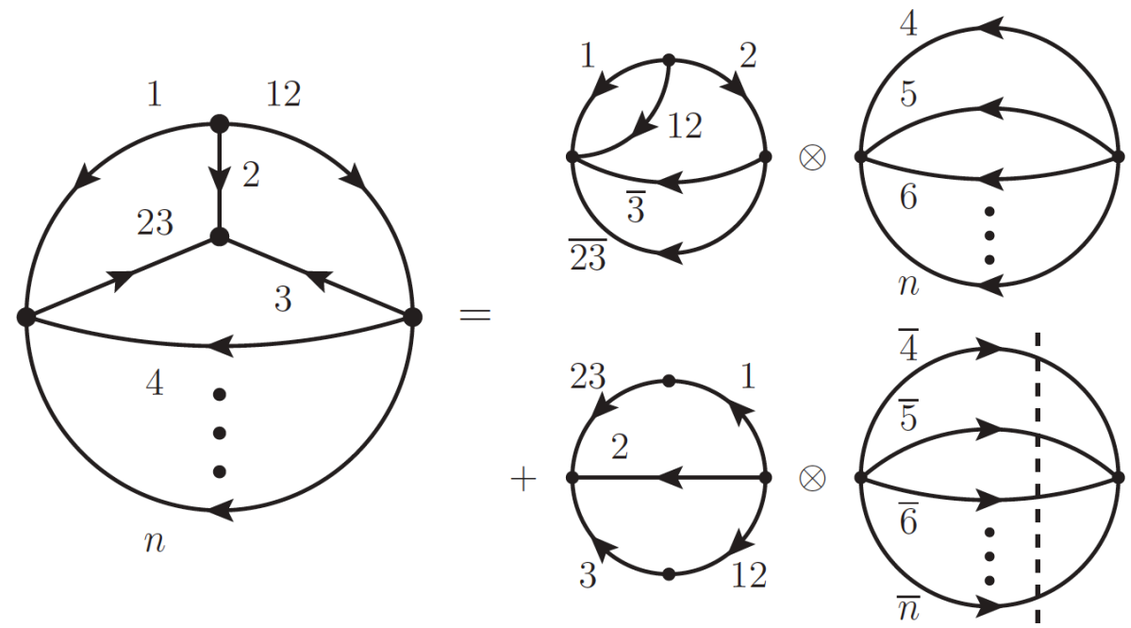
[Aguilera, Driencourt, Hernández, Plenter, Ramírez, Rentería, GR, Sborlini, Torres, Tracz, arXiv:2001:03564]

NMLT AND N2MLT: CASCADE FACTORIZATION



$$\mathscr{A}^{(L)}_{\text{NMLT}}(1,\ldots,n,12)$$
$$= \mathscr{A}^{(2)}_{\text{MLT}}(1,2,12) \otimes \mathscr{A}^{(L-2)}_{\text{MLT}}(3,\ldots,n)$$
$$+\mathscr{A}^{(1)}_{\text{MLT}}(1,2) \otimes \mathscr{A}^{(0)}(12) \otimes \mathscr{A}^{(L-1)}_{\text{MLT}}(\overline{3},\ldots,\overline{n})$$

- causal singularities determined by subtopologies
- direct and efficient application to **physical scattering processes**



$$\mathscr{A}^{(L)}_{\text{N}^2\text{MLT}}(1,\ldots,n,12,23)$$
$$= \mathscr{A}^{(3)}_{\text{NMLT}}(1,12,\overline{3},\overline{23},2) \otimes \mathscr{A}^{(L-3)}_{\text{MLT}}(4,\ldots,n)$$
$$+\mathscr{A}^{(2)}_{\text{MLT}}(1\cup 23,2,3\cup 12) \otimes \mathscr{A}^{(L-2)}_{\text{MLT}}(\overline{4},\ldots,\overline{n})$$

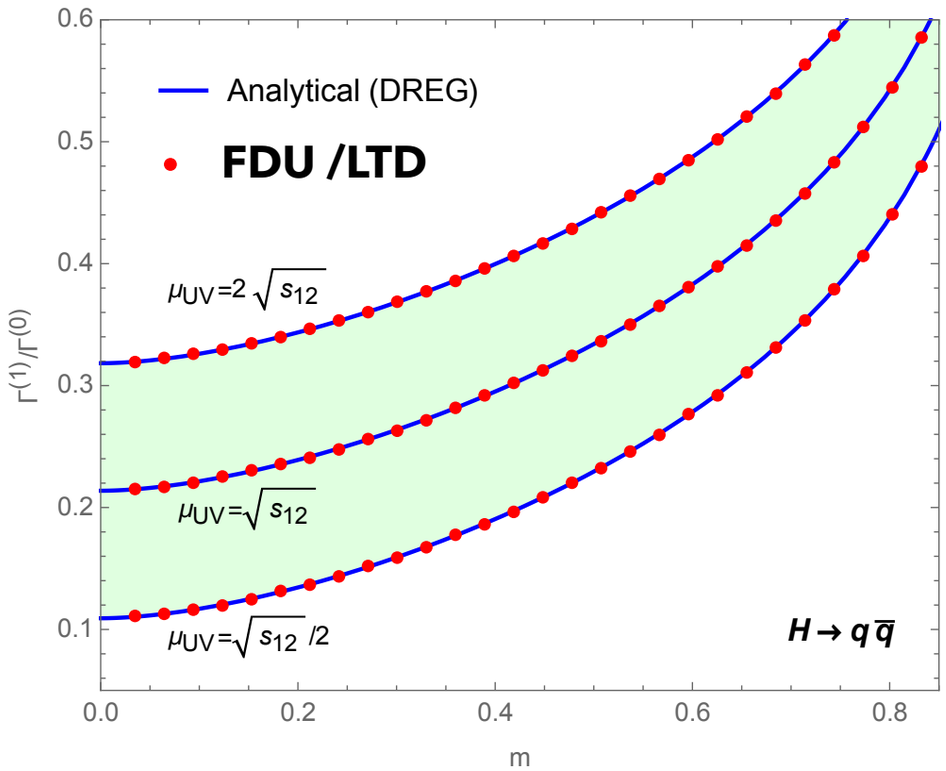
- sufficient up to **three loops**
- **factorization** conjectured to higher orders: advantageous to unveil **formal aspects** of multi-loop scattering amplitudes

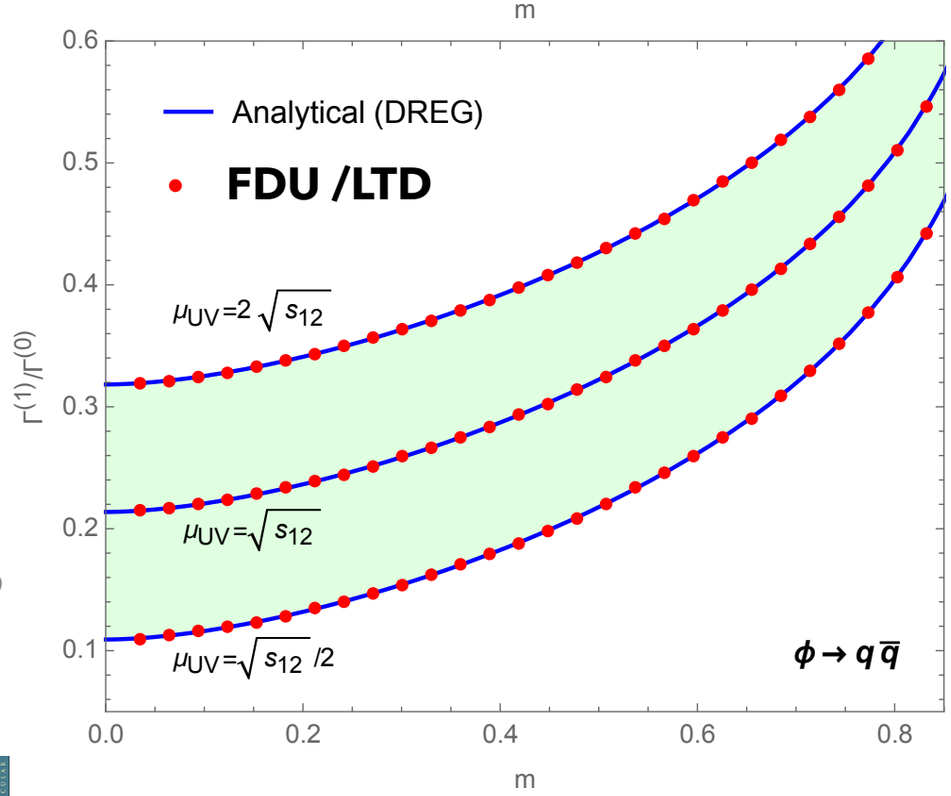
CONCLUSIONS

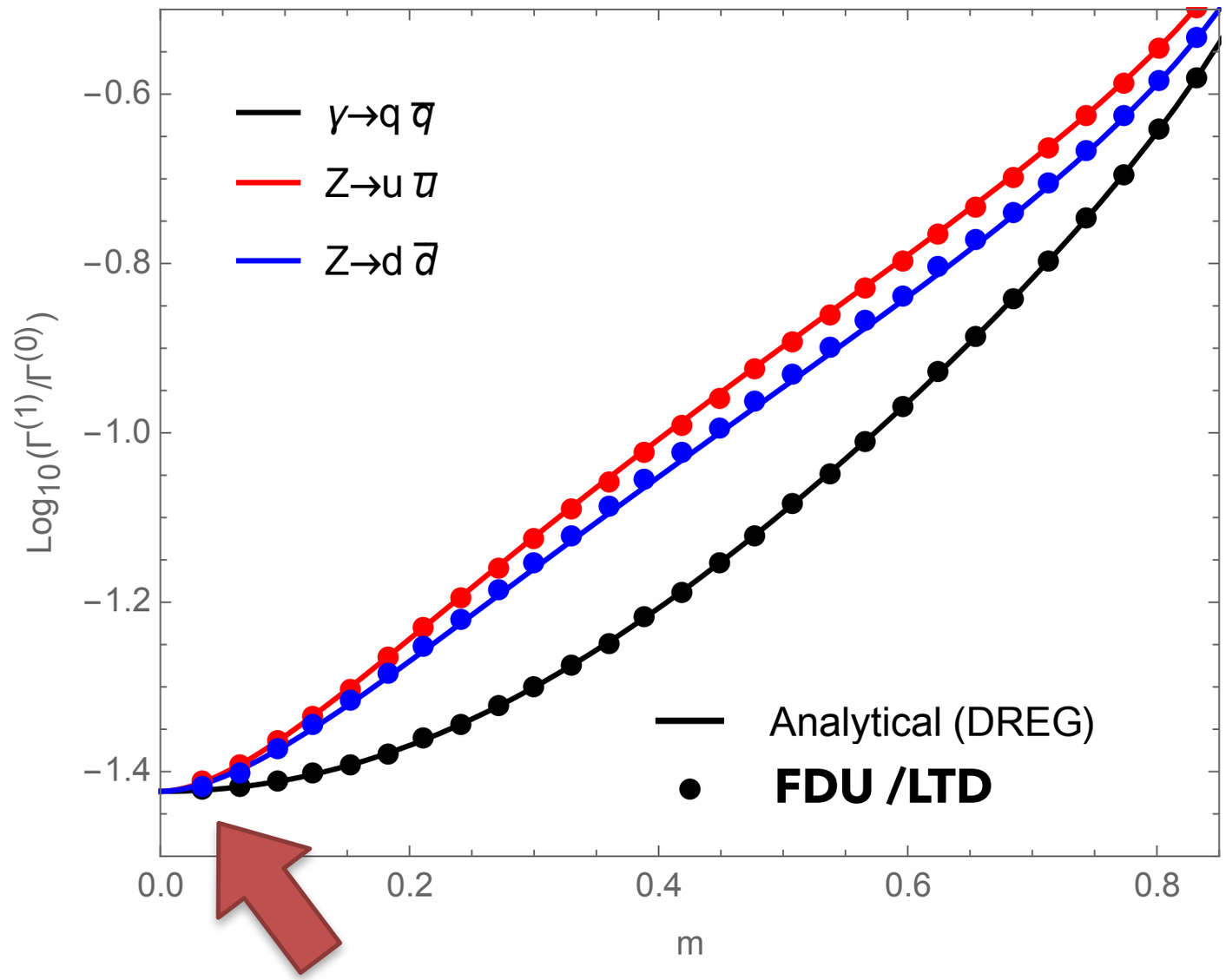
- **Theory is already the limiting factor** in many LHC analysis
- Current techniques insufficient to match the expected accuracy at future colliders (HL-LHC, FCC, HE-LHC, ILC/CLIC, CEPC-SPPC)
- New theoretical developments needed: numeric, semi-numeric or analytic [Duhr's talk]
- **pQFT Plattform ?**
- Back to four space-time dimensions and fully local
- **Loop-tree duality** to all orders and powers reformulated in terms of the original Lorentz invariant dual prescription, new benchmark loop topologies defined
- Direct application to physical scattering processes (numeric), useful to unveil formal aspects of multi-loop scattering amplitudes (analytic)

Benchmark application: $A^* \to q\bar{q}(g)$

[Sborlini, Driencourt-Mangin, GR, JHEP **1610**, 162]

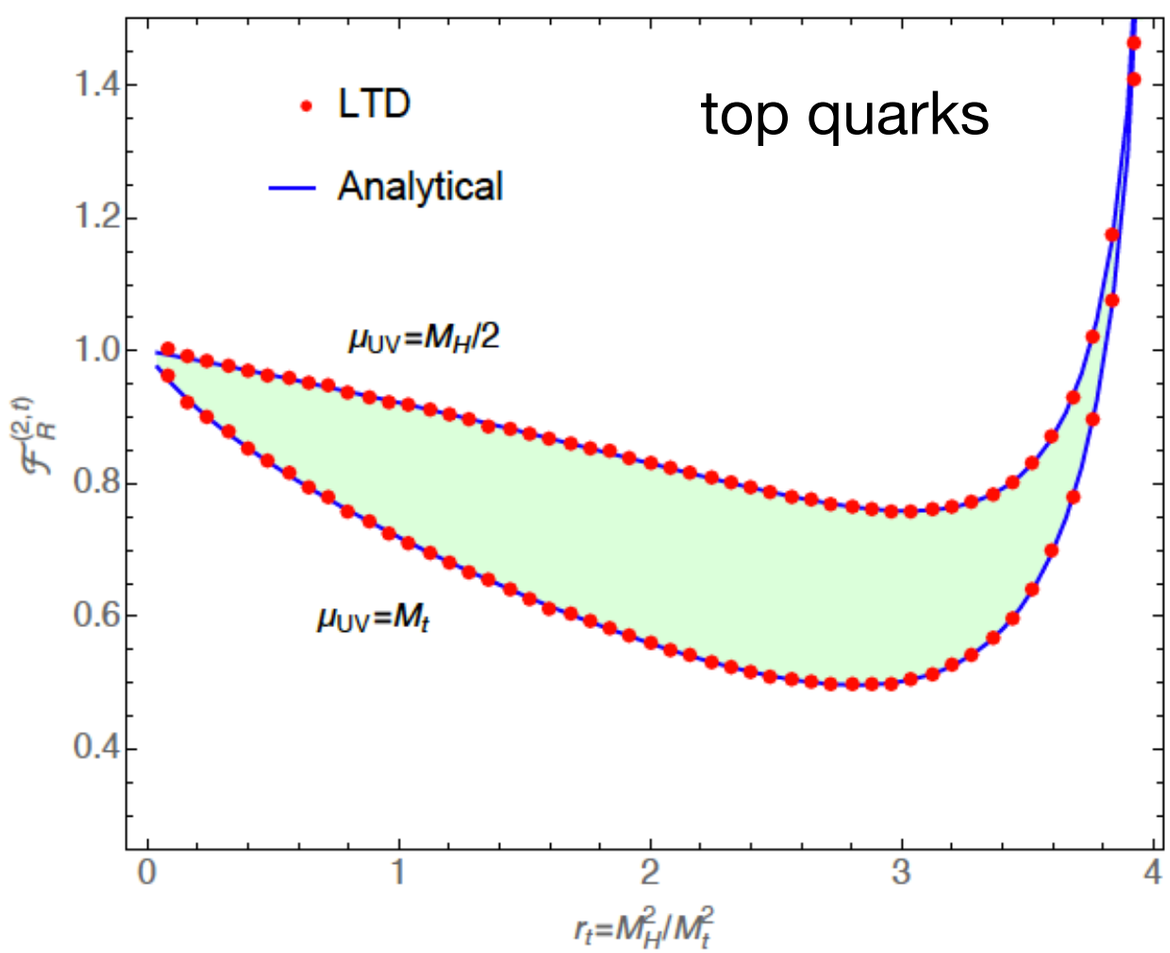


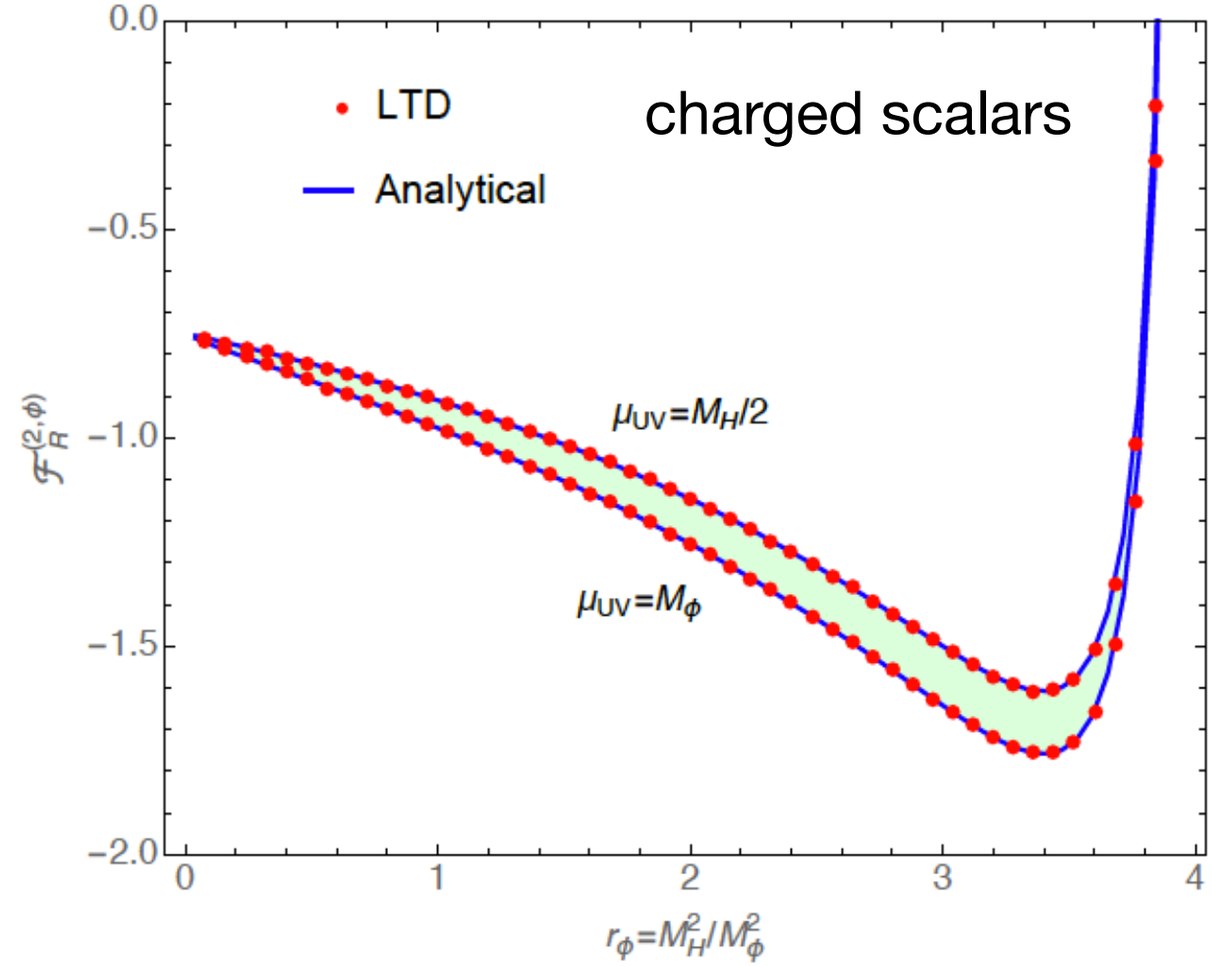




- Excellent agreement with analytic DREG
- Efficient numerical implementation
- **Smooth massless limit**

DUAL AMPLITUDE FOR $H \to \gamma\gamma$ AT TWO-LOOPS





Analytic expressions from Aglietti, Bonciani, Degrassi, Vicini, JHEP **0701** (2007) 021