Numerical approach to multiloop calculations: overview

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THE $N^3$LO ERA

$N^3$LO HADRON-COLLIDER CALCULATIONS VS. TIME

Higgs (TH, app.) C. Anastasiou, C. Duhr, F. Dulat, F. Herzog and B. Mistlberger
Higgs (VBF) F. A. Dreyer and A. Karlberg
Higgs (Diff in TH app.) F. Dulat, B. Mistlberger and A. Pelloni
Higgs, B. Mistlberger
Higgs (Diff in TH app.) F. Dulat, B. Mistlberger and A. Pelloni
HH (VBF) F. A. Dreyer and A. Karlberg
bb→H, Duhr, Dulat, Mistlberger

First calculations

L. Cieri
all options aimed at **attobarn**⁻¹ physics requires to go **far beyond** NNLO for theory

- Even conservative estimates not reachable with current techniques
some (semi-) numerical methods for loop integrals

- numerical solution of differential equations  
  [Caffo, Czyz, Laporta, Remiddi ’98; Czakon, Mitov ‘07 …]

- dispersion relations  
  [Baubberger et al ’94, Bauberger Freitas ‘17 …]

- use Bernstein-Sato-Tkachov theorem  
  [Passarino, Uccirati et al ’01- …]

- numerical evaluation of Mellin-Barnes representations  
  [Czakon ’05; … Dubovyk, Freitas, Gluza, Riemann, Usovitsch ‘16]

- numerical extrapolation  
  [De Doncker, Yuasa, Kato, Fujimoto, Kurihara, Ishikawa, Olagbemi, Shimizu]

- direct numerical integration in momentum space  
  [Soper ’99; Gong, Soper, Nagy ‘09; Weinzierl, Reuschle et al. ’10- …]

- loop-tree duality (4-dim)  
  [Rodrigo, Buchta, Chachamis, Sborlini, Driencourt-Mangin et al. ’08- …]

- sector decomposition  
  [Hepp ’66; Denner, Roth 96; Binoth, GH ’00; …]
<table>
<thead>
<tr>
<th>pro’s and con’s</th>
<th>analytic</th>
<th>numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>pole cancellation</strong></td>
<td>exact</td>
<td>with numerical uncertainty</td>
</tr>
<tr>
<td><strong>fast evaluation</strong></td>
<td>✓ (mostly)</td>
<td>depends</td>
</tr>
<tr>
<td><strong>control of integrable</strong></td>
<td>✓ control of</td>
<td>difficult</td>
</tr>
<tr>
<td><strong>singularities</strong></td>
<td>analytic regions</td>
<td></td>
</tr>
<tr>
<td><strong>extension to more scales</strong></td>
<td>difficult</td>
<td>less difficult</td>
</tr>
<tr>
<td><strong>automation</strong></td>
<td>difficult</td>
<td>less difficult</td>
</tr>
</tbody>
</table>

[G. Heinrich]
WG1 Meeting: Workstop / Thinkstart 3.0
Paving the way for alternative NNLO strategies

Galileo Galilei Institute
4-6 November 2019
To d, or not to d?

**traditional dimensional schemes**
- 't Hooft / Veltman (HV) '72
- conventional dim. reg. (CDR) '73
- dim. reduction (DRED) '79
- four-dim. helicity (FDH) '92

**reformulations of dimensional schemes**
- six-dim. formalism (SDF) '09
- four-dim. formalism (FDF) '14

**non-dimensional schemes**
- implicit reg. (IREG) '98
- loop regularization (LORE) '03
- four-dim. reg. / ren. (FDR) '12
- four-dim. unsubtraction (FDU) '16

- mathematical consistency
- unitarity, causality (equivalence to MS/BPHZ)
- symmetries (gauge invariance, SUSY, ...)

- computational efficiency (analytical/numerical automation)
Overview of schemes

- DREG and variants (CDR, HV, FDH, DRED)  
  standard, well developed, $Q_S^{[d_s]} = Q_S^{[d]} \oplus Q_S^{[n_e]}$  
- FDF, SFD (4-, 6-dimensional formulation)  
  =DREG/FDH, exploit properties of evanescent quantities  
- Implicit regularization and  
  FDR (Four-dimensional renormalization)  
  stay in 4-dim! regularize by “replacement rule”  
  use constraints/make choices for divergent integrals  
- FDU (4-dimensional unsubtraction)  
  loop-tree duality, cutting rules, combine real+virtual
Schemes and tricks to deal with the IR

Few schemes available at NLO:

- **Slicing**: [Giele, Glover], ...
- **Subtraction**: dipole [Catani, Seymour 9602277], FKS [Frixione et al. 9512328], NS [Nagy, Soper 0308127]

Many schemes available at NNLO:

- **Slicing**: $q_{\perp}$ [Catani, Grazzini 0703012], N-Jettiness [Boughezal et al. 1505.03893, Gaunt et al. 1505.04794]
- **Subtraction**: Antenna [Gehrmann-DeRidder et al. 0505111], ColorfullNNLO [Del Duca et al. 1603.08927], Nested soft-collinear [Caola et al. 1702.01352], Geometric IR subtraction [Herzog 1804.07949], $\epsilon$-prescription [Frixione, Grazzini 0411399], Sector decomposition [Bonoth et al. 0402265, Anastasiou et al. 0311311], residue subtraction [Czakon 1005.0274]
- **New strategies**: Unsubtraction [Sborlini et al. 1608.01584], FDR [Pittau 1208.5457]

→ Many options, but still there is room for improvement according to the five criteria rule [Melnikov, talk@Amplitude2019]

A good subtraction scheme should be

1. physically transparent
2. general (scaleable)
3. local
4. analytic
5. efficient

¿?
✓
✓
numeric ?

🏆 🎉 🥇
The difficulties to reach higher orders arise because we have defined Quantum Field Theory not in the optimal way.
QFT = Quantum Mechanics + space-time

- Loops encode quantum fluctuations at infinite energy (zero distance): SM/BSM extrapolated at energies $\gg M_{\text{Plank}}$

- QED/QCD massless gauge bosons/quarks: quantum state with $N$ partons $\neq$ quantum state with zero energy emission (infinite distance) of extra partons

- Partons can be emitted in exactly the same direction (zero distance)
NEW DIRECTIONS IN PQFT

to d                not to d
not to local               to local
UV and IR
FINITE HELICITY AMPLITUDES

[Diriencourt, GR, Sborlini, Torres, 1911.11125]

LOCAL UV RENORMALIZATION

- **UV finite** helicity amplitudes, but unintegrated amplitudes locally singular
- \( \mathcal{A}_R^{(L)} = \mathcal{A}^{(L)} - \mathcal{A}^{(L)}_{\text{UV}} \bigg|_{d=4} \quad \mathcal{A}^{(L)}_{\text{UV}} \bigg|_{d} = 0 \)
- Subtract not only logarithmic UV singularities, but also linear and quadratic
- Disentangle the UV from the IR behaviour in scaleless integrals (e.g. self-energies)
LOCAL UV RENORMALISATION

- Expand propagators and numerators around a UV propagator [Reuschle et al., similar to FDR in the UV - Pittau’s talk]

\[ G_F(q_{UV}) = \frac{1}{q_{UV}^2 - \mu_{UV}^2 + i0} \quad \{ \ell_j^2 \mid \ell_j \cdot k_i \} \rightarrow \{ \lambda^2 q_{UV}^2 + (1 - \lambda^2) \mu_{UV}^2 \mid \lambda q_{UV} \cdot k_i \} \]

- and adjust subleading terms, \( c_{UV} \), to subtract only the pole (\( \overline{MS} \) scheme), or to define any other renormalisation scheme. For the scalar two point function

\[ I_{cnt}^{UV} = \int \frac{1}{\ell \left( q_{UV}^2 - \mu_{UV}^2 + i0 \right)^2} \left( 1 + c_{UV} \frac{\mu_{UV}^2}{q_{UV}^2 - \mu_{UV}^2 + i0} \right) \]

- dual representation needs to deal with multiple poles [Bierenbaum et al.]

\[ I_{cnt}^{UV} = \int \frac{\tilde{\delta}(q_{UV})}{\ell \left( q_{UV,0}^{(+)2} \right)^2} \left( 1 - \frac{3 c_{UV} \mu_{UV}^2}{4 \left( q_{UV,0}^{(+)2} \right)^2} \right) q_{UV,0}^{(+)} = \sqrt{q_{UV}^2 + \mu_{UV}^2 - i0} \]

- Integration on the UV on-shell hyperboloid: loop three-momentum unconstrained, but loop contributions suppressed for loop energies larger than \( \mu_{UV} \)
LOCAL UV RENORMALISATION: MULTiloop

\[
\begin{cases}
|\ell_1| \to \infty , & |\ell_2| \\text{fixed} , \\
|\ell_2| \\text{fixed} , & |\ell_2| \to \infty , \\
|\ell_1| \to \infty . & |\ell_2| \to \infty .
\end{cases}
\]

- Multiple UV limit

\[
\text{UV}^2 : \{ \ell_j^2 \mid \ell_j \cdot \ell_k \mid \ell_j \cdot k_i \}
\]
\[
\to \{ \lambda^2 q_{j,\text{UV}}^2 + (1 - \lambda^2) \mu^2_{\text{UV}} \mid \lambda^2 q_{j,\text{UV}} \cdot q_{k,\text{UV}} + (1 - \lambda^2) \mu^2_{\text{UV}}/2 \mid \lambda q_{j,\text{UV}} \cdot k_i \}
\]

- Most subtle steep the adjustment of the subleading terms, \( d_{\text{UV}2} \), to be in agreement with e.g. the \( \overline{MS} \) scheme

\[
\left( \mathcal{A}^{(L)}_1,_{\text{UV}} - \mathcal{A}^{(L)}_2,_{\text{UV}} \right)_{\text{UV}^2} - d_{\text{UV}2} \mu^4_{\text{UV}} \int_{\ell_1, \ell_2} \left( G_F(q_{1,\text{UV}}) \right)^3 \left( G_F(q_{2,\text{UV}}) \right)^3
\]
**THE LOOP–TREE DUALITY (LTD)**

**Feynman Propagator** $+i0$:
Positive frequencies are propagated forward in time, and negative backward.

$$G_{F}(q_{i}) = \frac{1}{q_{i}^{2} - m_{i}^{2} + i0}$$

**Cauchy residue theorem**
In the loop energy complex plane

Selects residues with definite **positive energy and negative imaginary part** (indeed in any other coordinate system)

[Catani et al. JHEP 0809, 065]
THE LOOP–TREE DUALITY (LTD)

One-loop amplitudes in any relativistic, local and unitary QFT represented as a linear combination of $N$ single-cut phase-space/dual amplitudes | non-disjoint trees (at higher orders: number of cuts equal to the number of loops)

\[
\int_{\ell_1} \mathcal{N}(\ell_1) \prod G_F(q_i) = - \int_{\ell_1} \mathcal{N}(\ell_1) \otimes \sum \tilde{\delta}(q_i) \prod G_D(q_i; q_j)
\]

- $\tilde{\delta}(q_i) = i 2\pi \theta(q_{i,0}) \delta(q_i^2 - m_i^2)$ sets internal line on-shell, positive energy mode

- $G_D(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - i 0 \eta k_{ji}}$ dual propagator \( k_{ji} = q_j - q_i \) \( q_{i,0}^{(+)} = \sqrt{q_i^2 + m_i^2 - i 0} \)

- LTD realised by modifying the customary $+i0$ prescription of the Feynman propagators (only the sign matters), it encodes in a compact way the effect of multiple-cut contributions that appear in the Feynman’s Tree Theorem

- Lorentz invariant, best choice $\eta^\mu = (1,0)$: energy component integrated out, remaining integration in Euclidean space
LTD singular scheme

- non-causal singularities (forward-forward in blue): undergo **dual cancellations** among dual pairs
- causal singularities (forward-backward in orange): bounded to a **compact region**, which is of the size of the **hard scale**, collapse to a finite segment for **infrared singularities** (→ FDU)
- Numerical integration in the **Euclidean** space of the loop three-momenta, CPU/GPU time do not scale significantly with the number of legs
Integrable singularities / causal and anomalous thresholds through contour deformation of the loop three-momentum

\[ \ell \rightarrow \ell' = \ell + i\kappa \]

such that it matches the +i0 prescription

\[ q_{i,0}^{(+)} = \sqrt{-\kappa^2 + 2i\kappa \cdot q_i + q_i^2 + m_i^2} - i0 \quad \kappa \cdot q_i < 0 \]

e.g.

\[ \kappa = \sum_{i,j \in \text{group}} \lambda_{ij} \left( \frac{q_i}{\sqrt{q_i^2}} + \frac{q_j}{\sqrt{q_j^2}} \right) \exp \left( -\frac{G_D^{-2}(q_i; q_j)}{2\sigma_{ij}^2} \right) \]

Figure 9: Energy-scan of a scalar pentagon. The red curve is done with LoopTools and the blue points are obtained with the LTD method.
Table 5: Tensor hexagons involving numerators of rank one to three.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Tensor Hexagon</th>
<th>Real Part</th>
<th>Imaginary Part</th>
<th>Time[s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>P20</td>
<td>1 SecDec LTD</td>
<td>$-1.21585(12) \times 10^{-15}$</td>
<td>$-1.21552(354) \times 10^{-15}$</td>
<td>36</td>
</tr>
<tr>
<td>P21</td>
<td>3 SecDec LTD</td>
<td>$4.46117(37) \times 10^{-9}$</td>
<td>$4.461369(3) \times 10^{-9}$</td>
<td>5498</td>
</tr>
<tr>
<td>P22</td>
<td>1 SecDec LTD</td>
<td>$1.01359(23) \times 10^{-15}$</td>
<td>$+i , 2.68657(26) \times 10^{-15}$</td>
<td>33</td>
</tr>
<tr>
<td>P23</td>
<td>2 SecDec LTD</td>
<td>$2.45315(24) \times 10^{-12}$</td>
<td>$-i , 2.06087(20) \times 10^{-12}$</td>
<td>337</td>
</tr>
<tr>
<td>P24</td>
<td>3 SecDec LTD</td>
<td>$-2.07531(19) \times 10^{-6}$</td>
<td>$+i , 6.97158(56) \times 10^{-7}$</td>
<td>14280</td>
</tr>
</tbody>
</table>

Table 1: Scalar and tensor decagon with all internal masses different.

<table>
<thead>
<tr>
<th>Propagator</th>
<th>Real Part</th>
<th>Imaginary Part</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.530(4) \times 10^{-14}$</td>
<td>$+ i , 8.514(1) \times 10^{-14}$</td>
</tr>
<tr>
<td>$\ell.p_3 \times \ell.p_5$</td>
<td>$8.08(4) \times 10^{-15}$</td>
<td>$+ i , 6.144(5) \times 10^{-13}$</td>
</tr>
</tbody>
</table>
IR subtracted [Anastasiou, Sterman, JHEP 1907, 056]
After the first LTD round the position of the poles in the complex plane is momentum dependent:

1. Use a **general identity** to transform into Feynman propagators the dual propagators that enter the successive LTD rounds [Bierenbaum et al., 2010]
   - First **full two-loop** calculation ($H \to \gamma\gamma$) with local UV renormalization [Driencourt et al., 2019]
   - **Analytic proof** of the dual cancellation of unphysical (non-causal) singularities, causal and anomalous thresholds as well as infrared in a compact region ($\to$ FDU) [Aguilera et al., 2019]

2. **Average** over all possible momentum flows [Runkel et al., 2019]: cumbersome symmetry factors

3. Keep track of the position of the poles and close the Cauchy contour either from **above or from below** to cancel that dependence [Capatti et al., 2019]
   - **Numerical test** of dual cancellations
**LTD TO ALL ORDERS AND POWERS**

- **Multi-loop scattering amplitude:** \( n \) sets of momenta that depend on \( L \) loop momenta or a linear combination

\[
\mathcal{A}_N^{(L)}(1,\ldots,n) = \int \mathcal{N}({\ell}_i)_L, \{p_j\}_N) G_F(1,\ldots,n) = \prod_{i \in \cup \ldots \cup n} (G_F(q_i))^{a_i}
\]

- The **dual function** involving two sets that depend on the same loop momentum: momenta in the set \( t \) remain off-shell

\[
G_D(s; t) = -2\pi i \sum_{i_s \in s} \text{Res} \left( G_F(s,t), \text{Im}(\eta q_{i_s}) < 0 \right)
\]

- **Cauchy contour** always from **below** the real axis

- Valid for arbitrary powers and **Lorentz invariant** [Catani et al. JHEP 0809, 065]

- Reverse momenta, if necessary, to keep a coherent momentum flow

\[
t \to \bar{t} \quad (q_{i_t} \to -q_{i_t})
\]

- The **nested residue** involving several sets

\[
G_D(1,\ldots,r; n) = -2\pi i \sum_{i_r \in r} \text{Res} \left( G_D(1,\ldots,r - 1; r, n), \text{Im}(\eta q_{i_r}) < 0 \right)
\]
MULTILOOP TOPOLOGIES

MLT
Maximal Loop Topology
single topology at two loops

NMLT
Next-to-maximal Loop Topology
starting at three loops

N2MLT
Next-to-next-to-maximal Loop Topology

arbitrary number of external legs attached to each line
OPENING TO NON-DISJOINT TREES + CAUSALITY

MAXIMAL LOOP TOPOLOGY

\[ \mathcal{A}^{(L)}_{\text{MLT}}(1, \ldots, n) = \int \ell_1, \ldots, \ell_L \sum_{i=1}^{n} \mathcal{A}^{(L)} D(1, \ldots, i-1, i+1, \ldots, n; i) \]

- extremely simple and symmetric LTD representation, proven by induction and directly independent of the position of the poles in the complex plane

- causal singularities when on-shell momenta get aligned [Aguilera et al. JHEP 1912, 163]

\[ \mathcal{A}^{(L)} D(\bar{2}, \ldots, \bar{n}; 1) \rightarrow \mathcal{A}^{(L)} D(\bar{1}, \bar{2}, \ldots, \bar{n}) \]

- non-causal singularities (unphysical) entangled among dual pairs, they cancel

\[ \mathcal{A}^{(L)} D(\bar{2}, \bar{3}, \ldots, \bar{n}; 1) + \mathcal{A}^{(L)} D(1, \bar{3}, \ldots, \bar{n}; 2) \rightarrow \mathcal{A}^{(L)} D(1, \bar{2}, \bar{3}, \ldots, \bar{n}) - \mathcal{A}^{(L)} D(1, \bar{2}, \bar{3}, \ldots, \bar{n}) \]
OPENING TO NON-DISJOINT TREES + CAUSALITY

NMLT AND N2MLT: CASCADE FACTORIZATION

\[ \mathcal{A}_{\text{NMLT}}^{(L)}(1, \ldots, n, 12) = \mathcal{A}_{\text{MLT}}^{(2)}(1, 2, 12) \otimes \mathcal{A}_{\text{MLT}}^{(L-2)}(3, \ldots, n) + \mathcal{A}_{\text{MLT}}^{(1)}(1, 2) \otimes \mathcal{A}_{\text{MLT}}^{(0)}(12) \otimes \mathcal{A}_{\text{MLT}}^{(L-1)}(3, \ldots, \bar{n}) \]

\[ \mathcal{A}_{\text{N2MLT}}^{(L)}(1, \ldots, n, 12, 23) = \mathcal{A}_{\text{NMLT}}^{(3)}(1, 12, 23, 2) \otimes \mathcal{A}_{\text{MLT}}^{(L-3)}(4, \ldots, n) + \mathcal{A}_{\text{MLT}}^{(2)}(1 \cup 23, 2, 3 \cup 12) \otimes \mathcal{A}_{\text{MLT}}^{(L-2)}(4, \ldots, \bar{n}) \]

- causal singularities determined by subtopologies
- direct and efficient application to physical scattering processes
- sufficient up to three loops
- factorization conjectured to higher orders: advantageous to unveil formal aspects of multi-loop scattering amplitudes

CONCLUSIONS

- **Theory is already the limiting factor** in many LHC analysis
- Current techniques insufficient to match the expected accuracy at future colliders (HL-LHC, FCC, HE-LHC, ILC/CLIC, CEPC-SPPC)
- New theoretical developments needed: numeric, semi-numeric or analytic [Duhr’s talk]
- **pQFT Plattform** ?
- Back to four space-time dimensions and fully local
- **Loop-tree duality** to all orders and powers reformulated in terms of the original Lorentz invariant dual prescription, new benchmark loop topologies defined
- Direct application to physical scattering processes (numeric), useful to unveil formal aspects of multi-loop scattering amplitudes (analytic)
Benchmark application: $A^* \rightarrow q\bar{q}(g)$

- Excellent agreement with analytic DREG
- Efficient numerical implementation
- Smooth massless limit

[Sborlini, Driencourt-Mangin, GR, JHEP 1610, 162]
DUAL AMPLITUDE FOR $H \rightarrow \gamma\gamma$ AT TWO-LOOPS

Analytic expressions from Aglietti, Bonciani, Degrassi, Vicini, JHEP 0701 (2007) 021