Lattice calculations for strange processes

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LEVERHULME TRUST

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"Direct CP violation and the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decay from the Standard Model,"

R.Abbott,T.Blum, P.A.Boyle, M.Bruno, N.H.Christ, D.Hoying, C.Jung, **C.Kelly**, C.Lehner, R.D.Mawhinney, D.J.Murphy, C.T.S, A. Soni, M.Tomii and T.Wang, arXiv:2004.09440 [hep-lat].

The release of this paper allows me to tell a coherent story of RBC-UKQCD's long-standing project on $K \to \pi\pi$ decays.

Outline of talk

1. Directly computing $K \to \pi\pi$ decay amplitudes
2. Evaluation of A_2
3. Evaluation of A_0
4. Conclusions and Outlook

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- $K \to \pi\pi$ decays are a very important class of processes for standard model phenomenology with a long and noble history.
  - It is in these decays that both indirect and direct CP-violation was discovered.
- Bose Symmetry $\Rightarrow$ the two-pion state has isospin 0 or 2.

$$_{I=2}\langle \pi\pi | H_W | K^0 \rangle = A_2\, e^{i\delta_2}\,, \qquad _{I=0}\langle \pi\pi | H_W | K^0 \rangle = A_0\, e^{i\delta_0}\,.$$

- Among the very interesting issues are the origin of the $\Delta I = 1/2$ rule ($\mathrm{Re}\,A_0/\mathrm{Re}\,A_2 \simeq 22.5$) and an understanding of the experimental value of ε'/ε, the parameter which was the first experimental evidence of direct CP-violation.

- CP-violating experimental amplitudes:

$$\eta_{+-} = \frac{\langle \pi^+\pi^- | H_W | K_L \rangle}{\langle \pi^+\pi^- | H_W | K_S \rangle} = \epsilon + \epsilon'$$

$$\eta_{00} = \frac{\langle \pi^0\pi^0 | H_W | K_L \rangle}{\langle \pi^0\pi^0 | H_W | K_S \rangle} = \epsilon - 2\epsilon'$$

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = \frac{1}{6}\left(1 - \frac{|\eta_{00}|^2}{|\eta_{+-}|^2}\right)$$

- Theoretically (without isospin breaking corrections),

$$\epsilon' = \frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}} \left(\frac{\mathrm{Im}\,A_2}{\mathrm{Re}\,A_2} - \frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}\right)$$

where $\omega = \mathrm{Re}\,A_2/\mathrm{Re}A_0 \simeq 1/22$.

- Indirect CP-violation: $|\epsilon| = (2.228 \pm 0.011) \times 10^{-3}$
- Direct CP-violation: Re $(\epsilon'/\epsilon) = (16.6 \pm 2.3) \times 10^{-4}$

- The effective $\Delta S = 1$ Hamiltonian can be written in the standard form:

$$H_W = \frac{G_F}{\sqrt{2}}\, V_{us}^* V_{ud} \sum_{i=1}^{10} \left\{z_i(\mu) + \tau\, y_i(\mu)\right\} Q_i(\mu)\,,$$

  where

- G_F and V_{ij} are the Fermi Constant and CKM matrix elements respectively;
- τ is the ratio of CKM matrix elements

$$\tau = -\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \simeq \big(1.558(65) + 0.663(33)i\big) \times 10^{-3};$$

- $Q_i(\mu)$ are four-quark operators defined at the renormalisation scale μ with Wilson Coefficients $z_i(\mu)$ and $y_i(\mu)$.
- Role of lattice computations is to evaluate the hadronic matrix elements $\langle \pi\pi | Q_i(\mu) | K \rangle$.

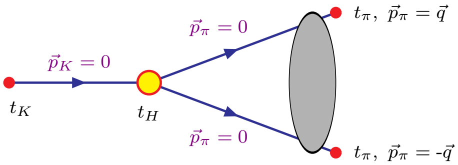
Introduction to $K \to \pi\pi$ decays (cont.)

$$H_W = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{10} \{z_i(\mu) + \tau\, y_i(\mu)\}\, Q_i(\mu)$$

Schematic structure of the calculation:

$$\begin{aligned} A_I &= F \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{10} \sum_{j,k=1}^{7} \{z_i(\mu) + \tau\, y_i(\mu)\}\, Z_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}\, Z_{jk}^{\mathrm{Latt}\to\mathrm{RI}} \langle(\pi\pi)_I | Q_k^{\mathrm{Latt}} | K\rangle \\ &= F \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{10} \sum_{j=1}^{7} \{z_i(\mu) + \tau\, y_i(\mu)\}\, Z_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}} \langle(\pi\pi)_I | Q_j^{\mathrm{RI}} | K\rangle \end{aligned}$$

- F is the Lellouch-Lüscher factor, necessary because the computations are performed in a finite-volume.
- RI is a “Regularisation Independent" renormalisation scheme which can be defined non-perturbatively (not $\overline{\mathrm{MS}}$).
- Lattice computations provide $\langle(\pi\pi)_I | Q_j^{\mathrm{RI}} | K\rangle$ and F.
- The Wilson coefficients z_i, y_i and the matching matrix $Z_{ij}^{\mathrm{RI}\to\overline{\mathrm{MS}}}$, necessarily calculated in perturbation theory.

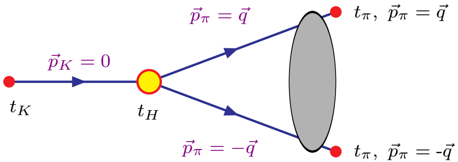


- $K \to \pi\pi$ correlation function is dominated by the lightest state, i.e. the state with two-pions at rest (or the vacuum for $I = 0$). Maiani and Testa, PL B245 (1990) 585

$$C(t_\pi) = A + B_1 e^{-2m_\pi t_\pi} + B_2 e^{-2E_\pi t_\pi} + \cdots$$

  - Solution 1: Study an excited state. Lellouch and Lüscher, hep-lat/0003023
  - Solution 2: Introduce suitable boundary conditions such that the $\pi\pi$ ground state is $|\pi(\vec{q})\pi(-\vec{q})\rangle$. RBC-UKQCD, C.h.Kim hep-lat/0311003
    N.Christ, C.Kelly, D.Zhang, arXiv:1908.08640

For B-decays, with so many intermediate states below threshold, this is the main obstacle to producing reliable calculations.



- Imagine now that we chosen the boundary conditions so that the ground state is $|\pi(\vec{q})\pi(-\vec{q})\rangle$.
  - In a finite volume each component of $\vec{q}$ is quantised, with allowed values separated by $2\pi/L$.
  - Thus in order to obtain the physical value of $|\vec{q}|$ the volume must be chosen appropriately.
  - Moreover, the s-wave, $I=0$ and $I=2$ channels are attractive and repulsive respectively and so the two cases must be studied on lattices of different volumes.

- For A_2, there is no vacuum subtraction and we can use the Wigner-Eckart theorem to write

$$\underbrace{\langle(\pi\pi)^{I=2}_{I_3=1}\,|}_{\frac{1}{\sqrt{2}}(\langle\pi^+\pi^0|+\langle\pi^0\pi^+|)}\quad Q^{\Delta I=3/2}_{\Delta I_3=1/2,i}\,|\,K^+\rangle=\frac{3}{2}\underbrace{\langle(\pi\pi)^{I=2}_{I_3=2}\,|}_{\langle\pi^+\pi^+|}Q^{\Delta I=3/2}_{\Delta I_3=3/2,i}\,|\,K^+\rangle\,,$$

  and impose anti-periodic conditions on the d-quark in one or more directions.

- If we impose the anti-periodic boundary conditions in all 3 directions then the ground state is

$$\left|\pi\left(\frac{\pi}{L},\frac{\pi}{L},\frac{\pi}{L}\right)\,\pi\left(\frac{-\pi}{L},\frac{-\pi}{L},\frac{-\pi}{L}\right)\right\rangle.$$

- With an appropriate choice of L and the number of directions, we can arrange that $E_{\pi\pi}=m_K$.
- Isospin breaking by the boundary conditions is harmless here.

CTS & G.Villadoro, hep-lat/0411033

- The amplitude A_2 is considerably simpler to evaluate that A_0.
- Our first results for A_2 at physical kinematics were obtained at a single, rather coarse, value of the lattice spacing ($a \simeq 0.14$ fm). Estimated discretization errors at 15%. arXiv:1111.1699, arXiv:1206.5142
- Our latest results were obtained on two new ensembles, 48^3 with $a \simeq 0.11$ fm and 64^3 with $a \simeq 0.084$ fm so that we can make a continuum extrapolation:

$$\begin{aligned} \mathrm{Re}(A_2) &= 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8}\ \mathrm{GeV}. \\ \mathrm{Im}(A_2) &= -6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13}\ \mathrm{GeV}\,. \end{aligned}$$

arXiv:1502.00263

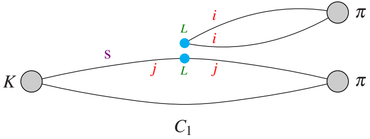
- The experimentally measured value is $\mathrm{Re}(A_2) = 1.479(4) \times 10^{-8}$ GeV.
- Although the precision can still be significantly improved (partly by perturbative calculations), the calculation of A_2 at physical kinematics can now be considered as standard.
- We are not currently working towards improving this result.

RBC-UKQCD Collaboration, arXiv:1212.1474

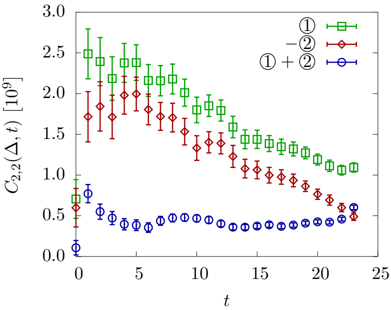
- Re A_2 is dominated by a simple operator:

$$O^{3/2}_{(27,1)} = (\bar{s}^i d^i)_L \left\{(\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L\right\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_L$$

and two diagrams:



- Re A_2 is proportional to $C_1 + C_2$.
- The contribution to Re A_0 from Q_2 is proportional to $2C_1 - C_2$ and that from Q_1 is proportional to $C_1 - 2C_2$ with the same overall sign.
- Colour counting might suggest that $C_2 \simeq \frac{1}{3}C_1$.
- We find instead that $C_2 \approx -C_1$ so that A_2 is significantly suppressed!
- The strong suppression of Re A_2 is a major factor in the $\Delta I = 1/2$ rule.



Physical Kinematics

- Notation ⓘ $\equiv C_i,\ i = 1, 2.$

3. Evaluation of A_0 and ϵ'/ϵ

- In 2015 RBC-UKQCD published our first result for ϵ'/ϵ computed at physical quark masses and kinematics, albeit still with large relative errors:

  Z.Bai et al. (RBC-UKQCD), arXiv:1505.07863

$$\left.\frac{\epsilon'}{\epsilon}\right|_{\text{RBC-UKQCD}} = (1.38 \pm 5.15 \pm 4.59) \times 10^{-4}$$

  to be compared with

$$\left.\frac{\epsilon'}{\epsilon}\right|_{\text{Exp}} = (16.6 \pm 2.3) \times 10^{-4}\,.$$

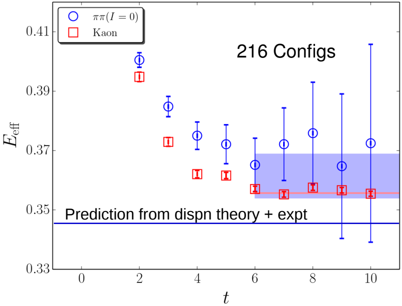
  - Is this 2.1σ deviation real? $\Rightarrow$ must reduce the uncertainties.
  - The matrix elements themselves are calculated with a smaller relative error.
- This is by far the most complicated project that I have ever been involved with.
- Puzzle: For the $I = 0$ s-wave $\pi\pi$ phase shift we obtained $\delta_0 = (23.8 \pm 4.9 \pm 2.2)^\circ$, to be compared with the dispersive results of about 34°. G.Colangelo et al.

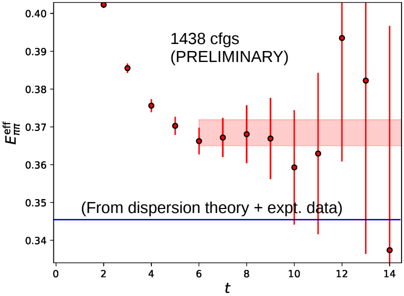
2015

- $32^3 \times 64$ ensemble (Möbius DWF and Iwasaki + DSDR gauge action)
- $a^{-1} = 1.3784(68)$GeV, $L = 4.53$ fm.
- G-parity boundary conditions in 3-directions
- 216 configurations
- Almost physical kinematics: ($m_\pi = 143.1(2.0)$ MeV, $m_K = 490.6(2.2)$ MeV, $E_{\pi\pi} = 498(11)$ MeV).

Extension and Improvement in 2020

- Increase the statistics: 216 $\rightarrow$ 1438 configurations.
  - Reduce the statistical error;
  - Improved statistics allows for an in-depth study of the systematics.
- Use an expanded set of operators to create the $\pi\pi$ state. (741 configurations)
- Improve the non-perturbative renormalisation, including step-scaling to match at a higher energy.
- Significantly improve the analysis techniques. C.Kelly and T.Wang, arXiv:1911.04582





- Increasing the statistics from 216 to 1438 configurations, the $\pi\pi$ correlation function is still well described by a single $\pi\pi$ state.
  - It does not solve the δ_0 puzzle however:

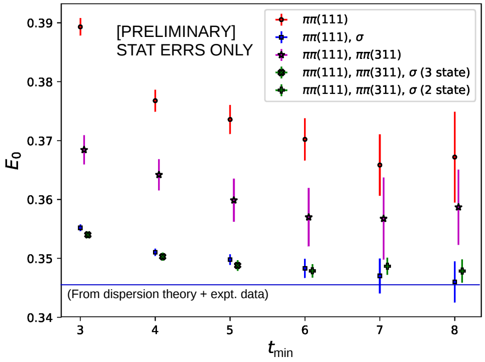
$$\delta_0 = (23.8 \pm 4.9 \pm 2.2)^\circ \quad \rightarrow \quad \delta_0 = (19.1 \pm 2.5 \pm 1.2)^\circ \qquad (\chi^2/\text{dof} = 1.6)$$

Adding more $\pi\pi$ interpolating operators

- The δ_0-puzzle has been resolved by adding more interpolating operators for the $\pi\pi$ states.
  Originally we only had a single $\pi\pi$ operator with each pion being given a momentum $\pm(1,1,1)\pi/L$ (with total momentum $\vec{0}$).
- In particular the inclusion of a σ-like two-quark operator $(\bar{u}u + \bar{d}d)$ has exposed a second state, e.g. for $t_f - t_i = 5$

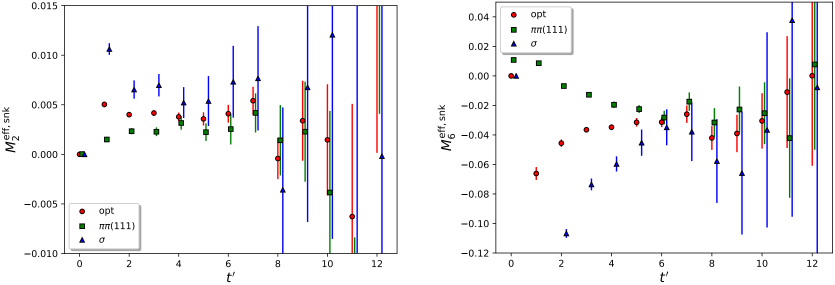
$$\det \begin{pmatrix} \langle \pi\pi(t_f)\pi\pi(t_i)\rangle & \langle \pi\pi(t_f)\sigma(t_i)\rangle \\ \langle \sigma(t_f)\pi\pi(t_i)\rangle & \langle \sigma(t_f)\sigma(t_i)\rangle \end{pmatrix} = 0.439(50) \neq 0$$

- We have also included a third operator giving each pion a larger momentum $\pm(3,1,1)\pi/L$.
- At present we have only analysed 741 configurations with the additional operators. Remainder will be done in the future.

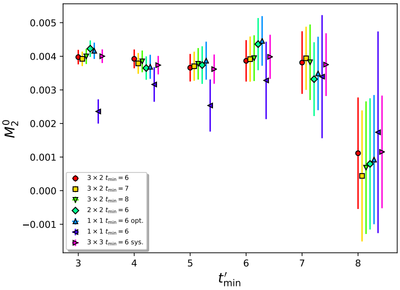


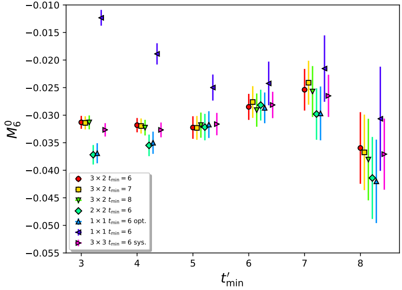
- $\delta_0 = (32.3 \pm 1.0 \pm 1.8)^\circ$ from a fit in the range $t = 5$ - 15 (statistical error only).
  - The fit from dispersion theory at this value of $E_{\pi\pi}$ and m_π is about 35.9°.
- The $\pi\pi(3,1,1)$ operator turns out not to be very important.

Multiple operators and $K \to \pi\pi$ matrix elements



- It is convenient to visualise the data by taking “optimal" linear combinations of operators which best projects into the ground state.





- Examples of results of fits with different numbers of operators, states, $t_{\rm min}=(t_H-t_K)_{\rm min}$ and $t'_{\rm min}=(t_{\pi\pi}-t_H)_{\rm min}$.
  - Adopt uniform fit with $t'_{\rm min}=5$ which is stable for all operators.
  - Evidence that excited state errors significantly underestimated in 2015.

arXiv:1505.07863

Description	2015 Error	2020 Error
Operator normalisation	15%	5%[1]
Wilson coefficients	12%	unchanged
Finite lattice spacing	12%	unchanged
Lellouch - Lüscher factor	11%	1.5%[2]
Residual FV corrections	7%	unchanged
Parametric errors	5%	6%[3]
Excited state contamination	5%	negligible[4]
Unphysical kinematics	3%	5%
Total	**27%**	**21%**

- [1] As a result of step scaling from $\mu = 1.53$ GeV $\rightarrow$ 4.00 GeV.
- [2] Better control of $\pi\pi$ system due to additional operators.
- [3] Largest uncertainty is due to $\tau \sim 5\%$.
- [4] Significantly underestimated in 2015.

- For leptonic decays, the precision of lattice calculations is such that $O(1\%)$ isospin breaking corrections (including electromagnetism) are becoming important. (I wish that I had the time to discuss our framework and applications on this.)
- The extension of this framework to $K \to \pi\pi$ decays is considerably more complicated.
  - Some first steps, towards including electromagnetism in $K \to \pi\pi$ decays we taken by N.Christ and X.Feng. arXiv:1711.09339
- At the current stage of precision we are not concerned with including $O(1\%)$ corrections.
  - However, because of the $\Delta I = \frac{1}{2}$ rule, the corrections are expected to be amplified.

- Recently a detailed updated study of isospin corrections was presented in the framework of ChPT and the large N_C approximation.

  V.Cirigliano, H.Gisbert, A.Pich, A.Rodriguez-Sanchez, arXiv:1911.01359

  The authors write the formula for ϵ' in the form:

$$\epsilon' = \frac{i\omega_+ e^{i(\delta_2-\delta_0)}}{\sqrt{2}} \left[\frac{\mathrm{Im}\,A_2^{\mathrm{ewp}}}{\mathrm{Re}\,A_2} - \frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}\left(1-\hat{\Omega}_{\mathrm{eff}}\right)\right],$$

  where $\omega_+ = \mathrm{Re}\,A_2^+/\mathrm{Re}\,A_0$ and A_2^+ is A_2 obtained from the physical decay $K^+ \to \pi^+\pi^0$ at NLO

$$\hat{\Omega}_{\mathrm{eff}} = \left(17.0^{+9.1}_{-9.0}\right) \times 10^{-2}\,.$$

  (At LO the corresponding number is 19.5 ± 3.9.)

- As expected, this is a very significant effect, certainly requiring further investigation.

- A careful discussion of arXiv:1911.01359, and the determination of the LECs at NLO in particular, is beyond the scope of our work and we include the central value as a further 23% systematic error on our result.

- Re $A_0 = 2.99(0.32)(0.59) \times 10^{-7}$ GeV (Experiment: 3.3201(18) GeV)

  Im $A_0 = -6.98(0.62)(1.44) \times 10^{-11}$ GeV .
- Combining our new result for Re A_0 with our 2015 one for ReA_2 we find:

$$\frac{\mathrm{Re}\, A_0}{\mathrm{Re}\, A_2} = 19.9 \pm 2.3 \pm 4.4\,,$$

  in good agreement with the experimental result of 22.45(6).
- Combining our new result for Im A_0 with our 2015 result for Im A_2 and using the experimental results for the real parts, we find

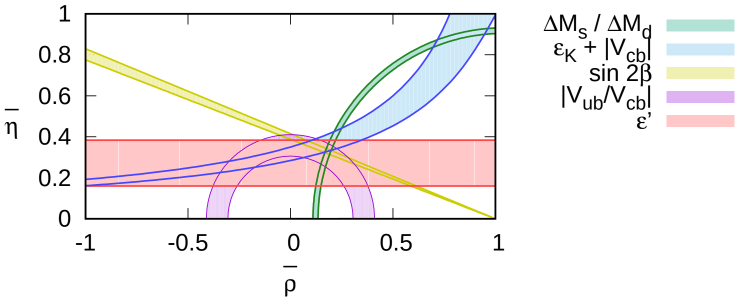
$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = 0.00217(26)(62)(50),$$

  where the third uncertainty is due to isospin breaking effects. This result is consistent with the experimental value of $0.00166(23)$.
  - Note that if, instead of treating the isospin correction from arXiv:1911.01359 as a component of the systematic uncertainty, we were to implement on our result, we would obtain a central value Re$(\epsilon'/\epsilon) = 0.00167$, coincidentally identical to the experimental result.

- We have completed the update on our 2015 lattice determination of A_0 and ϵ'/ϵ with:
  - a 3.2 times increase in statistics;
  - the use of multi-operator techniques in order to essentially remove the systematic error due to excited state contamination;
  - the use of step-scaling to reduce significantly the systematic error in the renormalisation.
- We reproduce the experimental value of ReA_0/ReA_2 demonstrating that, within our uncertainties, QCD is sufficient to solve this decades-old puzzle.
- Our result for Re ϵ'/ϵ is consistent with the experimental value, with an error which is about 3.5 times larger. This quantity remains a promising avenue in which to search for new physics but more precision is required.

- The collaboration intends to perform measurements on two larger lattices with different lattice spacings to the perform continuum limit. This will require the next generation of supercomputers.
- A project is currently underway to perform the $4 \to 3$ flavour matching in the Wilson coefficients non-perturbatively. M.Tomii, arXiv:1901.04107
- We are also working on laying the groundwork for the lattice calculation of isospin breaking and electromagnetic effects. N.Christ and Xu Feng, arXiv:1711.09339
- The collaboration is actively investigating the potential for multi-operator fits to circumvent need for G-parity BCs, allowing for more reuse of ensembles and eigenvectors from other RBC&UKQCD projects. D.Hoying, PoS LATTICE2018 (2019) 064

Effect on Unitarity Triangle



UNIVERSITY OF Southampton

1. A_0 and A_2 amplitudes with unphysical quark masses and with the pions at rest.

   "K to $\pi\pi$ decay amplitudes from lattice QCD,"
   T.Blum, P.A.Boyle, N.H.Christ, N.Garron, E.Goode, T.Izubuchi, C.Lehner, Q.Liu, R.D. Mawhinney, C.T.S, A.Soni, C.Sturm, H.Yin and R. Zhou, Phys. Rev. D **84** (2011) 114503 [arXiv:1106.2714 [hep-lat]].

   "Kaon to two pions decay from lattice QCD, $\Delta I = 1/2$ rule and CP violation"
   Q.Liu, Ph.D. thesis, Columbia University (2010)

2. A_2 at physical kinematics and a single coarse lattice spacing.
   "The $K \to (\pi\pi)_{I=2}$ Decay Amplitude from Lattice QCD,"
   T.Blum, P.A.Boyle, N.H.Christ, N.Garron, E.Goode, T.Izubuchi, C.Jung, C.Kelly, C.Lehner, M.Lightman, Q.Liu, A.T.Lytle, R.D.Mawhinney, C.T.S., A.Soni, and C.Sturm
   Phys. Rev. Lett. **108** (2012) 141601 [arXiv:1111.1699 [hep-lat]],

   "Lattice determination of the $K \to (\pi\pi)_{I=2}$ Decay Amplitude A_2"
   Phys. Rev. D **86** (2012) 074513 [arXiv:1206.5142 [hep-lat]]

   "Emerging understanding of the $\Delta I = 1/2$ Rule from Lattice QCD,"
   P.A. Boyle, N.H. Christ, N. Garron, E.J. Goode, T. Janowski, C. Lehner, Q. Liu, A.T. Lytle, C.T. Sachrajda, A. Soni, and D.Zhang, Phys. Rev. Lett. **110** (2013) 15, 152001 [arXiv:1212.1474 [hep-lat]].

3. A_2 at physical kinematics on two finer lattices $\Rightarrow$ continuum limit taken.
   "$K \to \pi\pi$ $\Delta I = 3/2$ decay amplitude in the continuum limit,"
   T.Blum, P.A.Boyle, N.H.Christ, J.Frison, N.Garron, T.Janowski, C.Jung, C.Kelly, C.Lehner, A.Lytle, R.D.Mawhinney, C.T.S., A.Soni, H.Yin, and D.Zhang
   Phys. Rev. D **91** (2015) 7, 074502 [arXiv:1502.00263 [hep-lat]].

4. A_0 at physical kinematics and a single coarse lattice spacing.
   "Standard-model prediction for direct CP violation in $K \to \pi\pi$ decay,"
   Z.Bai, T.Blum, P.A.Boyle, N.H.Christ, J.Frison, N.Garron, T.Izubuchi, C.Jung, C.Kelly, C.Lehner, R.D.Mawhinney, C.T.S, A. Soni, and D. Zhang,
   Phys. Rev. Lett. **115** (2015) 21, 212001 [arXiv:1505.07863 [hep-lat]].

5. Improved and Updated version of Item 4.
   "Direct CP violation and the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decay from the Standard Model,"
   R.Abbott,T.Blum, P.A.Boyle, M.Bruno, N.H.Christ, D.Hoying, C.Jung, C.Kelly, C.Lehner, R.D.Mawhinney, D.J.Murphy, C.T.S, A. Soni, M.Tomii and T.Wang, arXiv:2004.09440 [hep-lat].