EM corrections to leptonic and semileptonic decays


DIPARTIMENTO DI FISICA

CERN January 31 2020
PLAN OF THE TALK

1) Physics Motivations

2) Introduction: QED corrections to the hadron spectrum and to the hadronic amplitudes

3) Leptonic decays e.g. $\pi^+ \rightarrow \mu^+ \nu_\mu (\gamma) \text{ & heavier}$

4) Real Emissions (RM123 method) and Heavy Flavours

5) Semileptonic decays (only theory so far)

5) Conclusion & Outlook
Physics Motivations: Flavour and New Physics

Flavour phenomenology plays a fundamental role in indirect searches of New Physics (NP):

- looks for deviation from the SM whatever the origin is;

- needs good theoretical control of the SM contribution only.

The path leading to NP@ the TeV scale is much narrower after the results from LHC.
The SM CKM pattern represents the principal part of the flavor structure and of CP violation.

2018 results

\[ \rho = 0.148 \pm 0.013 \quad \eta = 0.348 \pm 0.010 \]

\[ \alpha = (90.9 \pm 2.0)^0 \]
\[ \sin 2\beta = 0.699 \pm 0.016 \]
\[ \beta = (22.3 \pm 0.7)^0 \]
\[ \gamma = (66.8 \pm 2.0)^0 \]
\[ A = 0.826 \pm 0.012 \]
\[ \lambda = 0.22500 \pm 0.00100 \]

Consistence on an over constrained fit of the CKM parameters

CKM matrix is the dominant source of flavour mixing and CP violation.
PROGRESS SINCE 1988

1988

1995

2000

2003

2016
The accuracy of lattice calculations of the hadron spectrum (and hence of the quark masses) and of the decay constants and form factors is such that isospin breaking and em effects cannot be neglected anymore:

\[
\begin{align*}
    f_\pi &= 130.2(0.8) \text{ MeV} \quad \varepsilon = 0.6\% \\
    f_K &= 155.7(0.3) \text{ MeV} \quad \varepsilon = 0.2\% \\
    f_K/f_\pi &= 1.1932(19) \quad \varepsilon = 0.16\% \\
    F_{K\pi}^{(0)} &= 0.9706(27) \quad \varepsilon = 0.3\%
\end{align*}
\]

A remark on useful and useless precision of lattice calculations:

1) \(\varepsilon_K\) and long distance charm contributions

2) \(f_K\) and \(f_\pi\)
B meson real photon emissions
Factorization at leading power in an expansion of the decay amplitude in $\Lambda_{QCD}/E_\gamma$ and $\Lambda_{QCD}/mb$ has been established to all orders in the strong coupling $\alpha_s$. In this approximation, the branching fraction depends only on the leading-twist B-meson light-cone distribution amplitude (LCDA) $\phi_+ (\omega, \mu)$.

More precisely, it is proportional to $1/\lambda_B$, the most important LCDA parameter in exclusive decays, is uncertain by a large factor ranging from 200 MeV favoured by non-leptonic decays to 460 MeV from QCD sum rules.

The radiative leptonic decay has therefore been suggested as a measurement of $\lambda_B$.

![Diagram](image)

**Figure 1.** Leading contribution to $B \to \gamma \ell \nu_\ell$.

For large photon energies the form factors can be written as [9]

$$F_V (E_\gamma) = \frac{e_u f_B m_B}{2 E_\gamma \lambda_B (\mu)} R (E_\gamma, \mu) + \xi (E_\gamma) + \Delta \xi (E_\gamma),$$

$$F_A (E_\gamma) = \frac{e_u f_B m_B}{2 E_\gamma \lambda_B (\mu)} R (E_\gamma, \mu) + \xi (E_\gamma) - \Delta \xi (E_\gamma).$$

(2.7)

The first term is equal in both expressions and represents the leading-power contribution in the heavy-quark expansion (HQE). It originates only from photon emission from the light spectator quark in B meson (Fig. 1). In the above, $f_B$ is the decay constant of B meson, and the quantity $\lambda_B$ is the first inverse moment of the B-meson LCDA,

$$\frac{1}{\lambda_B (\mu)} = \int_0^\infty \frac{d \omega}{\omega} \phi_+ (\omega, \mu).$$

(2.8)
Further applications in decays of heavy neutral B mesons: Virtual corrections (*some questions still open*)

Enhanced electromagnetic correction to the rare $B$-meson decay $B_{s,d} \rightarrow \mu^+\mu^-$

Martin Beneke,¹ Christoph Bobeth,¹,² and Robert Szafron¹

Is this really reabsorbed in the coefficient of $O_9$?
Further applications in decays of heavy neutral $B$ mesons: real corrections (some questions still open)

$B^0_s \to \mu^+\mu^-\gamma$ from $B^0_s \to \mu^+\mu^-$

Francesco Dettori$^a$, Diego Guadagnoli$^b$ and Méril Reboud$^{b,c}$

Figure 3: Dimuon invariant mass distribution from LHCb’s measurement of $B(B^0_s \to \mu^+\mu^-)$ [52] overlayed with the contribution expected from $B^0_s \to \mu^+\mu^-\gamma$ decays (ISR only). Assumes flat efficiency versus $m_{\mu^+\mu^-}$. The line denoted as ‘$B^0_s \to \mu^+\mu^-\gamma$ NP’ refers to the $V - A$ case with $\delta C_0 = -12\% C_0^{SM}$ (see also Fig. 2). The two filled curves are not stacked onto each other.
Particle(s) from weak vertex with momenta $q$

- **FCNC** $Q_b = Q_q$ (need long distance in addition):
  \[
  H^{\text{weak}} \sim O_{9,10} : B_{d,s} \rightarrow \ell^+\ell^-\gamma
  \]
  \[
  F(q^2, k^2) = F(q^2,0)
  \]
  Bobeth’s talk

- **FCNC** $Q_b \neq Q_q$:
  \[
  H^{\text{weak}} \sim V_{ub} \bar{u} \gamma_\mu b_L \ell_L \gamma\mu \nu_L : B_u \rightarrow \ell^+ \nu \gamma
  \]
  \[
  F(m_a^2, k^2) \rightarrow F^*(k^2)
  \]
  Ziegler’s talk

- Flavoured axion or dark photon, scalar DM, ...

- Physics: helicity suppression of $B \rightarrow f_i \bar{f}_j$ relieved in radiative decay!

Roman Zwicky@ Tenerife
Status of Lattice Calculations of $g_A$

**Neutron lifetime and the axial coupling**

\[
\frac{1}{\tau_n} = \frac{G^2_{\mu} |V_{ud}|^2}{2\pi^3} m_e^5 (1 + 3g_A^2)(1 + RC) f_{V,A}
\]

- The neutron lifetime and $g_A$ (neutron decay) are used to probe the limits of the Standard Model.
- We should have a (meaningful) Standard Model prediction for $g_A$ - LQCD (lattice QCD).
- To gain confidence in the application of LQCD to nuclear physics, we must benchmark (calibrate) our calculations against well known quantities of interest, such as $g_A$.
- In order for the theoretical uncertainty on $g_A$ to match the larger uncertainty in the neutron lifetime measurements, we must determine $g_A$ with < 0.2% uncertainty - **is this crazy?**

\[
\tau_n^{\text{beam}} = 888.0(2.0) \text{s}
\]

\[
\tau_n^{\text{bottle}} = 879.4(0.6) \text{s}
\]
nucleon axial coupling from LQCD

- To gain confidence in the application of Lattice QCD to nuclear physics, we must benchmark (calibrate) our calculations against well known quantities of interest.
- $g_A$ was supposed to be a good benchmark calculation for single nucleon structure - but it proved to have significant systematic challenges, preventing results with the precision anticipated.
- FLAG 2019 has included single nucleon quantities in their averaging for the first time.
- Notice one result is significantly more precise than the others.
improving the determination of $g_A$

**Final result**

<table>
<thead>
<tr>
<th>Source</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>statistical</td>
<td>0.81%</td>
</tr>
<tr>
<td>chiral extrapolation</td>
<td>0.31%</td>
</tr>
<tr>
<td>$a \to 0$</td>
<td>0.12%</td>
</tr>
<tr>
<td>$L \to \infty$</td>
<td>0.15%</td>
</tr>
<tr>
<td>isospin</td>
<td>0.03%</td>
</tr>
<tr>
<td>model selection</td>
<td>0.43%</td>
</tr>
<tr>
<td><strong>total</strong></td>
<td><strong>0.99%</strong></td>
</tr>
</tbody>
</table>

$g_A^{QCD} = 1.2711(103)^s(39)^\chi(15)^a(19)^{V}(04)^{I}(55)^M$

- More precise results at the physical pion mass will improve the three largest uncertainties:
  - statistical ($s$), extrapolation ($\chi$) and model selection ($M$)
  - **NOTE**, a12m130 has 2.3% uncertainty
- Following our existing strategy, we anticipate getting to 0.5% by the end of this year
- Getting below (or maybe to 0.5%) will require a 4th lattice spacing as well (~0.06fm)
- Adding a FV study on additional pion mass points will improve the FV uncertainty
- The isospin uncertainty seems unnecessary...
Isospin Symmetry Breaking

In the isospin symmetric lattice world up and down have the same mass and the electric charge is switched off.

1) Isospin is explicitly broken by the up and down mass difference

$$\frac{m_d - m_u}{\Lambda_{QCD}} \sim 0.01$$

2) Electromagnetic interaction

$$\alpha \sim 0.0073$$
Non-compact lattice QED

naively discretised Maxwell action

\[ S[A_\mu] = \frac{1}{4} \sum_{\mu\nu} \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right)^2 \]

pure gauge is free, it can be solved exactly
gauge invariance is preserved

QED_L: \( D(k^2) = 0 \) \( \vec{k} = 0 \)

where \( D(k^2) \) is the photon propagator

subtleties related to QED on a finite volume not discussed here
Masses and matrix elements at lowest order

\[ C_0^{12}(t)|_{t>0} = \sum_{\tilde{x}} \langle 0 | T \{O_2(\tilde{x}, t) \, O_1^+(\tilde{0}, 0) \} | 0 \rangle = \frac{\langle 0 | O_2 | \pi \rangle \langle \pi | O_1^+ | 0 \rangle}{2m_\pi} e^{-m_\pi t} + \ldots = \frac{Z_0^{12}}{2m_\pi} e^{-m_\pi t} + \ldots \]

excited states (exponentially suppressed at large distances)
Masses and matrix elements at $O(\alpha_{\text{em}})$

$O_{1}^{+}(0) \quad O_{2}(x)$

connected

$\bar{d}$

$u$

$\bar{d}$

$u$

$\bar{d}$

$\bar{d}$

$\bar{d}$

$u$

$\bar{d}$

disconnected
\[
C_{\alpha_{em}}^{12}(t) = \frac{(Z_0^{12} + \delta Z^{12})}{2 (m_\pi + \delta m_\pi)} e^{-(m_\pi + \delta m_\pi) t} \\
= C_0^{12}(t) \left(1 - \delta m_\pi t - \frac{\delta m_\pi}{m_\pi} + \delta Z^{12}\right) + O(\delta^2) \\
\delta = O\left(\frac{m_d - m_u}{\Lambda_{QCD}}, \alpha_{em}\right)
\]

\[
\frac{C_{\alpha_{em}}^{12}(t)}{C_0^{12}(t)} \sim -\delta m_\pi t + \text{const.}
\]
QED & Isospin Corrections
To Hadron Masses and Amplitudes

The renormalization of the SU(3)$_c$ × U(1) Lagrangian requires the definition of the renormalised couplings. At first order in $\alpha_{em}$, we only have to renormalise $\alpha_s$ and the quark masses.

Without QED, in the 4-flavour theory, for each value of $\alpha_s$ we can fix four dimensionless quantities related to the quark masses:

$$\frac{(a_0 m_{\pi^0})}{(a_0 M_{\Omega})}, \quad \frac{(a_0 m_{K^0})}{(a_0 M_{\Omega})}, \quad \frac{(a_0 m_{K^+})}{(a_0 M_{\Omega})}, \quad \frac{(a_0 m_{D^0})}{(a_0 M_{\Omega})}$$

and one physical mass to determine the dimensionful scale

$$\alpha_s(a_0) \text{- dimensional trasmutation} \quad a_0 = \frac{(a_0 M_{\Omega})}{M_{\Omega}^{ph}}$$

The same quantities can be used to define the physical scale and the quark masses for $\alpha_{em}=0$. What you define to be the electromagnetic correction to a given quantity remains however convention dependent.
The masses are infrared finite

Finite volume corrections discussed later on

\[ m_{\text{QED}_L}(T, L) = m \left\{ 1 - q^2 \alpha \left[ \frac{k}{2mL} \left( 1 + \frac{2}{mL} \right) - \frac{3\pi}{(mL)^3} \right] \right\} \]

\[ k = 2.83729 \text{ universal} \]
Light Hadron Masses

[Dürr et al (Budapest-Marseille-Wuppertal collaboration (BMWc)), Science 322 (2008) 1224]
Hadron Masses & Isospin Violation

[Borsanyi et al (Budapest-Marseille-Wuppertal collaboration (BMWc)), Science 347 (2015) 1452]

(PDG '14)

Strong + Higgs + Electromagnetism = Experiment
QED (Isospin) Corrections in Hadronic Processes

After the renormalization of the SU(3)$_c \times$ U(1) Lagrangian and quark masses, you still need

1) The renormalization of the operators mediating the physical process of interest (e.g. the Weak effective Hamiltonian). But this is not a novelty;

2) A complex procedure to remove the infrared cutoff because in general the amplitudes, contrary to the masses, are infrared divergent.

A method to solve this problem has been presented. I will recall some important theoretical subtleties before discussing semileptonic decays.
How to solve the problem of the infrared divergences discussed through an explicit example

\[ \pi \to \ell + \nu_\ell + (\gamma) \]


NOTE: Chiral Perturbation Theory is NOT Used
\[ \Gamma(\Delta E) = \int_0^{\Delta E} dE_\gamma \frac{d\Gamma}{dE_\gamma} = \Gamma_0 + \Gamma_1(\Delta E) \]

\[ \Gamma(\Delta E') = \int d\Omega_2 \quad \begin{array}{c}
\text{1) Virtual photons} \\
\text{2) Real photons} \\
\end{array} \]

\[ + \int_{E_{\gamma} \leq \Delta E} d\Omega_3 \quad \begin{array}{c}
\text{Electro-quenched} \\
\end{array} \]

\[ d\Omega_{2,3} = 2 - 3 \text{ body phase - space} \]
The Infrared Problem

\[ \Gamma(\Delta E) = \Gamma_0 + \Gamma_1(\Delta E) \]

**QED Corrections to Hadronic Processes in Lattice QCD,**
N.Carrasco, V.Lubicz, G.Martinelli, C.T.Sachrajda, N.Tantalo, C.Tarantino and M.Testa,

**Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD,**
V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,

**First Lattice Calculation of the QED Corrections to Leptonic Decay Rates,**
D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula, N.Tantalo and C.Tarantino,

**Light-meson leptonic decay rates in lattice QCD+QED**
**MASTER FORMULA for the rate at $O(\alpha)$**

\[ \Gamma(\Delta E) = \lim_{V \to \infty} \left( \Gamma_0 - \Gamma_0^{pt} \right) + \lim_{V \to \infty} \left( \Gamma_0^{pt} + \Gamma_1(\Delta E) \right) \]

- the infrared divergences in $\Gamma_0$ and $\Gamma_0^{pt}$ are exactly the same and cancel in the difference
- $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$ is infrared finite since it is a physical, well-defined quantity. \textit{F. Bloch, A. Nordsieck Phys. Rev. 52 (1937) T.D. Lee, M. Nauenberg Phys. Rev. 133 (1964)}
- the infrared divergences in $\Delta \Gamma_0(L) = \Gamma_0 - \Gamma_0^{pt}$ and $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$ cancel separately hence they can be regulated with different infrared cutoff
- $\Gamma_0$ and $\Gamma_0^{pt}$ are also ultraviolet finite

We now discuss the two terms, $\Delta \Gamma_0(L)$ and $\Gamma(\Delta E)$.
Leptonic decays at $O(\alpha)$ – The first term of the Master Formula

\[ \Delta \Gamma_0(L) = \Gamma_0 - \Gamma_0^{pt} \]

- Each of the two terms is U.V. finite but contains $\log(M_W)$
- Infrared divergences cancel in the difference
  
  \[ \pi^+ \rightarrow u d \rightarrow \ell^+ \nu_\ell \]
  \[ \pi^+ \rightarrow u d \rightarrow \ell^+ \nu_\ell \]
  \[ \pi^+ \rightarrow u d \rightarrow \ell^+ \nu_\ell \]

At this order we can take the difference of the amplitudes

*Can be computed as discussed in arXiv:1303.4896, Phys. Rev. D87(2013) RM123 method NOT by including the electromagnetic field in the action*
Certainly these diagrams are not simply a generalization of the evaluation of $f_\pi$; they are also infrared divergent.

We have to isolate the finite volume ground state (necessity of a mass gap – Minkowski $\leftrightarrow$ Euclidean continuation. J. Gasser and G.R.S. Zarnauskas, Phys. Lett. B 693 (2010) 122)

Finite volume effects are expected of $O(1/L)$ after the cancellation of the infrared divergences.
Calculation of the `nasty’ diagrams in a lattice simulation
Continuation from Minkowski to Euclidean

The starting point is the Minkowski Green function
\[
\int d^4x_1 d^4x_2 < 0 \left| T(j_\mu(x_1) J_\nu^\mu(0)) \right| \pi > iD_F(x_1 - x_2) \{ \bar{u}(p_\nu) \gamma^\nu (1 - \gamma^5)(iS_F(x_2)) \gamma^\mu v(p_\ell) \} e^{ip_\ell \cdot x_2}
\]
from which we can compute the on-shell amplitude
\[
\bar{u}_\alpha(p_\nu)(\bar{M}_1)_{\alpha \beta} v_\beta(p_\ell) = -i \lim_{k_0 \to m_\pi} (k_0^2 - m_\pi^2) \int d^4x_1 d^4x_2 d^4x e^{-ik^0 y^0} < 0 \left| T(j_\mu(x_1) J_\nu^\mu(0)\pi(x)) \right| 0 > \times iD_F(x_1 - x_2) \{ \bar{u}(p_\nu) \gamma^\nu (1 - \gamma^5)(iS_F(x_2)) \gamma^\mu v(p_\ell) \} e^{ip_\ell \cdot x_2}
\]
which in the Euclidean simulation becomes
\[
\bar{C}_1(t)_{\alpha \beta} = \int d^3x d^4x_1 d^4x_2 \left< 0 \left| T \{ J_\nu^\nu(0) j_\mu(x_1) \phi^\dagger(x, t) \} \right| 0 \right> \Delta(x_1 - x_2) \times (\gamma_\nu (1 - \gamma^5)S(x_2) \gamma^\mu)_{\alpha \beta} \epsilon_{E_\ell t_2} e^{-i p_\ell \cdot x_2}
\]
IMPORTANT slides:
the continuation from Minkowski to Euclidean

we need to ensure that the \( t_2 \) integration up to \( \infty \) converges in spite of the factor \( e^{E_1 t_2} \) where \( E_1 = \sqrt{m_1^2 + p_1^2} \) is the energy of the outgoing charged lepton.

1) Momentum conservation:
since we integrate over \( x_2 \)
\[ p_1 = k_1 + k_\gamma \]

2) The integrations over the energies \( k_{41} \) and \( k_{4\gamma} \) lead to the exponential factor \( e^{-(\omega_1 + \omega_\gamma - E_1) t_2} \) where
\[ \omega_1 = \sqrt{m_1^2 + k_1^2} \]
\[ \omega_\gamma = \sqrt{m_\gamma^2 + k_\gamma^2} \]
and \( m_\gamma \) is the mass of the photon introduced as an infra-red cut-off.
3) … but \((\omega_1 + \omega_\gamma) \geq \sqrt{(m_1 + m_\gamma)^2 + p_1^2} > E_1 = \sqrt{m_1^2 + p_1^2}\)

thus the argument of the exponent \(e^{-(\omega_1 + \omega_\gamma - E_1)t_2}\) is negative for every term appearing in the sum over the intermediate states and the integral over \(t_2\) converges

4) note that the integration over \(t_2\) is also convergent if we set \(m_\gamma = 0\) but remove photon zero mode in finite volume. In this case \((\omega_1 + \omega_\gamma) > E_1 + [1 - (p_1/E_1)](k_\gamma)_{\text{min}}\)

- necessity of a mass gap
- absence of a lighter intermediate state
Leptonic decays at $O(\alpha)$ – Perturbative Calculation of

$$\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$$

U.V. & Infrared finite but contains $\log(M_W) & \log(\Delta E)$

$$\Delta E << \Lambda_{\text{QCD}}$$

$$\Gamma(\Delta E) = \Gamma_{\text{tree}}^0 \times \left( 1 + \frac{\alpha}{4\pi} \left\{ 3 \log\left( \frac{m_{\pi}^2}{M_W^2} \right) + \log\left( r^2_\ell \right) - 4 \log(r_E^2) + \frac{2 - 10r^2_\ell}{1 - r^2_\ell} \log(r^2_\ell) \right\} 
- 2 \frac{1 + r^2_\ell}{1 - r^2_\ell} \log(r^2_E) \log(r^2_\ell) 
- 4 \frac{1 + r^2_\ell}{1 - r^2_\ell} \text{Li}_2(1 - r^2_\ell) - 3 
+ \left[ \frac{3 + r^2_E - 6r^2_\ell + 4r_E(-1 + r^2_\ell)}{(1 - r^2_\ell)^2} \log(1 - r_E) + \frac{r_E(4 - r_E - 4r^2_\ell)}{(1 - r^2_\ell)^2} \log(r^2_\ell) \right. 
\left. - \frac{r_E(-22 + 3r_E + 28r^2_\ell)}{2(1 - r^2_\ell)^2} - 4 \frac{1 + r^2_\ell}{1 - r^2_\ell} \text{Li}_2(r_E) \right] \right\}$$

We think that this is a new result;

$$\Gamma(\Delta E_1) \ T.Kinoshita, \ PRL \ 2 \ (1959) \ 477$$

$$r_E = \frac{2\Delta E}{m_{\pi}} \quad r_\ell = \frac{m_\ell}{m_{\pi}}$$
With our definition of the photon propagator we use the finite volume as infrared regulator. Finite volume effects have the form

$$\Gamma_0^{pt}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell) \log(m_\pi L) + \frac{C_1(r_\ell)}{m_\pi L} + O\left(\frac{1}{(m_\pi L)^2}\right)$$

where \( r_1 = m_1 / m_\pi \) and \( m_1 \) is the mass of the final lepton. The terms above are universal, namely independent of the structure of the pion. Thus in \( \Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt} \) the infrared divergence is cancelled and the finite volume corrections start at \( O(1/L^2) \).


\( M_\pi \sim 320 \text{ MeV} \)

![Graph showing finite volume corrections starting at \( O(1/L^2) \)]
with this method, our result for
\[
\Gamma_P(E) = \Gamma_P^0 \left\{ 1 + \delta R_P(E) \right\},
\]
\[
\delta R_{K\pi} = \delta R_K(E_{K}^\text{max}) - \delta R_\pi(E_\pi^\text{max})
\]
is the following
\[
\delta R_{K\pi} = -0.0122(10)^{st}(2)^{\text{turn}}(8)^{X}(5)^{L}(4)^{a}(\ldots)\text{O}\text{N}\text{E}\text{D}
\]
\[
\frac{|V_{us}|}{|V_{ud}|} = 0.23134 \text{ (24)}_{\text{exp}} \text{ (30)}_{\text{th}} = 0.23134 \text{ (38)}
\]

this can (remember the caveat concerning the definition of QCD) be compared with the result currently quoted by the PDG and obtained in v.cirigliano and h.neufeld, PLB 700 (2011)
\[
\delta R_{K\pi} = -0.0112(21)
\]

\[
\delta R_{K\pi}^{\text{phys}} - \delta R_{\pi}^{\text{phys}} = -0.0126 (14)
\]
\[
\frac{|V_{us}|}{|V_{ud}|} = 0.22538 \text{ (24)}_{\text{exp}} \text{ (30)}_{\text{th}} = 0.22538 \text{ (38)}
\]
MASTER FORMULA for the rate at $O(\alpha)$

\[
\Gamma(\Delta E) = \lim_{V \to \infty} (\Gamma_0 - \Gamma_0^{pt}) + \lim_{V \to \infty} (\Gamma_0^{pt} + \Gamma_1(\Delta E))
\]

\[
\Gamma_0 + \Gamma_1 = \lim_{V \to \infty} (\Gamma_0 - \Gamma_0^{pt}) + \lim_{V \to \infty} (\Gamma_0^{pt} + \Gamma_1^{pt}) + \lim_{V \to \infty} (\Gamma_1 - \Gamma_1^{pt})
\]

1) Each of the three terms is infrared finite
2) The last term is important for heavy meson phenomenology where $\chi$PT cannot be applied

We now discuss the non-perturbative determination of $\Gamma_1(\Delta E)$
Real photons

- the leptonic part factorizes;
- the photon is not really there, we just give a pinch that carries away some momentum at the vertex where the (conserved) electromagnetic current is inserted;
- therefore finite volume effects are exponentially suppressed.

- can be computed in perturbation theory;
- it has no relation with the non perturbative structure of the hadron.
The relevant hadronic amplitude (T-product)

\[ H^r_W(k, p) = \varepsilon^r_\mu(k) H^{\alpha, \mu}_W(k, p) = \varepsilon^r_\mu(k) \int d^4y \ e^{i k \cdot y} \langle 0 | T \{ j^\alpha_W(0) j^\mu_{em}(y) \} | P(p) \rangle \]

- \( j^\mu_{em}(y) \) electromagnetic current
- \( j^\alpha_W(0) \) weak current
- \( P(p) \) pseudoscalar meson with momentum \( p \)
- \( \varepsilon^r_\mu(k) \) polarisation vector of the photon with momentum \( k \)
Decomposition of the amplitude in form-factors

\[ H_W^{\alpha \mu}(k, p) = H_1 \left[ k^2 g^{\alpha \mu} - k^\alpha k^\mu \right] + H_2 \left[ (p \cdot k - k^2) k^\mu - k^2 (p - k)^\mu \right] (p - k)^\alpha \]

\[ - i \frac{F_V}{m_P} \varepsilon^{\mu \alpha \gamma \beta} k_\gamma p_\beta + \frac{F_A}{m_P} \left[ (p \cdot k - k^2) g^{\alpha \mu} - k^\alpha (p - k)^\mu \right] \]

\[ + f_P \left[ g^{\alpha \mu} + \frac{(2p - k)^\mu (p - k)^\alpha}{2p \cdot k - k^2} \right] \]

The last term is dictated from the Ward id

\[ k_\mu H_W^{\alpha \mu}(k, p) = i \langle 0 | j_W^\alpha | P(p) \rangle = f_P p^\alpha \]

it is the same of a point-like scalar particle

- 4 independent form-factors;
- the form factors only depend on \( k^2 e p.k \);
- we will only discuss the real photon case, \( k^2 = 0 \)  \( \varepsilon^r.k = 0 \);
- in the future it may be interesting to consider

\[ \Gamma^{\text{exp}}(K \rightarrow \ell \nu_\ell \ell^+ \ell^-) \]
Amplitude: decomposition in scalar form-factors

\[ H_{W}^{\alpha \mu}(k, p) = H_{1} \left[ k^{2} g^{\alpha \mu} - k^{\alpha} k^{\mu} \right] + H_{2} \left[ (p \cdot k - k^{2}) k^{\mu} - k^{2} (p - k)^{\mu} \right] (p - k)^{\alpha} \]

\[ - \frac{F_{V}}{m_{P}} \epsilon_{\mu \alpha \gamma \beta} k_{\gamma} p_{\beta} + \frac{F_{A}}{m_{P}} \left[ (p \cdot k - k^{2}) g^{\alpha \mu} - k^{\alpha} (p - k)^{\mu} \right] \]

\[ + f_{P} \left[ g^{\alpha \mu} + \frac{(2p - k)^{\mu} (p - k)^{\alpha}}{2p \cdot k - k^{2}} \right] \]

Real Photon Emission

\[ H_{W}^{\alpha \mu}(k, p) = - \frac{F_{V}}{m_{P}} \epsilon_{\mu \alpha \gamma \beta} k_{\gamma} p_{\beta} + \frac{F_{A}}{m_{P}} \left( \frac{f_{P}}{p \cdot k} - \frac{p}{m_{P}} \right) (p \cdot k) g^{\alpha \mu} - k^{\alpha} p^{\mu} \]

\[ + \frac{f_{P}}{p \cdot k} p^{\alpha} p^{\mu} \]

1. different tensors can be separated by using suitable projectors, we then have to disentangle \( F_{A} \) from \( f_{P} \);
2. the point-like term is related to the infrared divergence in \( 1/p.k \) that has to cancel the corresponding divergence of the virtual correction in the rate;

It is \( F_{A} \) (together with \( F_{V} \)) the relevant structure dependent quantity to be determined.
Lattice Calculation

**Relevant Correlator**

\[ C_{\alpha \gamma}^{\alpha \gamma}(t, p, k) = \epsilon_{\mu}^T(k) \int d^4 y \, d^3 x \, e^{i y E_j - i k \cdot y + i p \cdot x} \, T\langle 0| j_{W}^{\alpha}(t) j_{em}^{\mu}(y) P(0, x)|0\rangle \]

- **Euclidean extraction of the photon momentum \( k \)**
- **source of the pseudoscalar meson with momentum \( p \)**
- **suitable projector on the different form factors**

\[ E_{\gamma} = |k| \]

The convergence of the integral over \( t_y \) is ensured by the safe analytic continuation from Minkowsky to Euclidean, namely by the absence of intermediate states lighter than the pseudoscalar meson. The physical form factors can be extracted directly from the Euclidean correlation functions

\[ \sum_n \int_{-\infty}^{0} dt \, e^{t E_{\gamma}} \langle 0| j_{W}^{\alpha}(0)|n\rangle \langle n| e^{t H} j_{em}^{\mu}(0, k) e^{-t H} |P\rangle \rightarrow e^{t(E_{\gamma} + E_n - E_P)} \]

\[ E_{\gamma} + E_n > E_P \]
Define
\[ R^{\alpha r}(t) = \frac{2E}{e^{-t(E-E_{\gamma})}} \langle P | P | 0 \rangle C^{\alpha r}_{\gamma W}(t, p, k) \]

then
\[ \lim_{t \to \infty} R^{\alpha r}(t) = H^{\alpha r}_{\gamma W}(p, k) \] 
\( (t \leq T/2) \)

depending on the projector we can isolate the vector or axial vector form factors.

example, taking \( \bar{p} = (0, 0, |\bar{p}|) \) and \( \bar{k} = (0, 0, E_{\gamma}) \) we can choose for the basis of polarisation vectors \( \varepsilon^\mu_{1,2} = (0, \mp |\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0) \) so that

\[ R^{jr}_A(t, \bar{p}, \bar{k}) \to \frac{\varepsilon^j_r m_p}{2} x_\gamma \left[ F_A(x_\gamma) + \frac{2f_p}{m_p x_\gamma} \right], \quad R^{jr}_V(t, \bar{p}, \bar{k}) \to \frac{i \left( E_{\gamma} \bar{\varepsilon}_r \times \bar{p} - E \bar{\varepsilon}_r \times \bar{k} \right)^j}{m_p} F_V(x_\gamma) \]

where the subscripts \( A, V \) on \( R^{jr} \) label the components of the weak current.
A subtle point due to discretization errors

A possible strategy would be the subtraction of the term proportional to \( f_P \) since this coefficient is fixed by the Ward identity of the electromagnetic current.

\[
G_A(x_\gamma) = x_\gamma F_A(x_\gamma) + \frac{2f_P}{m_P}
\]

\[
A55.32 \quad a = 0.0815 \text{ fm}
\]

\[
\frac{[H^r_A(k,p) - \epsilon_0 f_P]}{x_\gamma m_P^2} + \frac{2}{x_\gamma m_P^2} P_{\text{non}}^2 \left( \frac{x_\gamma^2}{r_0^2} \right)
\]

Apparent divergence at small \( x_\gamma \)

Non-perturbative subtraction of \( O(a^2) \) terms
Numerical Results
(all the results in the following are preliminary)

- $2+1+1$ twisted mass fermions
- Three different lattice spacings
- 3-4 different pion masses for each lattice spacing
- Pion masses down to $\sim 230$ MeV
- 100 different momentum configurations
for the plateaus we find rather good signals (we are optimizing them)
Axial-vector channel for the Kaon and $D_s$

$$R_A = \frac{F_A}{m_P} + \frac{f_P}{p \cdot k}$$

$m_{ud} = 11.7$ MeV

$a = 0.0619$ fm

For light mesons the SD form-factors are small.

$$x_\gamma = \frac{2p \cdot k}{m_P^2}$$
we may compare with $\chi$PT that at $O(p^4)$ gives without any momentum dependence

$F_A = \frac{8m_K}{f_\pi} (L_9^r + L_{10}^r)$

with an improved analysis we are able to extract the momentum dependence too

$F_A \propto C_0 + C_1 x_\gamma$

Kaon $F_A$ form factor

$X_\gamma$
we cannot compare this case with $\chi$PT

large discretization effects observed but the extrapolation to the continuum is safe

we are able to extract the momentum dependence of $F_A$
Fv form factor for the Kaon

with an improved analysis we are able to extract the momentum dependence too

Similar results were found for the other mesons π, D and Ds
Similar results were found for the D with slightly larger uncertainties; (not all the physical range in $x_\gamma$ is covered)
Semileptonic Decays: similarities and differences

Semileptonic decays are characterized by a vector and a scalar form factor, $f_+(q^2)$ and $f_0(q^2)$ respectively, corresponding to the hadronic quantity $f_\pi (f_K, f_D, ...)$ in the case of leptonic decays. $q=p_K-p_\pi=p_l-p_\nu$

$$\langle \pi^+(p_\pi)|\bar{u}\gamma_\mu s|\bar{K}^0(p_k)\rangle = f_+(q^2) \left[ (p_k + p_\pi)_\mu - \frac{(m_K^2 - m_\pi^2)}{q^2} q_\mu \right] + f_0(q^2) \frac{(m_K^2 - m_\pi^2)}{q^2} q_\mu$$

For our discussion of the analytic continuation, of the infrared divergences and of finite volume effects the useful variables are $q^2$ and the invariant mass of the final pion-charged lepton pair $s=(p_\pi+p_l)^2$. In terms of these variables we have

$$\frac{d^2\Gamma}{dq^2 ds} = G_F^2 |V_{us}|^2 \left( a_+(q^2, s)|f_+(q^2)|^2 + a_0(q^2, s)|f_0(q^2)|^2 + a_{+0}(q^2, s)f_+(q^2)f_0(q^2) \right)$$

where the kinematical coefficients $a_i(q^2, s)$ can be easily determined
Following the procedure used for leptonic decays we write

\[ \frac{d^2\Gamma}{dq^2 ds} = \lim_{V \to \infty} \left( \frac{d^2\Gamma_0}{dq^2 ds} - \frac{d^2\Gamma_0^{pt}}{dq^2 ds} \right) + \lim_{V \to \infty} \left( \frac{d^2\Gamma_0^{pt}}{dq^2 ds} + \frac{d^2\Gamma_1^{pt}(\Delta E)}{dq^2 ds} \right) \]

where the two terms are now infrared safe. For soft photons with \( \Delta E / \Lambda_{\text{QCD}} << 1 \) we can compute the real emission using the eikonal approximation and the virtual one using a simple effective model with suitable defined form factors (with many caveat however).

The analogy however is not completely true: analytic continuation from Minkowsky and 1/L corrections are different
Semileptonic Decays: New problems in the continuation from Minkowski to Euclidean

A new problem may arise if there are intermediate π-l (or π-l-γ, π-γ, l-γ) states which have a smaller energy than the external π-l system.

$$G(t_H, t_{\pi\ell}) \sim e^{\Delta E \left(t_{\pi\ell} - t_H\right)}$$

$$\Delta E = E^{ext}_{\pi\ell} - E^{int}_{\pi\ell}$$

Remember that in QEDL the photon will always have a non-zero minimum energy
A close inspection, up to hadronic form factors, shows that

\[
G(t_H, t_{\pi\ell}) = \frac{1}{8k\omega'_\pi\omega'_\ell} \left[ \frac{1}{\Delta E} \left( \frac{1}{\Delta \omega_\pi - k} + \frac{1}{\Delta \omega_\ell - k} \right) + \frac{1}{\Delta E} \left( \frac{1}{\Delta \omega_\pi + k} + \frac{1}{\Delta \omega_\ell + k} \right) e^{\Delta E (t_{\pi\ell} - t_H)} \right]
\]

\[
- \frac{1}{(\Delta \omega_\pi - k)(\Delta \omega_\ell + k)} e^{(\Delta \omega_\pi - k) (t_{\pi\ell} - t_H)} - \frac{1}{(\Delta \omega_\pi + k)(\Delta \omega_\ell - k)} e^{(\Delta \omega_\ell - k) (t_{\pi\ell} - t_H)}
\]

\times \text{(hadronic form factor)}

physical term

\[
\omega_{\pi,\ell} = \sqrt{m_{\pi,\ell}^2 + \vec{p}_{\pi,\ell}^2} \quad \omega'_{\pi,\ell} = \sqrt{m_{\pi,\ell}^2 + (\vec{p}_{\pi,\ell} \pm \vec{k})^2} \quad \Delta \omega_{\pi,\ell} = \omega_{\pi,\ell} - \omega'_{\pi,\ell}
\]

note that the last two terms are always well behaved at large \( t_{\pi\ell} - t_H \)

since, as in the leptonic case, \( \Delta \omega_1 - k < 0 \) and \( \Delta \omega_\pi - k < 0 \) when we remove photon zero mode in the finite volume (necessity of a mass gap).

The dangerous term is instead the term proportional to \( \Delta E \)
In general, depending on the volume and the pion-lepton invariant mass $s$, there are unphysical contributions from lighter intermediate $\pi l (\gamma)$ states, which grow exponentially with the temporal integration region and must be subtracted (as in the Lellouch-Luscher formula for $K\pi\pi$ decays). In fact this is a general feature in the calculation of long distance effects.

For semileptonic decays of heavy mesons however, for much of phase space there are too many lighter intermediate states to handle (and above the inelastic threshold). This is analogous to the fact that e.g. $B\pi\pi$ and $B\pi K$ decays amplitudes cannot be calculated whereas $K\pi\pi$ amplitudes can.

We also need to study the FV corrections due to the electromagnetic rescattering.
The relevant parameter is the invariant mass of the $\pi$-$\ell$ pair

$$(m_\mu + m_\pi)^2 \leq M_{\pi\ell}^2 \leq m_K^2$$

which, in turn, is determined by the neutrino momentum [or $q^2 = (p_\pi - p_\ell)^2$]. When the invariant mass $M_{\pi\ell}$ is large $\Delta E$ become positive.

$$- \Delta E = \sqrt{m_\pi^2 + (\vec{p}_\pi + \vec{k})^2} + \sqrt{m_\ell^2 + (\vec{p}_\ell - \vec{k})^2} - \sqrt{m_\pi^2 + \vec{p}_\pi^2} - \sqrt{m_\ell^2 + \vec{p}_\ell^2}$$

for large $M_{\pi\ell}$ max $\Delta E$ corresponds to $\vec{k} = -\vec{p}_\pi$
\[ p_{\nu \mu} = 0 \]
Is the presence of intermediate states with a lower energy really a problem?

Certainly is a complication
But it is not new…..

The energy non-conserving matrix elements with initial or final states having energy $E^{\text{int}}$ can also be calculated to aid in the subtraction of the exponentially growing terms.

- IV reconstruction method to be developed for the SL decays.

- $D$ and $B$ decays: the large number of such terms which need to be subtracted in most of phase space, makes it very difficult to implement the method.

- No such exponentially growing terms are present for leptonic decays.
$\Gamma_0^{pt}$ \textbf{Universality of the logarithmically divergent term and of the $1/L$ correction}

$$\Gamma_0^{pt}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell) \log[m_P L] + \frac{C_1(r_\ell)}{m_P L} + \ldots$$

**Does not depend on the IR regularization or on the internal structure of the hadron**

**Depends on the IR regularization. The regularization dependent part does not depend on the internal structure of the hadron**

Thus $\Delta \Gamma(L) = \Gamma_0 - \Gamma_0^{pt} = \text{Infrared finite, independent of the regularization up to } O(1/L^2)$

BMW, Science 347 (2015) 1452
B. Lucini et al., JHEP 1602 (2016)
**Universality demonstrated via skeleton expansion or effective theory**

\[ \Delta \Gamma(L) = \Gamma_0 - \Gamma_0^{pt} \]

Non-universal terms enter with higher powers of the photon momentum i.e. less singular terms and correspond to higher powers in \(1/L\).

A practical rule summarising the relation between the power of the finite-volume corrections and the leading singularity of the integrand at \(k=0\) is

\[ \Delta \text{Loop} = \int \frac{dk_0}{2\pi} \left( \frac{1}{L^3} \sum_{\bar{k} \neq 0} - \int \frac{d^3k}{(2\pi)^3} \right) \frac{1}{(k^2)^{n/2}} = O \left( \frac{1}{L^{4-n}} \right) \]
Using only gauge invariance (Ward ids) one has

\[ \Gamma^\mu(p, k) = (2p + k)^\mu + 4z_1 p^\mu p \cdot k + 4z_1 \epsilon^2 p^\mu + O(k^2, \epsilon^4, \epsilon^2 k) \]

\[ \Delta(p + k) = \frac{1 - 2z_1 p \cdot k - \epsilon^2 z_1 + O(k^2, \epsilon^4, \epsilon^2 k)}{\epsilon^2 + 2p \cdot k + k^2} \]

from

\[ \Delta_\gamma(k^2) \left\{ \Gamma^\mu(p, k)\Delta(p + k)\Gamma^\mu(p + k, -k) + \frac{1}{2} \Gamma^{\mu\nu}(p, k, -k) \right\} \]

we find the following contribution to the amplitude

\[ \Delta_\gamma(k^2) \left\{ \frac{4m_P^2 + O(k^2)}{(2p \cdot k + k^2)^2} - \frac{8z_1 m_P^2 + O(k)}{2p \cdot k + k^2} \right\} + O \left( \frac{1}{L^2} \right) \]

\[ z_1 \text{ is structure dependent} \]

\[ \text{infrared logarithmic divergent term} \]

\[ \text{non-universal } 1/L \text{ term} \]
Finite-volume corrections - Semileptonic Decays (Cont.)

- We have seen that, as a result of the Ward Identity, we do not need the derivatives of the pion form-factors to obtain the $O(1/L)$ corrections.

- However, we also need to expand the weak-vertex which, in QCD without QED, is a linear combination of two form-factors $f^\pm(q^2)$.

- Off-shell, the $K\pi\ell\bar{\nu}$ weak vertex is a linear combination of two functions $F^\pm(p_\pi^2, p_K^2, 2p_K \cdot p_\pi)$ (which reduce on-shell to the form-factors $f^\pm(q^2)$).

- The WI relates the $K\pi\ell\bar{\nu}$ and $K\pi\ell\bar{\nu}\gamma$ vertices and does lead to a partial, but not complete, cancelation of $O(\frac{1}{L})$ terms.

- The $O(\frac{1}{L})$ corrections are found to depend on $\frac{df^\pm(q^2)}{dq^2}$, as well as on $f^\pm(q^2)$.
  - Such derivative terms are generic (a consequence of the Low theorem); absent only in particularly simple cases, such as leptonic decays.

- These corrections are "universal" in the sense that the coefficients are physical and can be computed in lattice simulations.
  - There are no corrections of the form $\frac{df^\pm}{dm_\pi^2}$ or $\frac{df^\pm}{dm_K^2}$, which would not be physical.
The starting point for our study is the relation:

\[
\frac{d^2 \Gamma}{dq^2 ds_{\pi\ell}} = \lim_{V \to \infty} \left( \frac{d^2 \Gamma_0}{dq^2 ds_{\pi\ell}} - \frac{d^2 \Gamma_0^{pt}}{dq^2 ds_{\pi\ell}} \right) + \lim_{V \to \infty} \left( \frac{d^2 \Gamma_0^{pt}}{dq^2 ds_{\pi\ell}} + \frac{d^2 \Gamma_1 (\Delta E)}{dq^2 ds_{\pi\ell}} \right)
\]

and the second term on the r.h.s. needs to be calculated in perturbation theory.

- This has not been fully done (yet?).

The second term has been calculated in the soft-photon approximation in which all terms proportional to \(k^n\) (\(n \geq 1\)) are dropped in the numerator.

De Boer, Kitahari, Nišandžić, arXiv:1803.05881

- This work was motivated by the \(R(D)\) and \(R(D^*)\) anomalies in semileptonic \(B\)-decays and the suggestion that radiative corrections not present in photos may be the explanation. Appears to be not true.

The soft-photon approximation is sufficient to make both terms on the r.h.s. infrared finite, but not to eliminate the \(O\left(\frac{1}{k}\right)\) corrections in the first term.
\[
\frac{d^2 \Gamma}{dq^2 ds_{\pi \ell}} = \lim_{V \to \infty} \left( \frac{d^2 \Gamma_0}{dq^2 ds_{\pi \ell}} - \frac{d^2 \Gamma_0^{pt}}{dq^2 ds_{\pi \ell}} \right) + \lim_{V \to \infty} \left( \frac{d^2 \Gamma_0^{pt}}{dq^2 ds_{\pi \ell}} + \frac{d^2 \Gamma_1(\Delta E)}{dq^2 ds_{\pi \ell}} \right)
\]

- The calculation of the second term on the r.h.s. has also been performed at lowest non-trivial order in ChPT.  
  Cirigliano, Giannotti, Neufeld, arXiv:0807.4507
Present:

Full lattice calculations of radiative corrections to leptonic decays are possible.
The form factors for real emissions are accessible from Euclidean correlators.
Our preliminary results are very encouraging, still more work is needed and the analysis is ongoing.

Future:

B-mesons are also very interesting, we expect a dynamical enhancement, but they are more difficult on current lattices.
Neutron lifetime is also a fundamental piece of information in flavor physics.
THANKS FOR YOUR ATTENTION