EM corrections to leptonic and semileptonic decays

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DIPARTIMENTO DI FISICA

SAPIENZA
UNIVERSITÀ DI ROMA

CERN January 31 2020

Extended Twisted Mass Collaboration

UTfit

PRACE

INFN Istituto Nazionale di Fisica Nucleare

PLAN OF THE TALK

1) Physics Motivations

2) Introduction: QED corrections to the hadron spectrum and to the hadronic amplitudes

3) Leptonic decays e.g. $\pi^+ \rightarrow \mu^+ \nu_\mu (\gamma)$ *& heavier*

4) Real Emissions (RM123 method) and Heavy Flavours

5) Semileptonic decays (only theory so far)

5) Conclusion & Outlook

Physics Motivations: Flavour and New Physics

Flavour phenomenology plays a fundamental role in indirect searches of New Physics (NP):

- *looks for deviation from the SM whatever the origin is;*

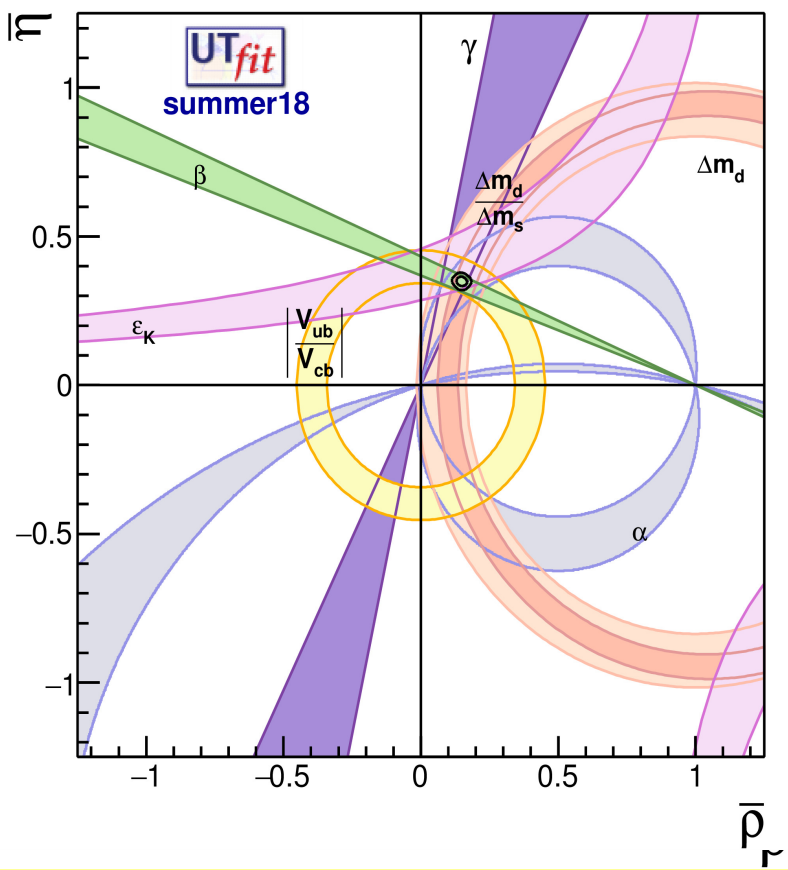
- *needs good theoretical control of the SM contribution only.*

the path leading to NP@ the TeV scale is much narrower after the results from LHC

The SM CKM pattern represents the principal part of the flavor structure and of CP violation

2018 results

$\rho = 0.148 \pm 0.013 \quad \eta = 0.348 \pm 0.010$



$\alpha = (90.9 \pm 2.0)^0$

$\sin 2\beta = 0.699 \pm 0.016$

$\beta = (22.3 \pm 0.7)^0$

$\gamma = (66.8 \pm 2.0)^0$

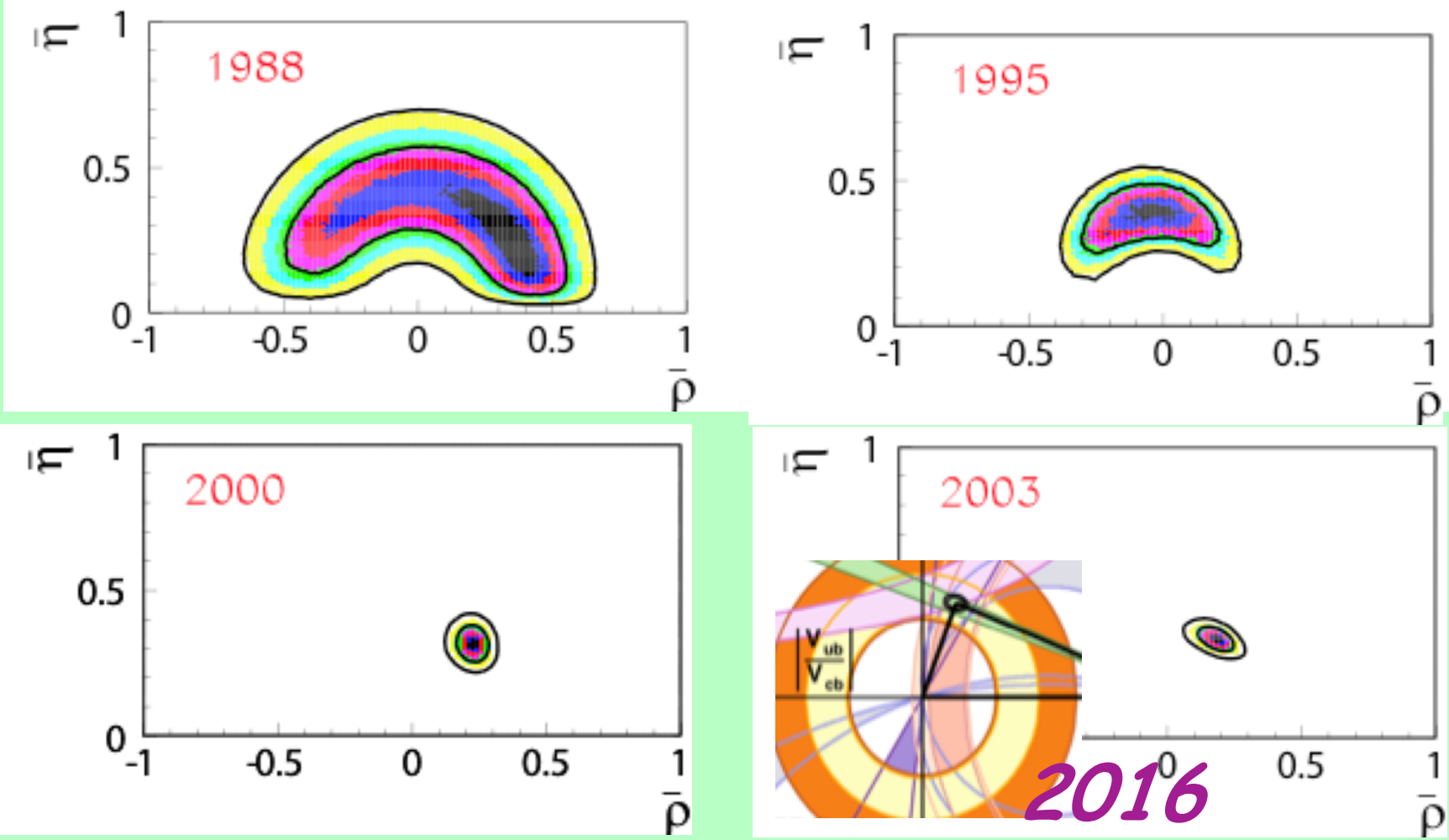
$A = 0.826 \pm 0.012$

$\lambda = 0.22500 \pm 0.00100$

Consistence on an over constrained fit of the CKM parameters

CKM matrix is the dominant source of flavour mixing and CP violation

PROGRESS SINCE 1988



The accuracy of lattice calculations of the hadron spectrum (and hence of the quark masses) and of the decay constants and form factors is such that isospin breaking and em effects cannot be neglected anymore:

f_π = 130.2(0.8) MeV ε =0.6% f_K = 155.7(0.3) MeV ε =0.2%

f_K/f_π = 1.1932(19) ε =0.16% $F^{K\pi}(0)$ =0.9706(27) ε =0.3%

A remark on useful and useless precision of lattice calculations:

1) ε_K and long distance charm contributions

2) f_K and f_π

B meson real photon emissions

Factorization at leading power in an expansion of the decay amplitude in Λ_{QCD}/E_γ and Λ_{QCD}/mb has been established to all orders in the strong coupling α_s. In this approximation, the branching fraction depends only on the leading-twist B-meson light-cone distribution amplitude (LCDA) $\phi_+(\omega,\mu)$

More precisely, it is proportional to $1/\lambda_B$, the most important LCDA parameter in exclusive decays, is uncertain by a large factor ranging from 200 MeV favoured by non-leptonic decays to 460 MeV from QCD sum rules.

The radiative leptonic decay has therefore been suggested as a measurement of λ_B

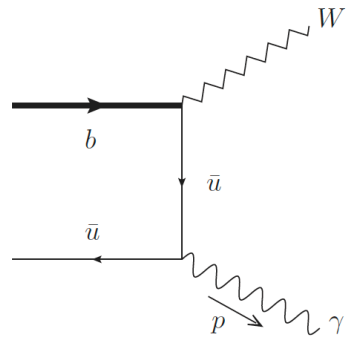


Figure 1. Leading contribution to $B \to \gamma \ell \nu_\ell$.

For large photon energies the form factors can be written as [9]

$$F_V(E_\gamma) = \frac{e_u f_B m_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma,\mu) + \xi(E_\gamma) + \Delta\xi(E_\gamma)\,,$$
$$F_A(E_\gamma) = \frac{e_u f_B m_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma,\mu) + \xi(E_\gamma) - \Delta\xi(E_\gamma)\,. \tag{2.7}$$

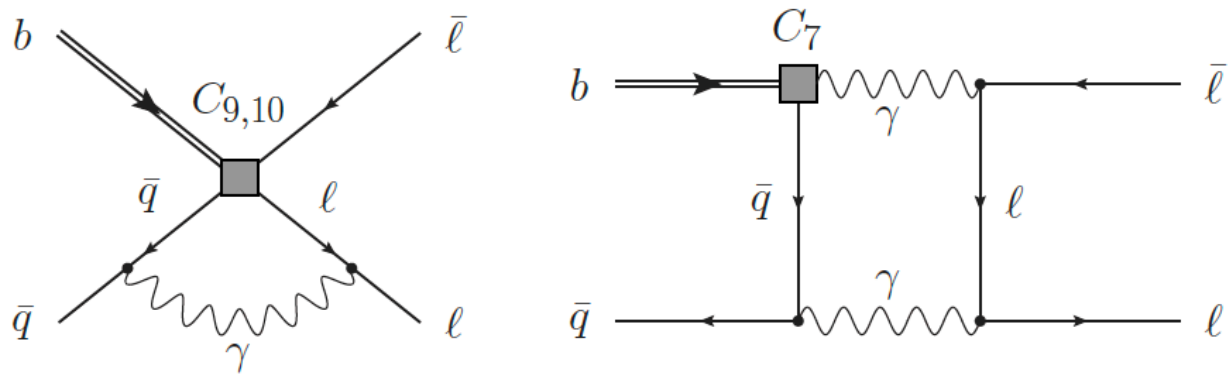
The first term is equal in both expressions and represents the leading-power contribution in the heavy-quark expansion (HQE). It originates only from photon emission from the light spectator quark in B meson (Fig. 1). In the above, f_B is the decay constant of B meson, and the quantity λ_B is the first inverse moment of the B-meson LCDA,

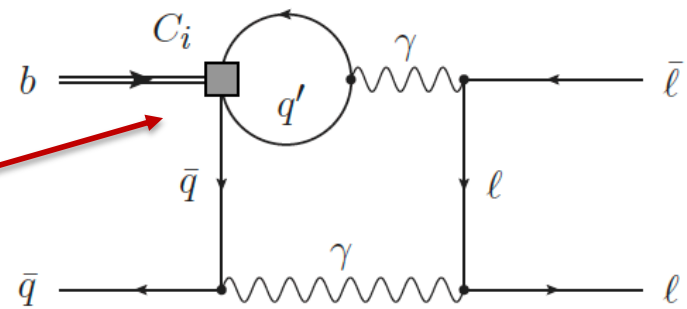
$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty \frac{d\omega}{\omega}\,\phi_+(\omega,\mu)\,. \tag{2.8}$$

Further applications in decays of heavy neutral B mesons: Virtual corrections (some questions still open)

Enhanced electromagnetic correction to the rare B-meson decay $B_{s,d} \to \mu^+\mu^-$

Martin Beneke,[1] Christoph Bobeth,[1,2] and Robert Szafron[1]





is this really reabsorbed in the coefficient of O_9?

Further applications in decays of heavy neutral B mesons: real corrections (some questions still open)

$B_s^0 \to \mu^+\mu^-\gamma$ from $B_s^0 \to \mu^+\mu^-$

Francesco Dettori[a], Diego Guadagnoli[b] and Méril Reboud[b,c]

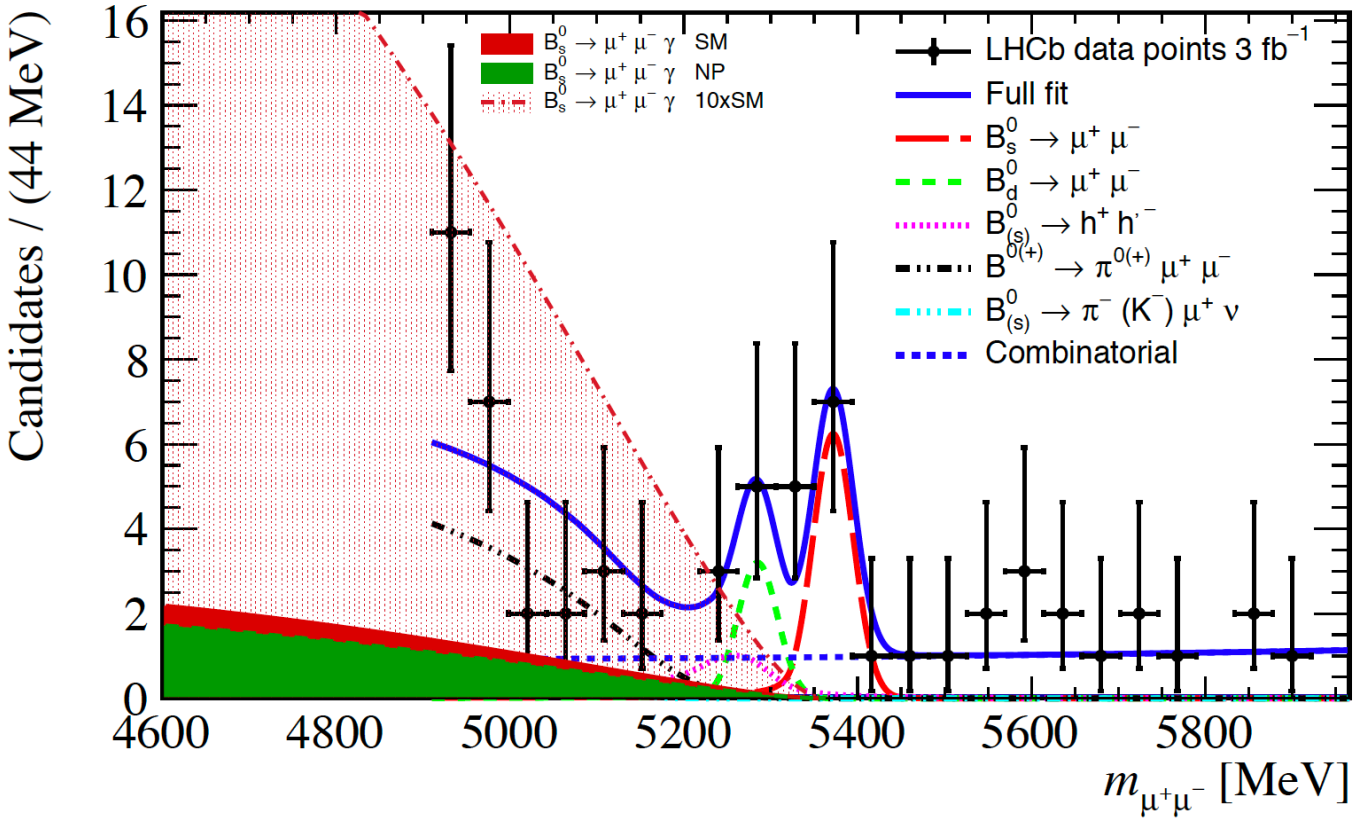
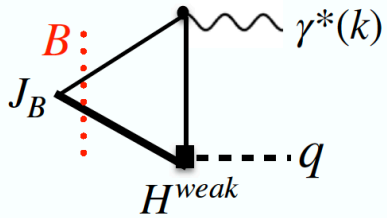


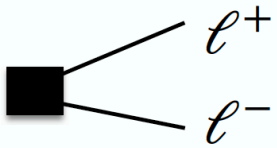
Figure 3: Dimuon invariant mass distribution from LHCb's measurement of $\mathcal{B}(B_s^0 \to \mu^+\mu^-)$ [52] overlayed with the contribution expected from $B_s^0 \to \mu^+\mu^-\gamma$ decays (ISR only). Assumes flat efficiency versus $m_{\mu^+\mu^-}$. The line denoted as '$B_s^0 \to \mu^+\mu^-\gamma$ NP' refers to the $V-A$ case with $\delta C_9 = -12\%\, C_9^{\rm SM}$ (see also Fig. 2). The two filled curves are not stacked onto each other.

Particle(s) from weak vertex with momenta q



$$F(q^2, k^2)$$

- **FCNC** Qb= Qq (need long distance in addition) :



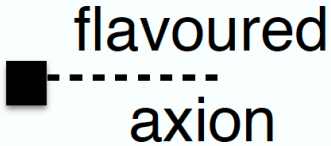
$H^{weak} \sim O_{9,10} : B_{d,s} \to \ell^+\ell^-\gamma$

$F(q^2) = F(q^2,0)$

Bobeth's talk

$H^{weak} \sim O_7 : B_{d,s} \to \ell^+\ell^-\gamma$

$F^*(k^2) = F(0,k^2)$



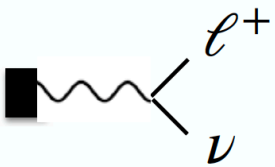
$H^{weak} \sim \bar{q}\gamma_\mu b_L \partial^\mu a : B_{d,s} \to \ell^+\ell^- a$

$F(m_a^2, k^2) \to F^*(k^2)$

Ziegler's talk

or dark photon, scalar DM, …

- **FCCC** Qb ≠ Qq :



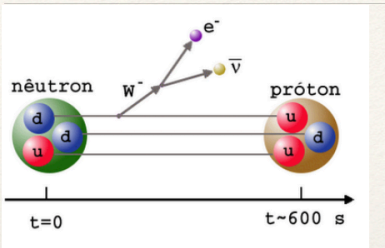
$H^{weak} \sim V_{ub}\, \bar{u}\gamma_\mu b_L \ell\gamma^\mu \nu_L : B_u \to \ell^+\nu\gamma$

- Physics: helicity suppression of $B \to f_i \bar{f}_j$ relieved in radiative decay!

Status of Lattice Calculations of g_A

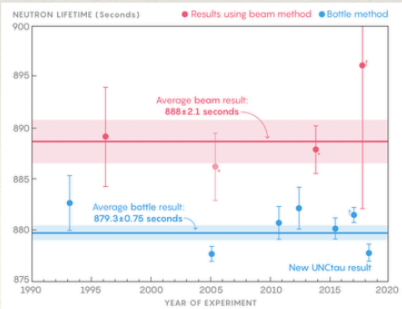
André Walker-Loud
BERKELEY LAB

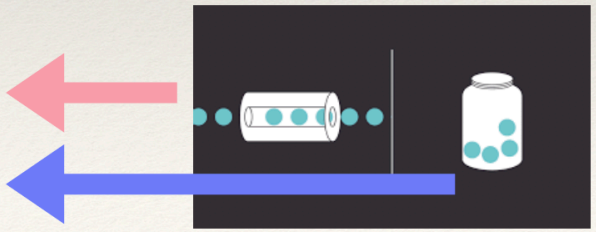
Neutron lifetime and the axial coupling



$$\frac{1}{\tau_n} = \frac{G_\mu^2 |V_{ud}|^2}{2\pi^3} m_e^5 (1 + 3g_A^2)(1 + RC) f_{V,A}$$

- The neutron lifetime and g_A (neutron decay) are used to probe the limits of the Standard Model
- We should have a (meaningful) Standard Model prediction for g_A - LQCD (lattice QCD)
- To gain confidence in the application of LQCD to nuclear physics, we must benchmark (calibrate) our calculations against well known quantities of interest, such as g_A
- In order for the theoretical uncertainty on g_A to match the larger uncertainty in the neutron lifetime measurements, we must determine g_A with < 0.2% uncertainty - **is this crazy?**

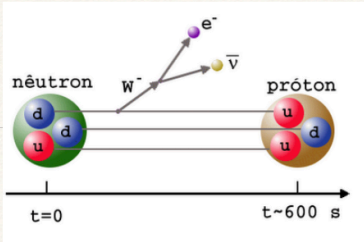


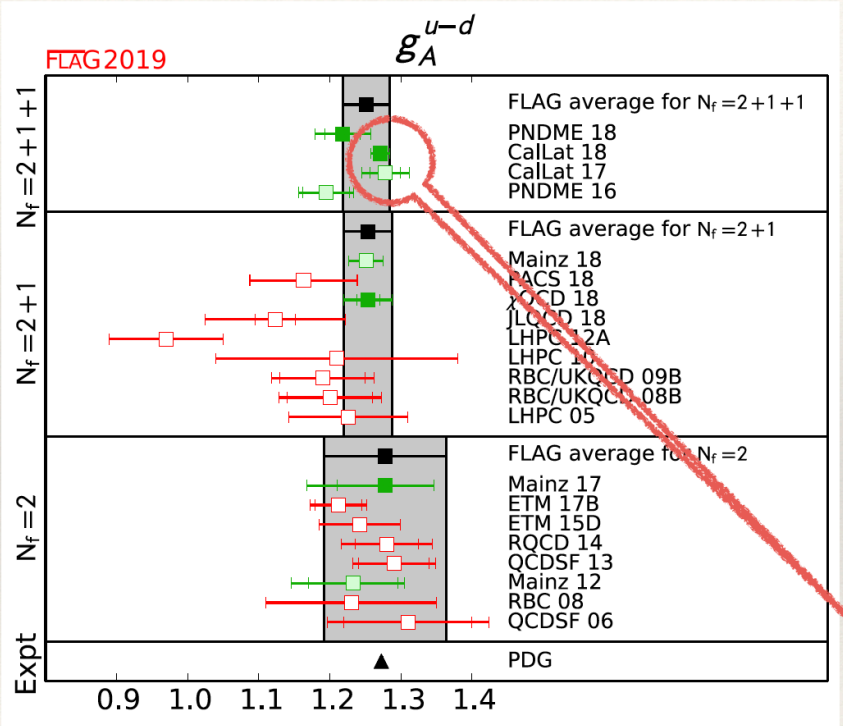


$$\tau_n^{\text{beam}} = 888.0(2.0)s$$

$$\tau_n^{\text{bottle}} = 879.4(0.6)s$$

nucleon axial coupling from LQCD





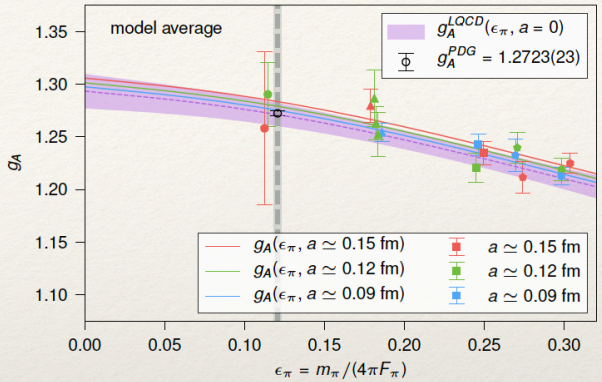
- To gain confidence in the application of Lattice QCD to nuclear physics, we must benchmark (calibrate) our calculations against well known quantities of interest
- g_A was supposed to be a good benchmark calculation for single nucleon structure - but it proved to have significant systematic challenges, preventing results with the precision anticipated
- FLAG 2019 has included single nucleon quantities in their averaging for the first time
- Notice one result is significantly more precise than the others

improving the determination of g_A

Nature 558 (2018) no.7708, 91-94
Chang et al.
[arXiv:1805.12130]

Final result

statistical	0.81%
chiral extrapolation	0.31%
$a \to 0$	0.12%
$L \to \infty$	0.15%
isospin	0.03%
model selection	0.43%
total	0.99%



$$g_A^{\rm QCD} = 1.2711(103)^s(39)^\chi(15)^a(19)^V(04)^I(55)^M$$

- More precise results at the physical pion mass will improve the three largest uncertainties:
  - statistical (s), extrapolation (χ) and model selection (M) **NOTE**, **a12m130** has 2.3% uncertainty
- Following our existing strategy, we anticipate getting to 0.5% by the end of this year
- Getting below (or maybe to 0.5%) will require a 4th lattice spacing as well (~0.06fm)
- Adding a FV study on additional pion mass points will improve the FV uncertainty
- The isospin uncertainty seems unnecessary…

Isospin Symmetry Breaking

In the isospin symmetric lattice world
up and down have the same mass
and the electric charge is switched off

1) Isospin is explicitly broken by
the up and down mass difference

$$\frac{m_d - m_u}{\Lambda_{QCD}} \sim 0.01$$

2) Electromagnetic interaction

$$\alpha \sim 0.0073$$

Non-compact lattice QED

naively discretised Maxwell action

$$S[A_\mu] = \frac{1}{4}\sum_{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$$

pure gauge is free, it can be solved exactly

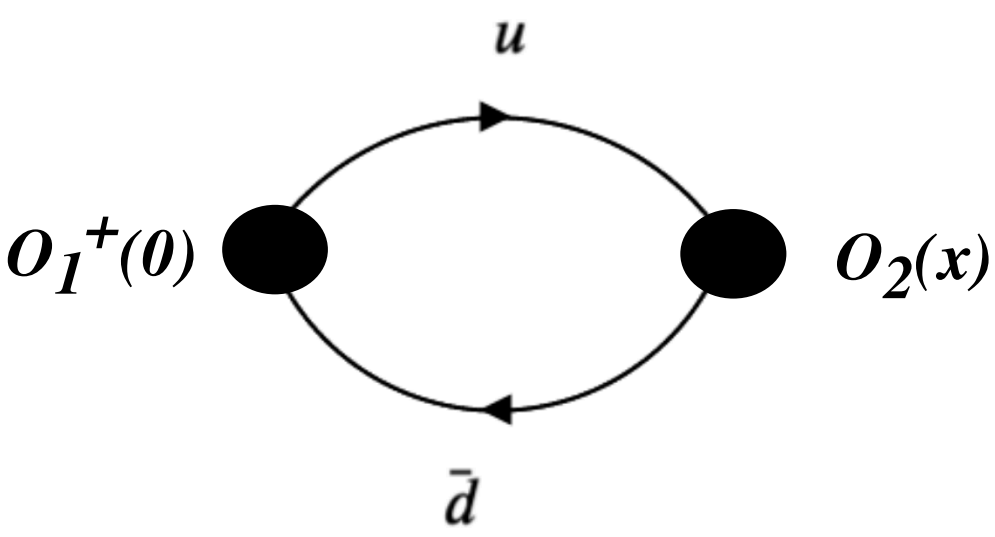
gauge invariance is preserved

QED_L: $D(k^2)=0 \quad \vec{k}=0$

where $D(k^2)$ is the photon propagator

subtleties related to QED on a finite volume not discussed here

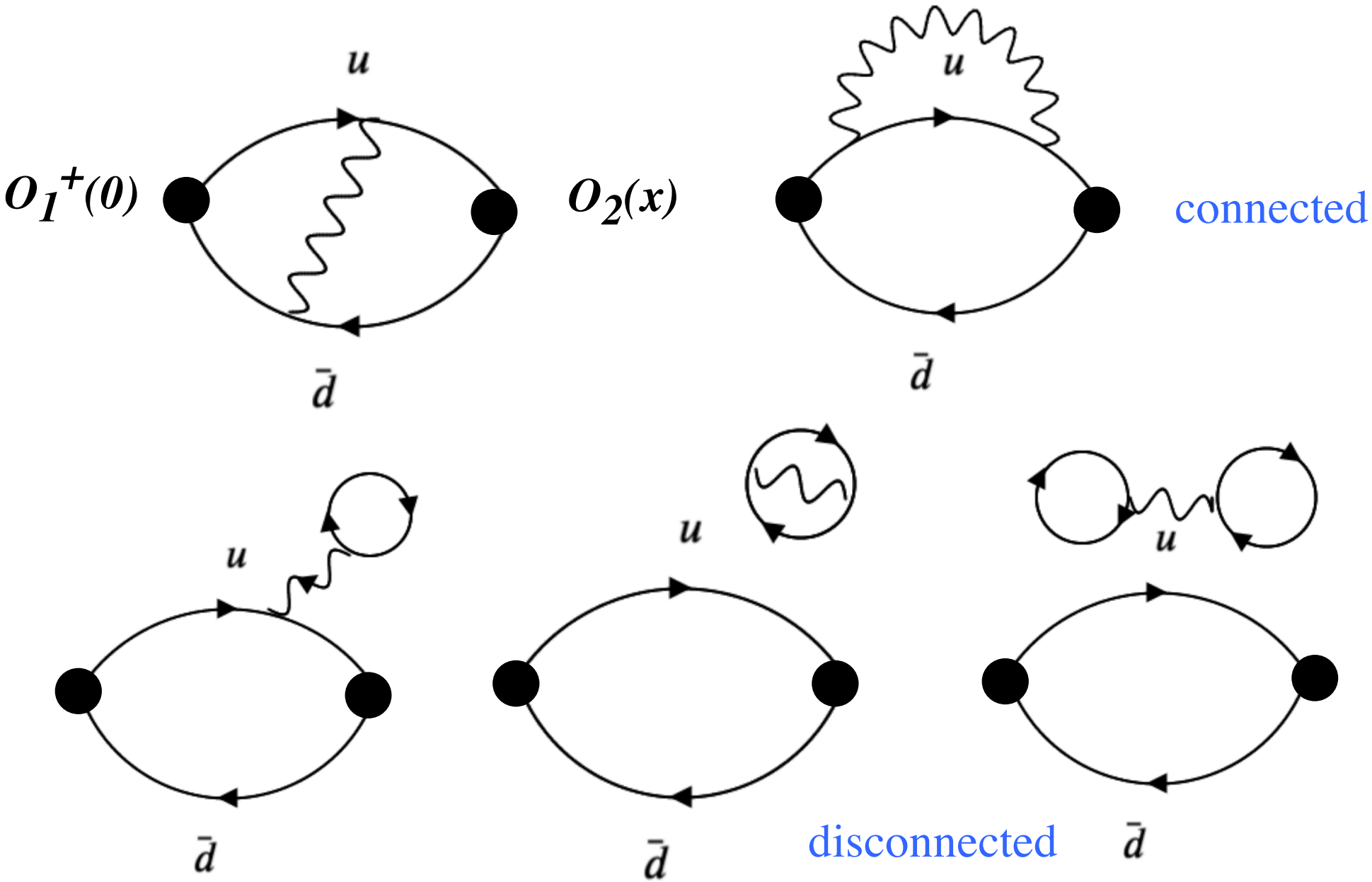
Masses and matrix elements at lowest order

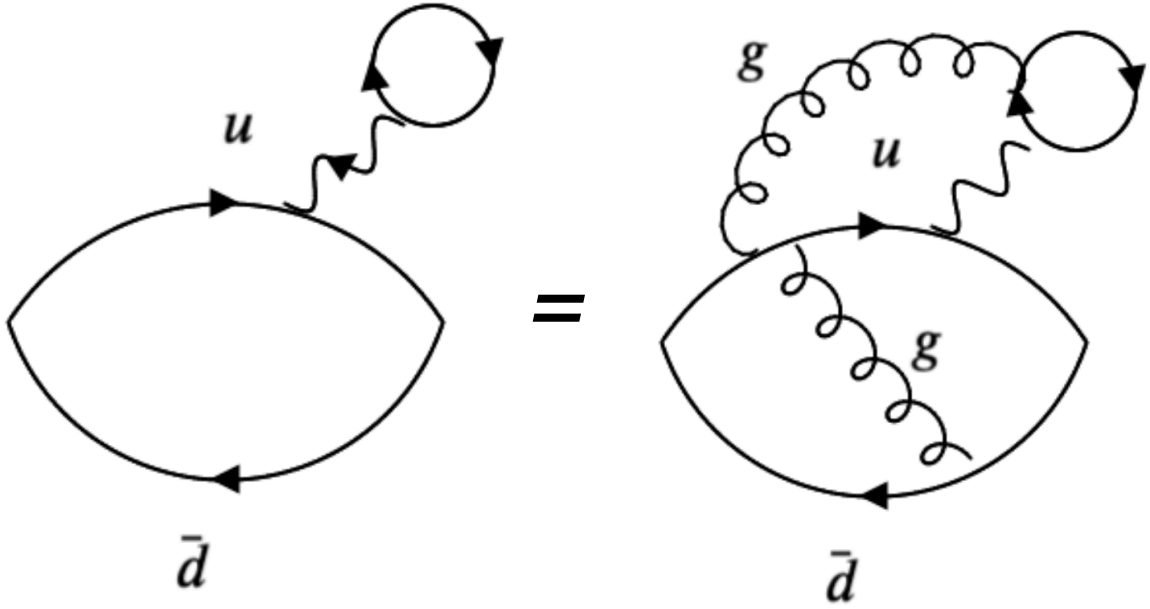


$$C_0^{12}(t)|_{t>0} = \sum_{\vec{x}} \langle 0|T\{O_2(\vec{x},t)\, O_1^\dagger(\vec{0},0)\}|0\rangle = \frac{\langle 0|O_2|\pi\rangle\langle\pi|O_1^\dagger|0\rangle}{2m_\pi} e^{-m_\pi t} + \ldots = \frac{Z_0^{12}}{2m_\pi} e^{-m_\pi t} + \ldots$$

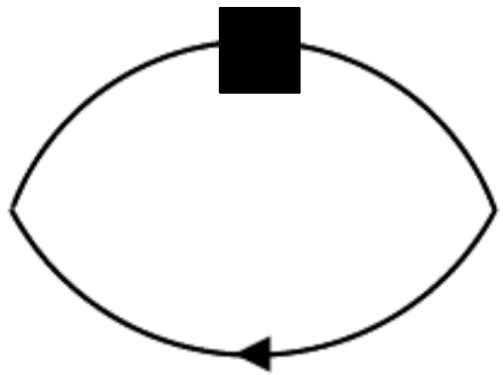
excited states (exponentially supressed at large distances)

Masses and matrix elements at $O(\alpha_{em})$





mass insertion



$$C^{12}_{\alpha_{em}}(t) = \frac{(Z_0^{12} + \delta Z^{12})}{2\,(m_\pi + \delta m_\pi)} e^{-(m_\pi + \delta m_\pi)t}$$

$$= C_0^{12}(t)\left(1 - \delta m_\pi t - \frac{\delta m_\pi}{m_\pi} + \delta Z^{12}\right) + O(\delta^2) \qquad \delta = O\left(\frac{m_d - m_u}{\Lambda_{QCD}}, \alpha_{em}\right)$$

$$\boxed{\frac{C^{12}_{\alpha_{em}}(t)}{C_0^{12}(t)} \sim -\delta m_\pi t + \text{const.}}$$

QED & Isospin Corrections To Hadron Masses and Amplitudes

The renormalization of the $SU(3)_c \times U(1)$ Lagrangian requires the definition of the renormalised couplings. At first order in α_{em}, we only have to renormalise α_s and the quark masses.

Without QED, in the 4-flavour theory, for each value of α_s we can fix four dimensionless quantities related to the quark masses

$$\frac{(a_0 m_{\pi^0})}{(a_0 M_\Omega)}, \quad \frac{(a_0 m_{K^0})}{(a_0 M_\Omega)}, \quad \frac{(a_0 m_{K^+})}{(a_0 M_\Omega)}, \quad \frac{(a_0 m_{D^0})}{(a_0 M_\Omega)}$$

and one physical mass to determine the dimensionful scale

($\alpha_s(a_0)$ - dimensional trasmutation) $$a_0 = \frac{(a_0 M_\Omega)}{M_\Omega^{ph}}$$

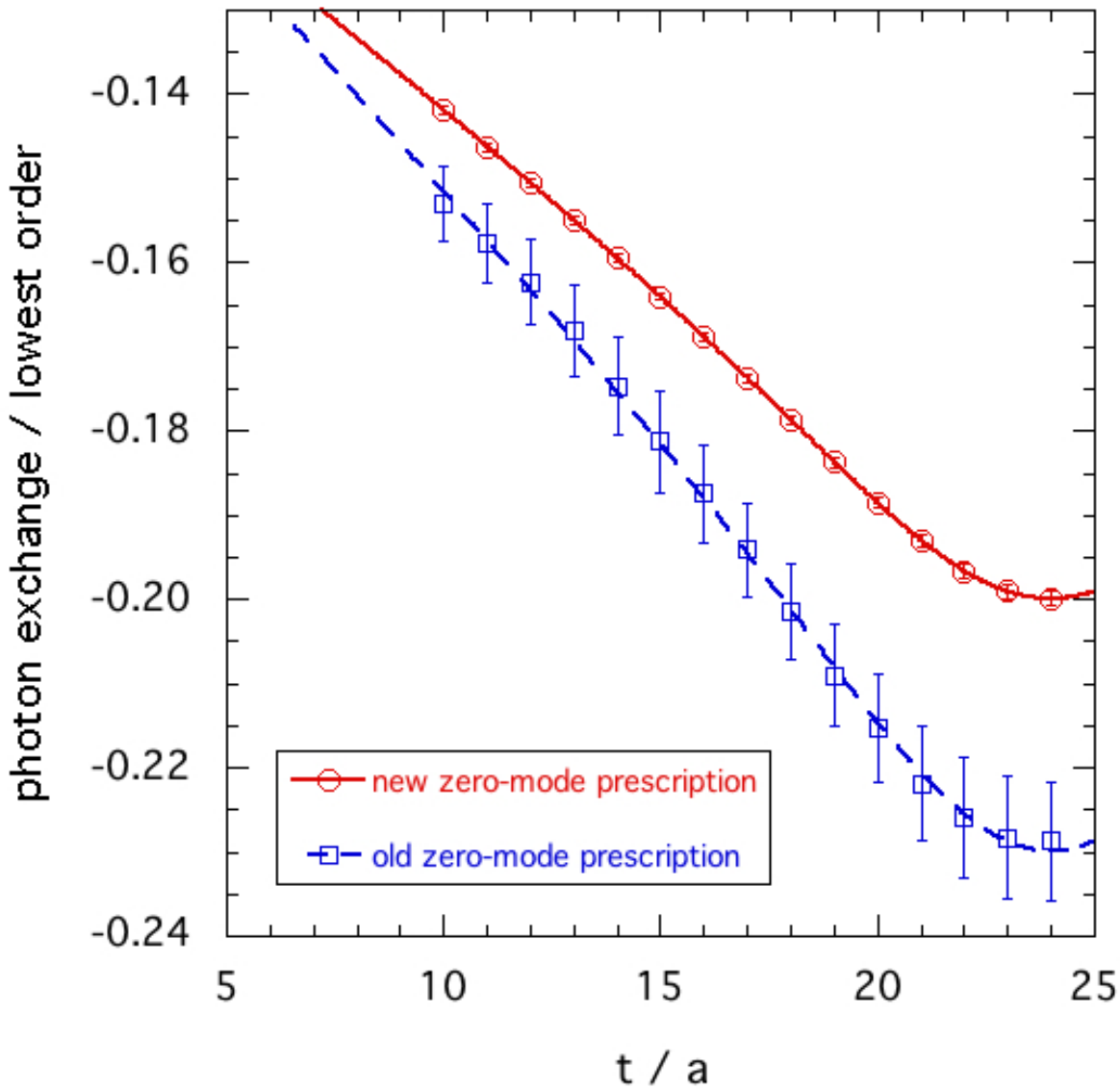
The same quantities can be used to define the physical scale and the quark masses for $\alpha_{em}=0$. Whatr you define to be the electromagnetic correction to a given quantity remains however convention dependent.

$$m_{\mathrm{QED_L}}(T,L) \underset{T,L\to+\infty}{=} m\left\{1-q^2\alpha\left[\frac{\kappa}{2mL}\left(1+\frac{2}{mL}\right)-\frac{3\pi}{(mL)^3}\right]\right\}$$

k=2.83729 universal

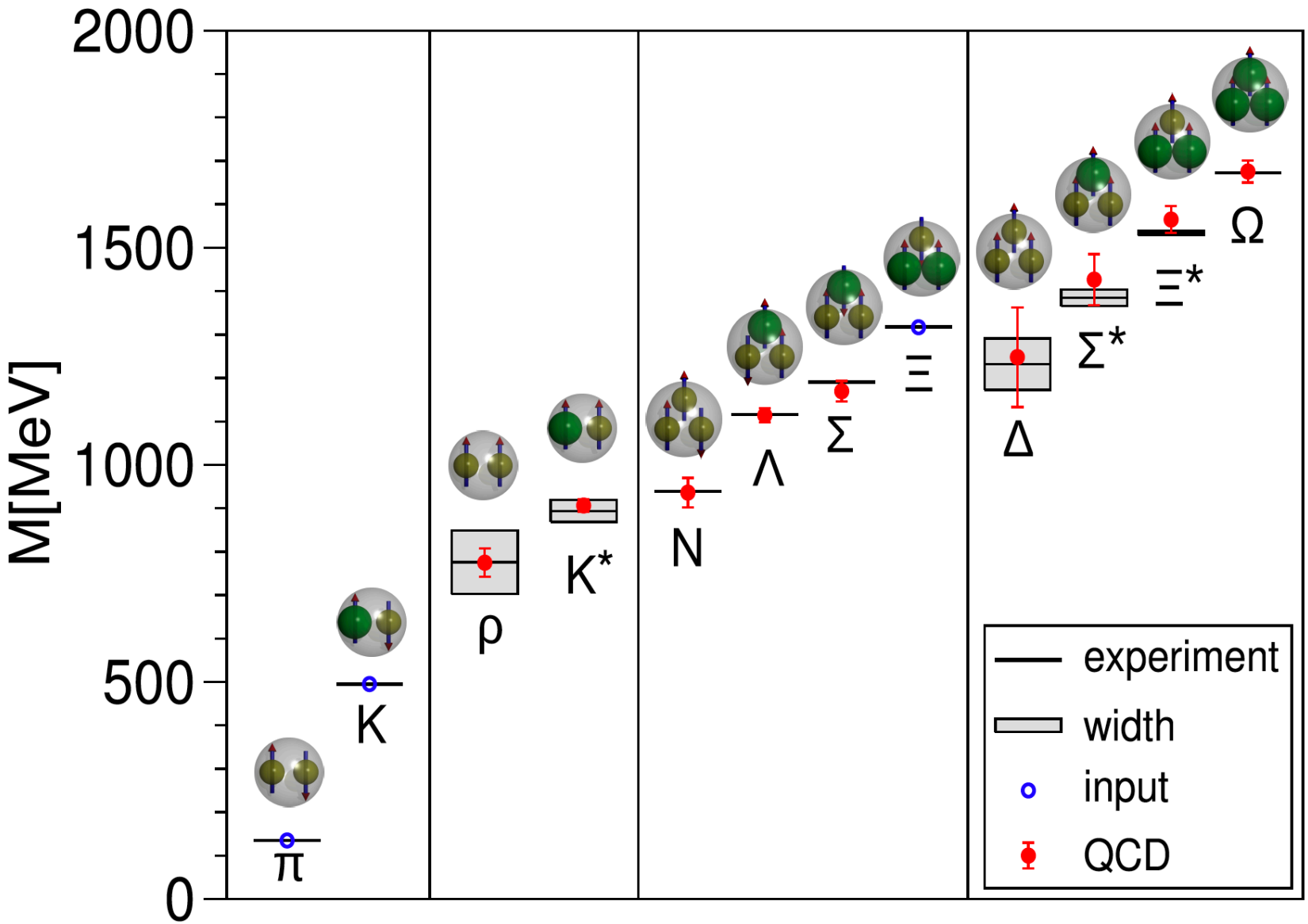
The masses are infrared finite

Finite volume corrections discussed later on

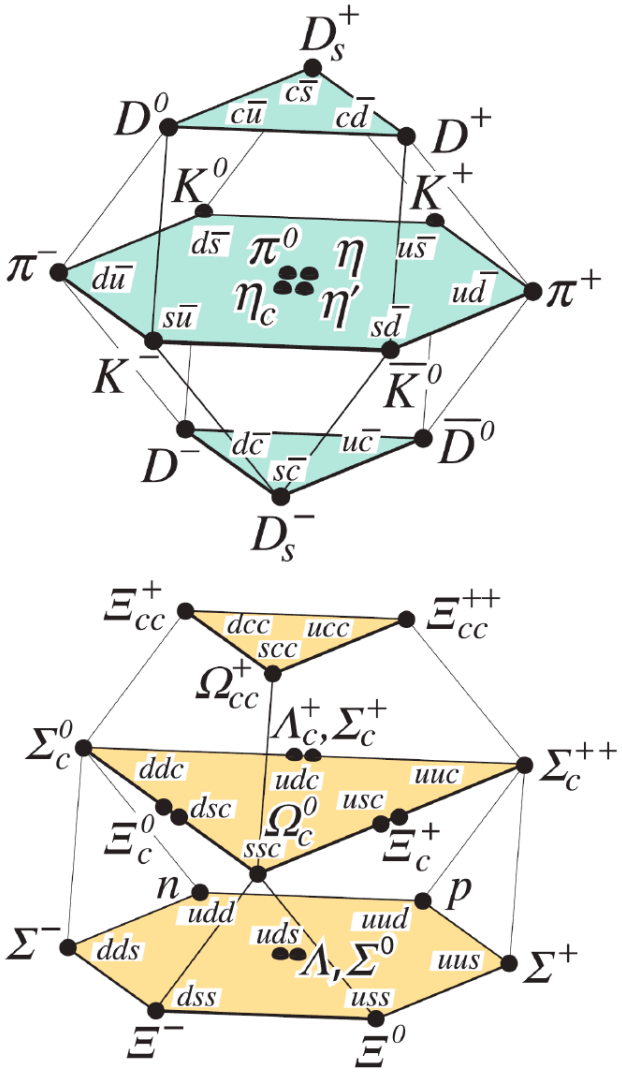


Light Hadron Masses

[Dürr et al (Budapest-Marseille-Wuppertal collaboration (BMWc)), Science 322 (2008) 1224]

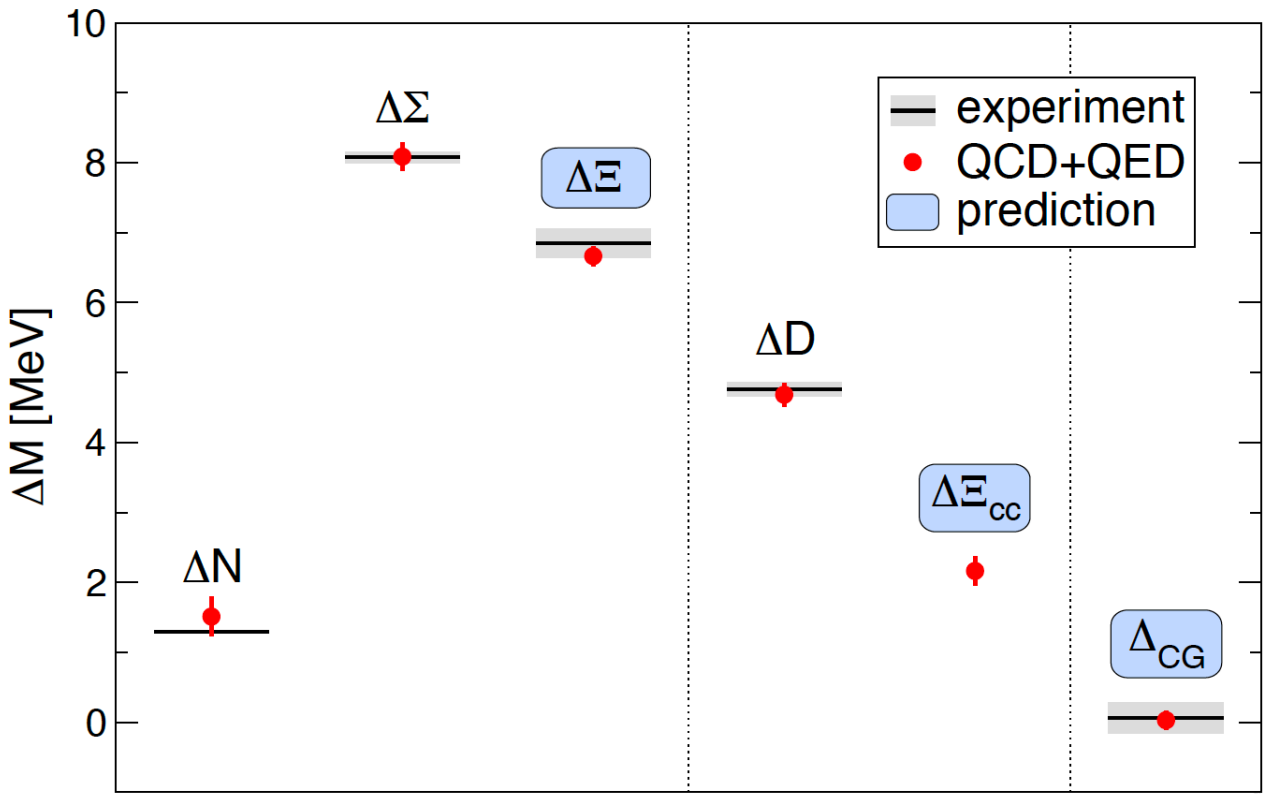


Hadron Masses & Isospin Violation



(PDG '14)

[Borsanyi et al (Budapest-Marseille-Wuppertal collaboration (BMWc)), Science 347 (2015) 1452]



Strong + Higgs + Electromagnetism = Experiment

QED (Isospin) Corrections in Hadronic Processes

After the renormalization of the $SU(3)_c \times \cancel{U(1)}$ Lagrangian and quark masses, you still need

1)The renormalization of the operators mediating the physical process of interest (e.g. the Weak effective Hamiltonian). But this is not a novelty;

2)A complex procedure to remove the infrared cutoff because in general the amplitudes, contrary to the masses, are infrared divergent.

A method to solve this problem has been presented . I will recall some important theoretical subtelties before discussing semileptonic decays

How to solve the problem of the infrared divergences discussed through an explicit example

$$\pi \to \ell + \nu_\ell + (\gamma)$$

N.Carrasco, V.Lubicz, G.M., C.T.Sachrajda, S. Simula, F.Sanfillippo, N.Tantalo, C.Tarantino, M.Testa

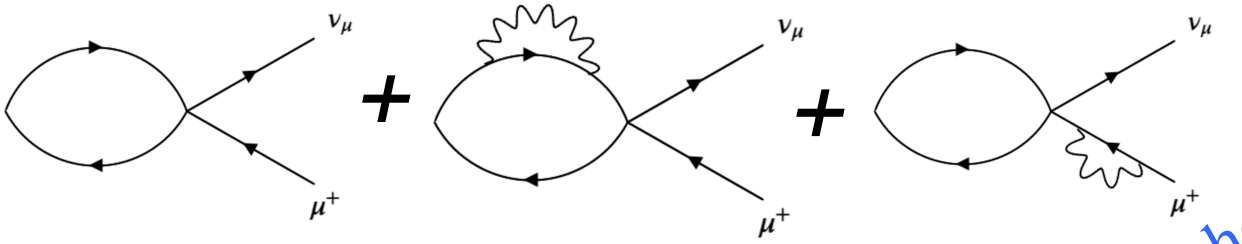
NOTE: Chiral Perturbation Theory is NOT Used

Rate at O(α)

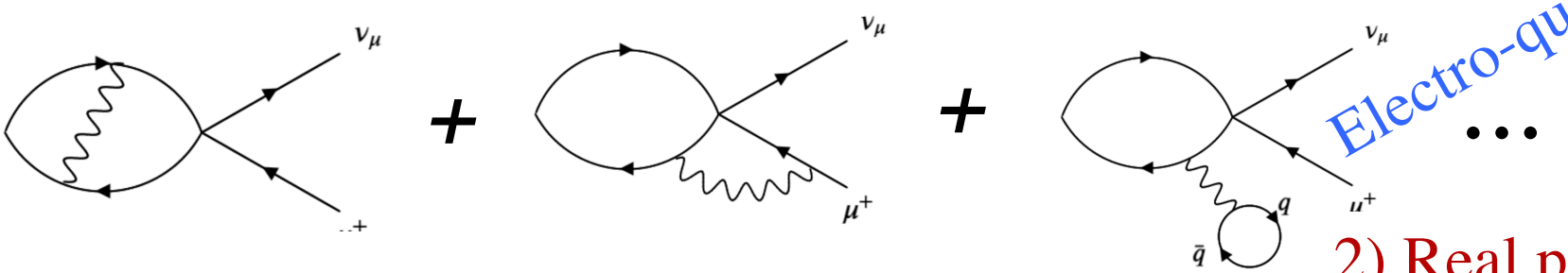
$$\Gamma(\Delta E) = \int_0^{\Delta E} dE_\gamma \frac{d\Gamma}{dE_\gamma} = \Gamma_0 + \Gamma_1(\Delta E)$$

1) Virtual photons

$$\Gamma(\Delta E) = \int d\Omega_2 \Bigg|$$



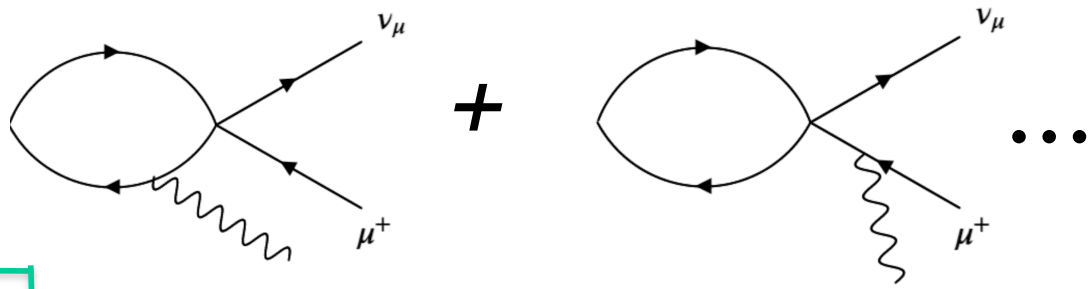
Electro-quenched



$$\Bigg|^2$$

2) Real photons

$$+ \int_{E_\gamma \leq \Delta E} d\Omega_3 \Bigg|$$



$$\Bigg|^2$$

$d\Omega_{2,3} = 2 - 3$ body phase − space

The Infrared Problem

$$\Gamma(\Delta E) = \Gamma_0 + \Gamma_1(\Delta E)$$

QED Corrections to Hadronic Processes in Lattice QCD,
N.Carrasco, V.Lubicz, G.Martinelli, C.T.Sachrajda, N.Tantalo, C.Tarantino and M.Testa,
Phys. Rev. D **91** (2015) no.7, 07450 [arXiv:1502.00257 [hep-lat]].

Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD,
V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,
Phys. Rev. D **95** (2017) no.3, 034504 [arXiv:1611.08497 [hep-lat]].

First Lattice Calculation of the QED Corrections to Leptonic Decay Rates,
D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula, N.Tantalo and C.Tarantino,
Phys. Rev. Lett. **120** (2018) 072001 [arXiv:1711:06537]

Light-meson leptonic decay rates in lattice QCD+QED
M.Di Carlo, D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,
Phys.Rev. D100 (2019) no. 3, 034514 arXiv:1904.08731

MASTER FORMULA for the rate at O(α)

$$\Gamma(\Delta E) = \lim_{V\to\infty}(\Gamma_0 - \Gamma_0^{\rm pt}) + \lim_{V\to\infty}(\Gamma_0^{\rm pt} + \Gamma_1(\Delta E))$$

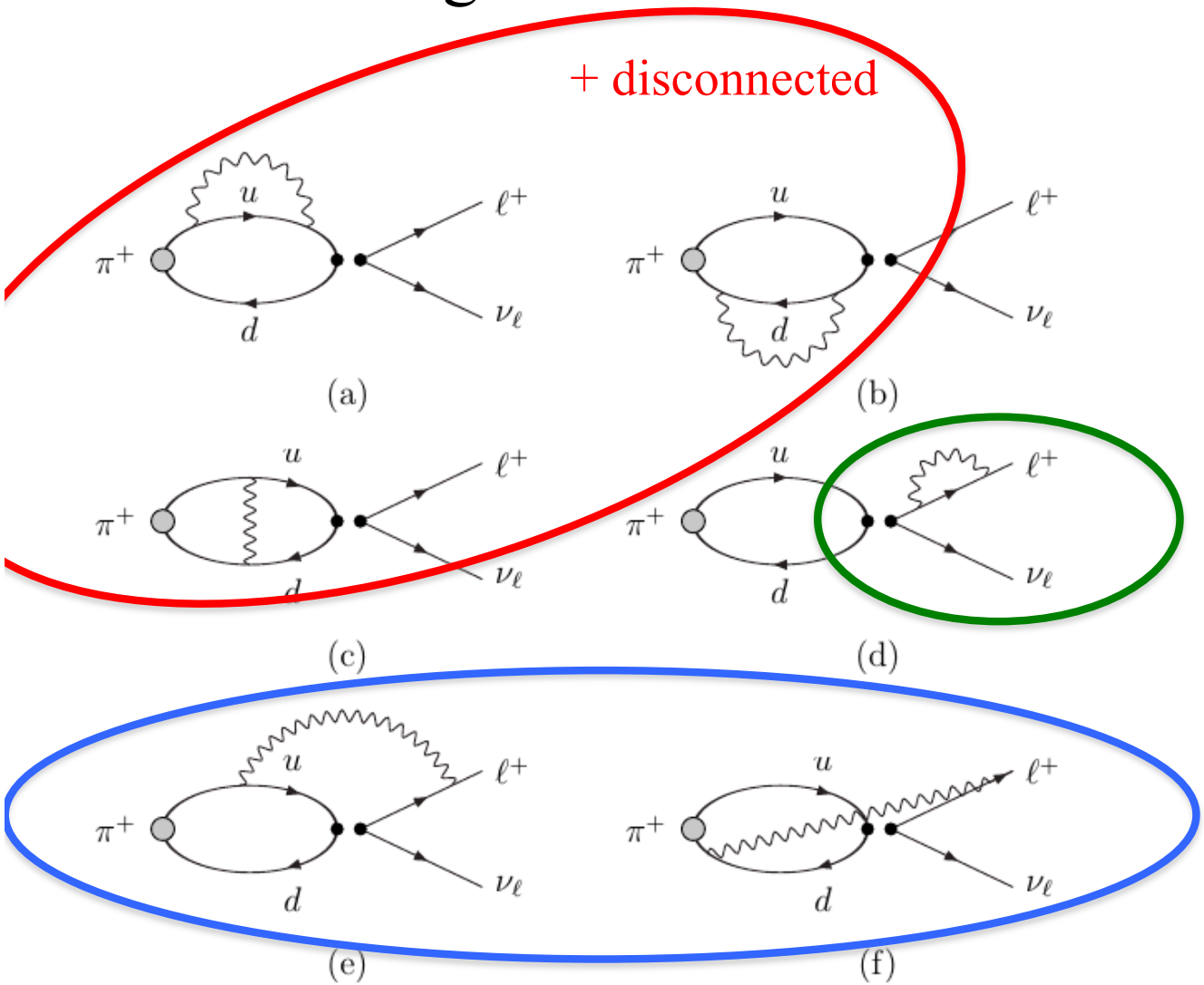
pt = point-like & perturbative

- the infrared divergences in Γ_0 and Γ_0^{pt} are exactly the same and cancel in the difference
- $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$ is infrared finite since is a physical, well defined quantity *F. Bloch, A. Nordsieck Phys.Rev. 52 (1937) T.D. Lee, M. Nauenberg Phys.Rev. 133 (1964)*
- the infrared divergences in $\Delta\Gamma_0(L) = \Gamma_0 - \Gamma_0^{pt}$ and $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$ cancel separately hence they can be regulated with different infrared cutoff
- Γ_0 and Γ_0^{pt} are also ultraviolet finite

We now discuss the two terms, $\Delta\Gamma_0(L)$ and $\Gamma(\Delta E)$

Leptonic decays at O(α) – The first term of the Master Formula $\Delta\Gamma_0(L) = \Gamma_0 - \Gamma_0^{pt}$

- Each of the two terms is U.V. finite but contains $\log(M_W)$
- Infrared divergences cancel in the difference

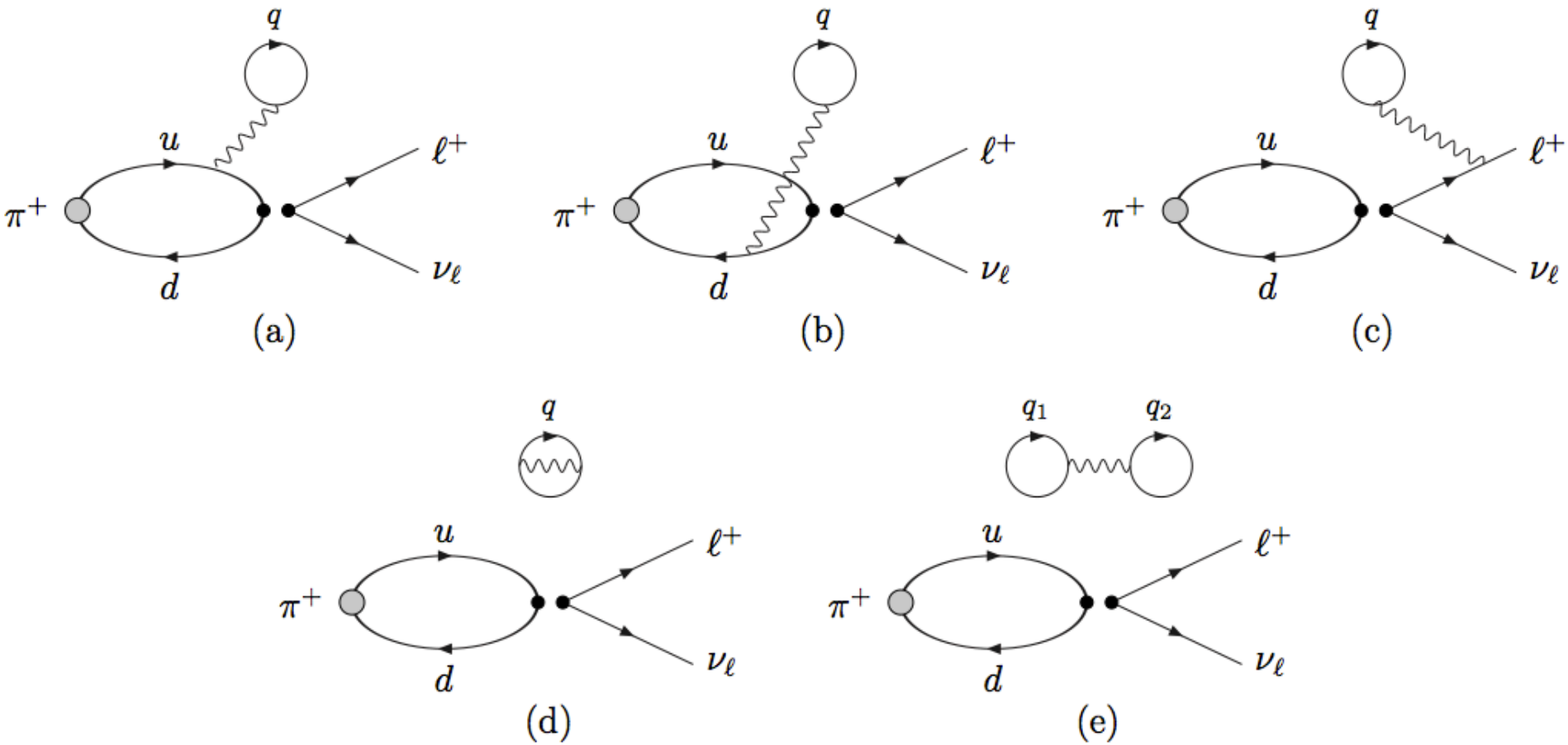


at this order we can take the difference of the amplitudes

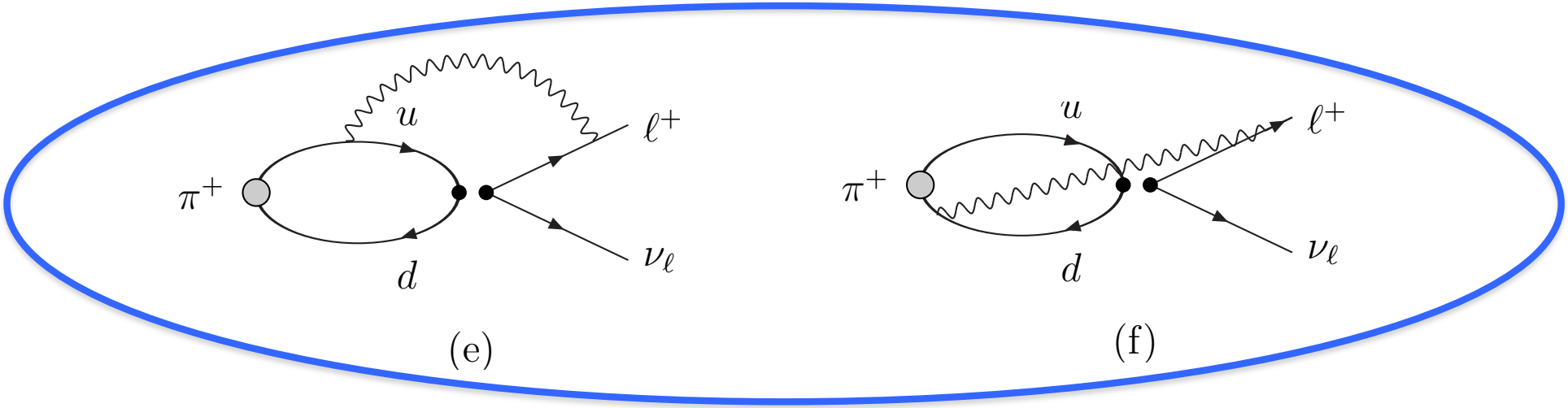
Can be computed as discussed in arXiv:1303.4896,Phys.Rev. D87(2013) RM123 method

NOT by including the electromagnetic field in the action

``DISCONNECTED" DIAGRAMS



NASTY DIAGRAMS



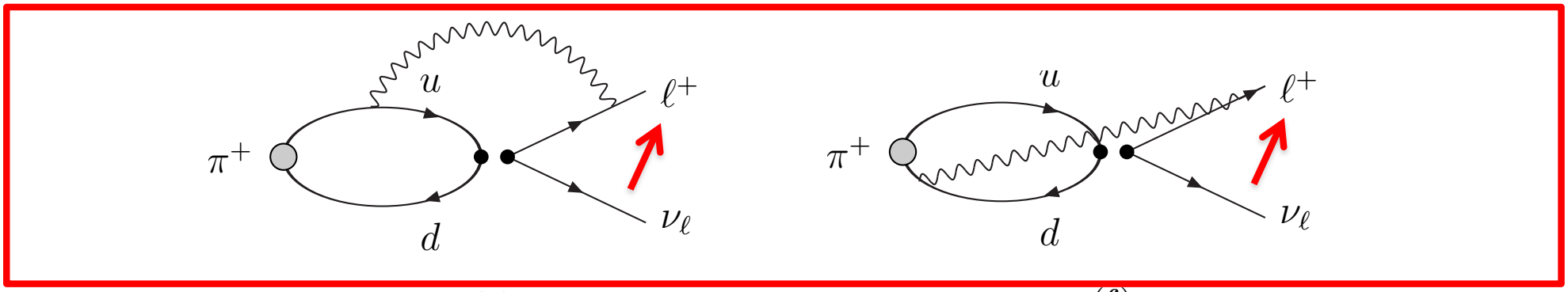
- *Certainly these diagrams are not simply a generalization of the evaluation of f_π ; they are also infrared divergent)*

- *We have to isolate the finite volume ground state (necessity of a mass gap – Minkowski $\longleftrightarrow$ Euclidean continuation* J. Gasser and G.R.S. Zarnauskas, Phys. Lett. B 693 (2010) 122 *)*

- *Finite volume effects are expected of O(1/L) after the cancellation of the infrared divergences*

Calculation of the `nasty' diagrams in a lattice simulation

Continuation from Minkowski to Euclidean



(e) (f)

The starting point is the Minkowski Green function

$$\int d^4x_1 d^4x_2 < 0|\, T(j_\mu(x_1) J_W^\nu(0))\, |\pi > \; iD_F(x_1 - x_2)\{\bar{u}(p_{\nu_\ell})\gamma^\nu(1-\gamma^5)(iS_F(x_2))\gamma^\mu v(p_\ell)\} e^{ip_\ell \cdot x_2}$$

from which we can compute the on-shell amplitude

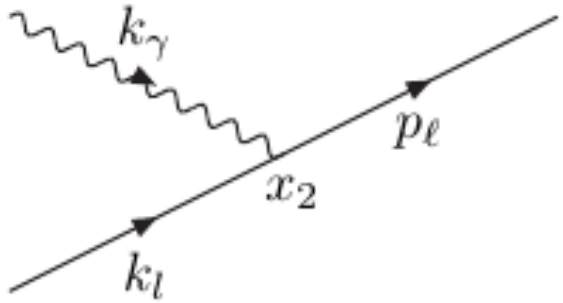
$$\bar{u}_\alpha(p_{\nu_\ell})(\bar{M}_1)_{\alpha\beta} v_\beta(p_\ell) = \; -i \lim_{k_0 \to m_\pi} ({k_0}^2 - m_\pi^2) \int d^4x_1 d^4x_2\, d^4x\, e^{-ik^0 y^0} < 0|T(j_\mu(x_1) J_W^\nu(0)\pi(x))|0 >$$
$$\times iD_F(x_1 - x_2)\{\bar{u}(p_{\nu_\ell})\gamma_\nu(1-\gamma^5)(iS_F(x_2))\gamma^\mu v(p_\ell)\} e^{ip_\ell \cdot x_2}$$

which in the Euclidean simulation becomes

$$\bar{C}_1(t)_{\alpha\beta} = \int d^3\mathbf{x}\, d^4x_1\, d^4x_2\, \langle 0|T\{J_W^\nu(0)\, j_\mu(x_1)\phi^\dagger(\mathbf{x}, t)\}|\, 0\rangle\, \Delta(x_1 - x_2)$$
$$\times \left(\gamma_\nu(1-\gamma^5) S(x_2)\gamma^\mu\right)_{\alpha\beta} e^{E_\ell\, t_2} e^{-i\mathbf{p}_\ell \cdot \mathbf{x}_2}$$

IMPORTANT slides:
the continuation from Minkowski to Euclidean

we need to ensure that the t_2 integration up to ∞ converges in spite of the factor $e^{E_l t_2}$ where $E_l = \sqrt{m_l^2 + p_l^2}$ is the energy of the outgoing charged lepton



1) Momentum conservation: since we integrate over x_2 $p_l = k_l + k_\gamma$

2) The integrations over the energies k_{4l} and $k_{4\gamma}$ lead to the exponential factor $e^{-(\omega_l + \omega_\gamma - E_l)\, t_2}$ where $\omega_l = \sqrt{m_l^2 + k_l^2}$, $\omega_\gamma = \sqrt{m_\gamma^2 + k_\gamma^2}$, and m_γ is the mass of the photon introduced as an infra-red cut-off.

IMPORTANT slides:
the continuation from Minkowski to Euclidean

3) … but $(\omega_l+\omega_\gamma) \geq \sqrt{(m_l+m_\gamma)^2 + p_l^2} > E_l = \sqrt{m_l^2 + p_l^2}$

thus the argument of the exponent $e^{-(\omega_l + \omega_\gamma - E_l)\, t_2}$ is negative for every term appearing in the sum over the intermediate states and the integral over t_2 converges

4) note that the integration over t_2 is also convergent if we set $m_\gamma=0$ but remove photon zero mode in finite volume. In this case $(\omega_l+\omega_\gamma) > E_l+[1-(p_l/E_l)]\ (k_\gamma)_{min}$

- necessity of a mass gap
- absence of a lighter intermediate state

Leptonic decays at O(α) – Perturbative Calculation of $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$

U.V. & Infrared finite but contains $\log(M_W)$ & $\log(\Delta E)$

$\Delta E << \Lambda_{QCD}$

$$\Gamma(\Delta E) = \Gamma_0^{\text{tree}} \times \left(1 + \frac{\alpha}{4\pi}\left\{3\log\left(\frac{m_\pi^2}{M_W^2}\right) + \log\left(r_\ell^2\right) - 4\log(r_E^2) + \frac{2-10r_\ell^2}{1-r_\ell^2}\log(r_\ell^2)\right.\right.$$
$$-2\frac{1+r_\ell^2}{1-r_\ell^2}\log(r_E^2)\log(r_\ell^2) - 4\frac{1+r_\ell^2}{1-r_\ell^2}\,\mathrm{Li}_2(1-r_\ell^2) - 3$$
$$+\left[\frac{3+r_E^2-6r_\ell^2+4r_E(-1+r_\ell^2)}{(1-r_\ell^2)^2}\log(1-r_E) + \frac{r_E(4-r_E-4r_\ell^2)}{(1-r_\ell^2)^2}\log(r_\ell^2)\right.$$
$$\left.\left.\left.-\frac{r_E(-22+3r_E+28r_\ell^2)}{2(1-r_\ell^2)^2} - 4\frac{1+r_\ell^2}{1-r_\ell^2}\,\mathrm{Li}_2(r_E)\right]\right\}\right)$$

$$r_E = \frac{2\Delta E}{m_\pi} \qquad r_\ell = \frac{m_\ell}{m_\pi}$$

We think that this is a new result;

$\Gamma(\Delta E_l)$ *T.Kinoshita, PRL 2 (1959) 477*

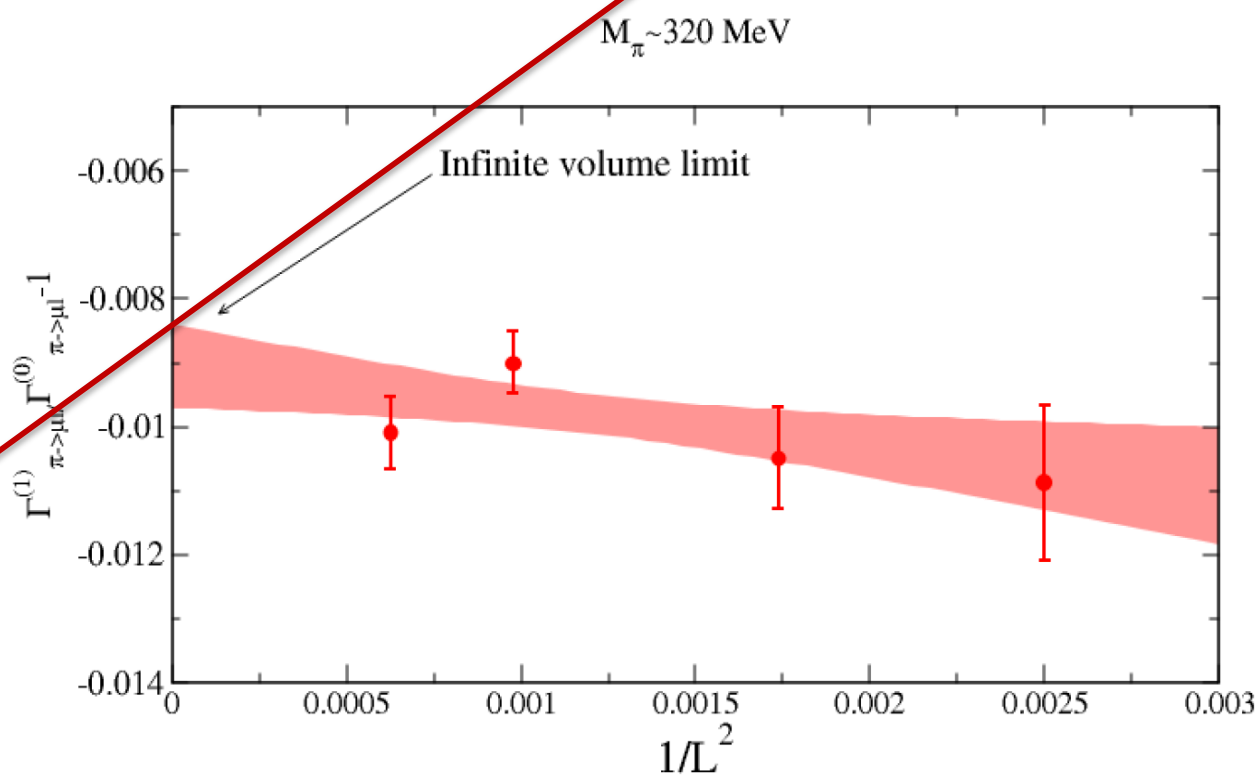
With our definition of the photon propagator we use the finite volume as infrared regulator. Finite volume effects have the form

$$\Gamma_0^{pt}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\,\log(m_\pi L) + \frac{C_1(r_\ell)}{m_\pi L} + O\left(\frac{1}{(m_\pi L)^2}\right)$$

where $r_l = m_l / m_\pi$ and m_l is the mass of the final lepton.

The terms above are universal, namely independent of the structure of the pion Thus in $\Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt}$ the infrared divergence is cancelled and the finite volume corrections start at O(1/L^2)

V.Lubicz, G.Martinelli, CTS, F.Sanfilippo, S.Simula, N.Tantalo, arXiv:1611.08497



RM123+SOTON, PRL 120 (2018)

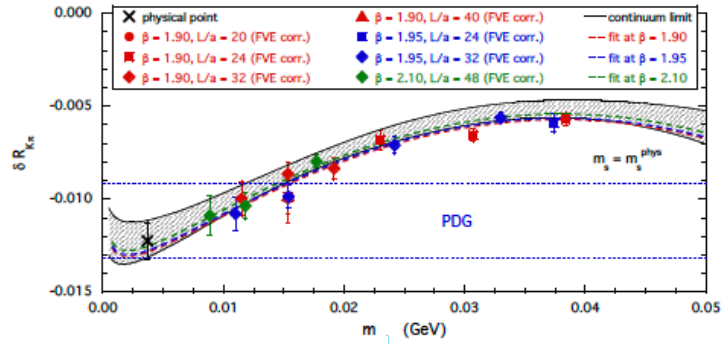
- with this method, our result for

$$\Gamma_P(E) = \Gamma_P^0 \{1 + \delta R_P(E)\} \ ,$$

$$\delta R_{K\pi} = \delta R_K(E_K^{max}) - \delta R_\pi(E_\pi^{max})$$

is the following

$$\delta \mathbf{R}_{\mathbf{K}\pi}$$
$$= -0.0122(10)^{st}(2)^{tun}(8)^{\chi}(5)^{L}(4)^{a}(\ldots^{aQED}$$
$$= \mathbf{-0.0122(16)}$$



$$\frac{|V_{us}|}{|V_{ud}|} = 0.23134\,(24)_{\rm exp}\,(30)_{\rm th} = 0.23134\,(38)$$

ETMC gauge configurations

$n_f = 1+1+1+1$

$a \geq 0.0619(18)$ fm

- this can (remember the caveat concerning the definition of QCD) be compared with the result currently quoted by the PDG and obtained in v.cirigliano and h.neufeld, PLB 700 (2011)

$$\delta R_{K\pi} = -0.0112(21)$$

NEW: Phys. Rev. D100 (2019) no. 3, 034514

$$\delta R_{K\pi}^{\rm phys} = \delta R_K^{\rm phys} - \delta R_\pi^{\rm phys} = -0.0126\,(14)$$

$$\frac{|V_{us}|}{|V_{ud}|} = 0.23134\,(24)_{\rm exp}\,(30)_{\rm th} = 0.23134\,(38)$$

$$|V_{us}| = 0.22538\,(24)_{\rm exp}\,(30)_{\rm th} = 0.22538\,(38)$$

MASTER FORMULA for the rate at O(α)

$$\Gamma(\Delta E) = \lim_{V\to\infty}(\Gamma_0 - \Gamma_0^{\rm pt}) + \lim_{V\to\infty}(\Gamma_0^{\rm pt} + \Gamma_1(\Delta E))$$

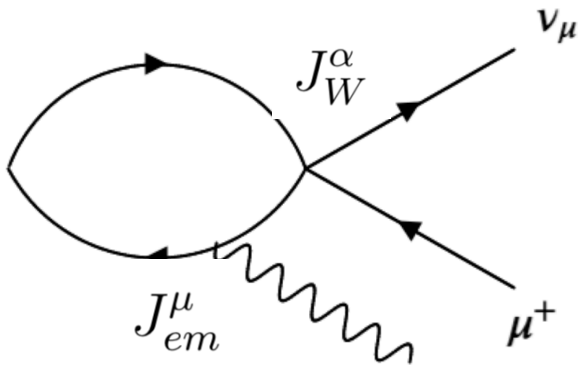
pt = point-like & perturbative

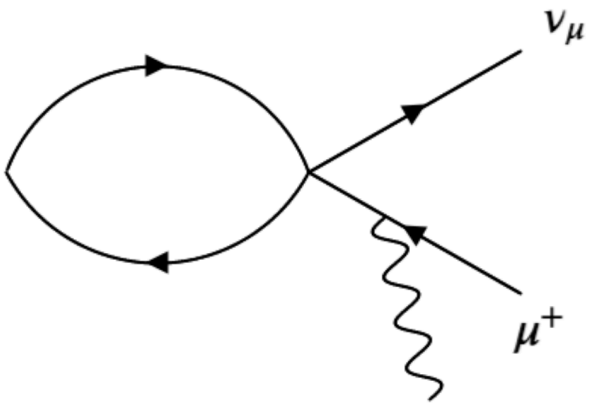
$$\Gamma_0 + \Gamma_1 = \lim_{V\to\infty}(\Gamma_0 - \Gamma_0^{\rm pt}) + \lim_{V\to\infty}(\Gamma_0^{\rm pt} + \Gamma_1^{\rm pt}) + \lim_{V\to\infty}(\Gamma_1 - \Gamma_1^{\rm pt})$$

1) *Each of the three terms is infrared finite*
2) *The last term is important for heavy meson phenomenology where χPT cannot be applied*

We now discuss the <u>non-perturbative</u> determination of $\Gamma_1(\Delta E)$

Real photons





- the leptonic part factorizes;
- the photon is not really there, we just give a pinch that carries away some momentum at the vertex where the (conserved) electromagnetic current is inserted;
- <u>therefore finite volume effects are exponentially suppressed</u>.

- can be computed in perturbation theory;
- it has no relation with the non perturbative structure of the hadron.

The relevant hadronic amplitude (T-product)

$$H_W^{\alpha\, r}(k,p) = \varepsilon_\mu^r(k)\, H_W^{\alpha\,\mu}(k,p) = \varepsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\, \langle 0|T\{j_W^\alpha(0) j_{em}^\mu(y)\}|P(p)\rangle$$

- $j_{em}^\mu(y)$ electromagnetic current
- $j_W^\alpha(0)$ weak current
- $P(p)$ pseudoscalar meson with momentum p
- $\epsilon_\mu^r(k)$ polarisation vector of the photon with momentum *k*

Decomposition of the amplitude in form-factors

$$
\begin{aligned}
H_W^{\alpha\,\mu}(k,p) &= H_1\left[k^2 g^{\alpha\,\mu} - k^\alpha k^\mu\right] + H_2\left[\left(p\cdot k - k^2\right)k^\mu - k^2\,(p-k)^\mu\right](p-k)^\alpha \\
&- i\frac{F_V}{m_P}\epsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \frac{F_A}{m_P}\left[\left(p\cdot k - k^2\right)g^{\alpha\,\mu} - k^\alpha\,(p-k)^\mu\right] \\
&+ f_P\left[g^{\alpha\,\mu} + \frac{(2p-k)^\mu\,(p-k)^\alpha}{2p\cdot k - k^2}\right]
\end{aligned}
$$

The last term is dictated from the Ward id

$$k_\mu H_W^{\alpha\,\mu}(k,p) = i\langle 0|j_W^\alpha|P(p)\rangle = f_P\,p^\alpha$$

it is the same of a point-like scalar particle

- 4 independent form-factors; it is related to the i.r. divergence
- the form factors only depend on k^2 e p.k;
- we will only discuss the real photon case, $k^2 = 0$ $\varepsilon^r.k = 0$;
- in the future it may be interesting to consider $\Gamma^{exp}(K \to \ell\nu_\ell\, l^+ l^-)$

Amplitude: decomposition in scalar form-factors

$$
\begin{aligned}
H_W^{\alpha\,\mu}(k,p) &= H_1\left[k^2 g^{\alpha\,\mu} - k^\alpha k^\mu\right] + H_2\left[\left(p\cdot k - k^2\right)k^\mu - k^2\,(p-k)^\mu\right](p-k)^\alpha \\
&- \frac{F_V}{m_P}\,\epsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \frac{F_A}{m_P}\left[\left(p\cdot k - k^2\right)g^{\alpha\,\mu} - k^\alpha\,(p-k)^\mu\right] \\
&+ f_P\left[g^{\alpha\,\mu} + \frac{(2p-k)^\mu\,(p-k)^\alpha}{2p\cdot k - k^2}\right]
\end{aligned}
$$

Real Photon Emission

$$
H_W^{\alpha\,\mu}(k,p) = -\imath\frac{F_V}{m_P}\epsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \left(\frac{F_A}{m_P} + \frac{f_P}{p\cdot k}\right)\left(p\cdot k\, g^{\alpha\,\mu} - k^\alpha p^\mu\right) + \frac{f_P}{p\cdot k}\,p^\alpha p^\mu
$$

1. different tensors can be separated by using suitable projectors, we then have to disentangle F_A from f_P ;
2. the point-like term is related to the infrared divergence in 1/p.k that has to cancel the corresponding divergence of the virtual correction in the rate;

It is F_A (together with F_V) the relevant structure dependent quantity to be determined.

Lattice Calculation

Relevant Correlator

$$E_\gamma = |\mathbf{k}|$$

$$C_W^{\alpha r}(t,\mathbf{p},\mathbf{k}) = \epsilon_\mu^r(\mathbf{k}) \int d^4y\; d^3x\; e^{t_y E_\gamma - i\mathbf{k}\cdot\mathbf{y} + i\mathbf{p}\cdot\mathbf{x}}\; \mathrm{T}\langle 0|j_W^\alpha(t) j_{em}^\mu(y) P(0,\mathbf{x})|0\rangle$$

suitable projector on the different form factors

Euclidean extraction of the photon momentum k

source of the pseudoscalar meson with momentum p

the convergence of the integral over t_y is ensured by the safe analytic continuation from Minkowsky to Euclidean, namely by the absence of intermediate states lighter than the pseudoscalar meson. The physical form factors can be extracted directly from the Euclidean correlation functions

$$\sum_n \int_{-\infty}^0 dt\; e^{tE_\gamma} \langle 0|j_W^\alpha(0)|n\rangle\langle n|e^{tH} j_{em}^\mu(0,\mathbf{k}) e^{-tH}|P\rangle \;\to e^{t(E_\gamma + E_n - E_P)}$$

$$E_\gamma + E_n > E_P$$

Define

$$R^{\alpha r}(t) = \frac{2E}{e^{-t(E-E_\gamma)}\langle P|P|0\rangle} C_W^{\alpha r}(t, \mathbf{p}, \mathbf{k})$$

then $(t \leq T/2)$

$$\lim_{t\to\infty} R^{\alpha r}(t) = H_W^{\alpha r}(p, k)$$

depending on the projector we can isolate the vector or axial vector form factors

example, taking $\vec{p} = (0,0,|\vec{p}|)$ and $\vec{k} = (0,0,E_\gamma)$ we can choose for the basis of polarisation vectors $\varepsilon^\mu_{1,2} = (0, \mp\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$ so that

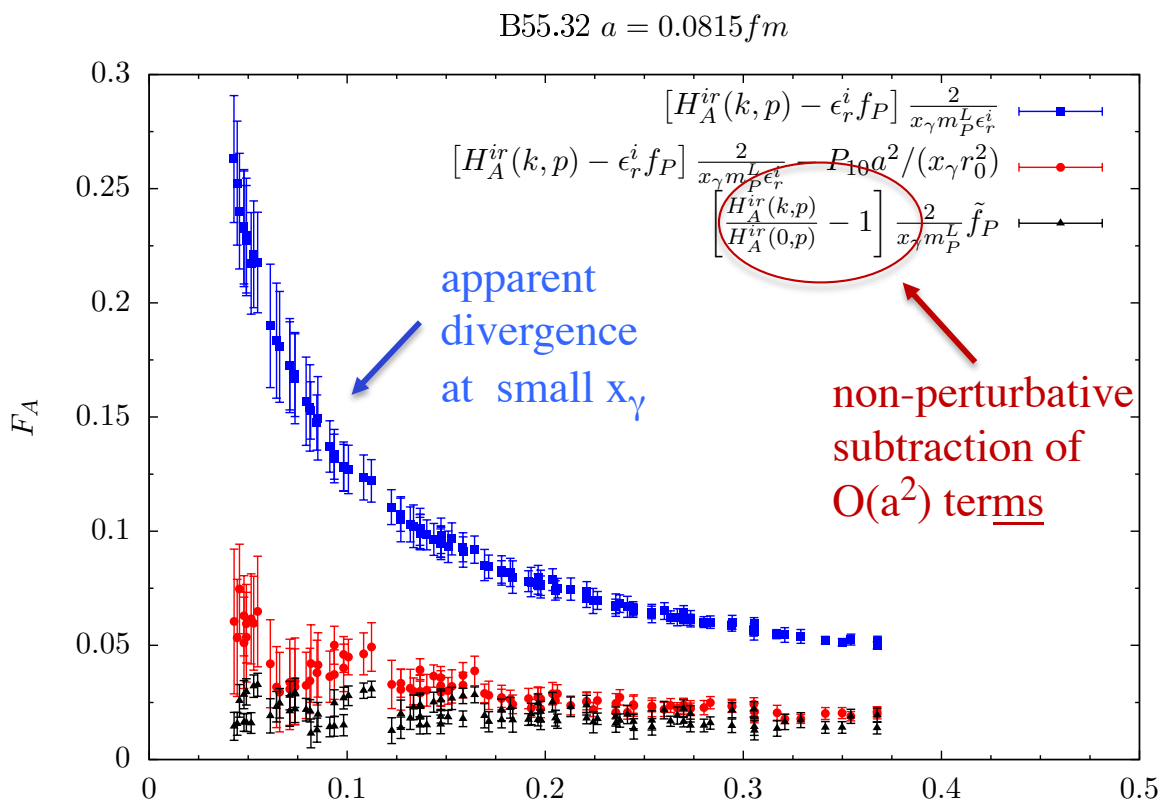
$$R_A^{jr}(t,\vec{p},\vec{k}) \to \frac{\varepsilon_r^j m_P}{2} x_\gamma \left[F_A(x_\gamma) + \frac{2f_P}{m_P x_\gamma}\right], \quad R_V^{jr}(t,\vec{p},\vec{k}) \to \frac{i\left(E_\gamma \vec{\varepsilon}_r \times \vec{p} - E\vec{\varepsilon}_r \times \vec{k}\right)^j}{m_P} F_V(x_\gamma)$$

where the subscripts A, V on R^{jr} label the components of the weak current.

A subtle point due to discretization errors

$$G_A(x_\gamma) = \left[x_\gamma F_A(x_\gamma) + \frac{2 f_P}{m_P} \right]$$

A possible strategy would be the subtraction of the term proportional to f_P since this coefficient is fixed by the Ward identity of the electromagnetic current



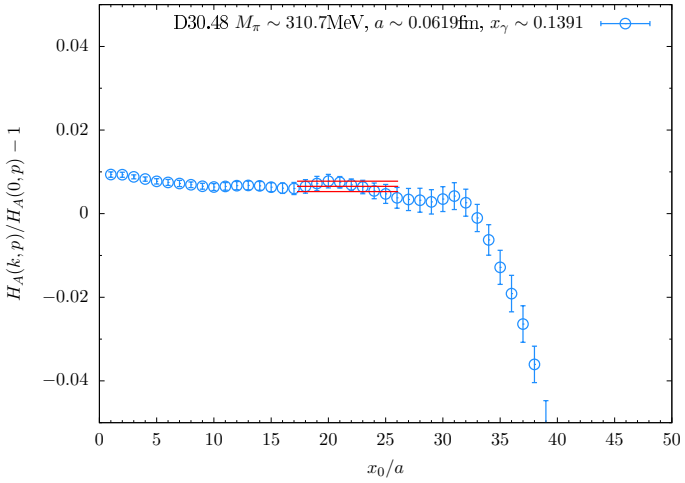
$$G_A(x_\gamma) = x_\gamma \left[F_A(x_\gamma) + a^2 \Delta F_A(x_\gamma) \right] + \frac{2}{m_P} \left(f_P + a^2 \Delta f_P \right) + \cdots$$

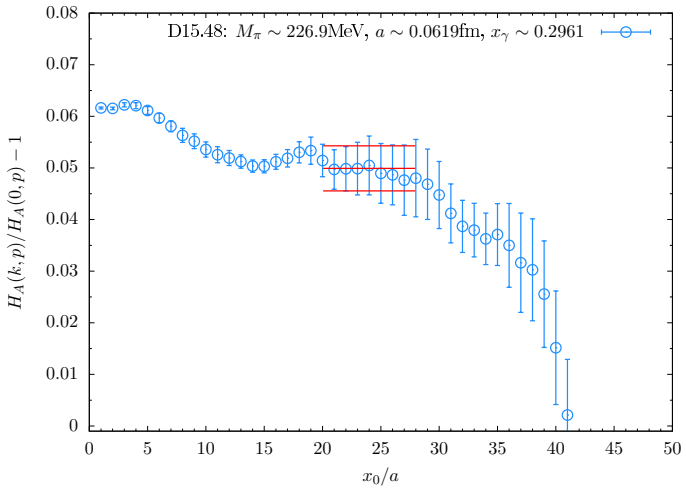
$$F_A^{sub}(x_\gamma) = \frac{1}{x_\gamma} \left[G_A(x_\gamma) - \frac{2 f_P}{m_P} \right] = F_A(x_\gamma) + a^2 \Delta F_A(x_\gamma) + \frac{O(a^2)}{x_\gamma}$$

Numerical Results

(all the results in the following are preliminary)

- *2+1+1 twisted mass fermions*
- *Three different lattice spacings*
- *3-4 different pion masses for each lattice spacing*
- *Pion masses down to ~230 MeV*
- *100 different momentum configurations*





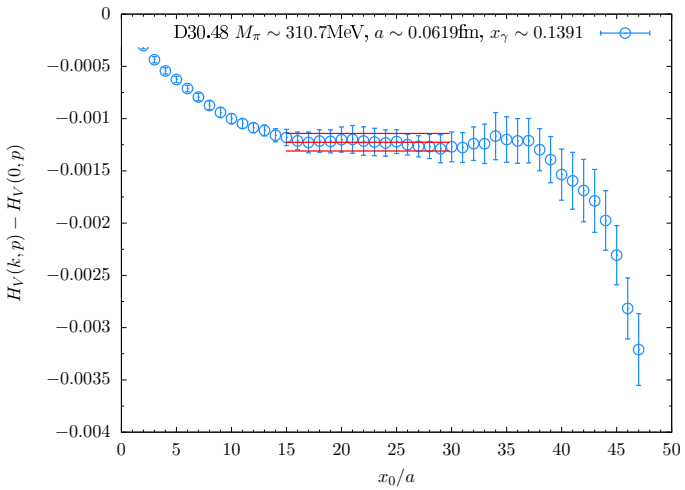
Axial-Vectors (F_A)

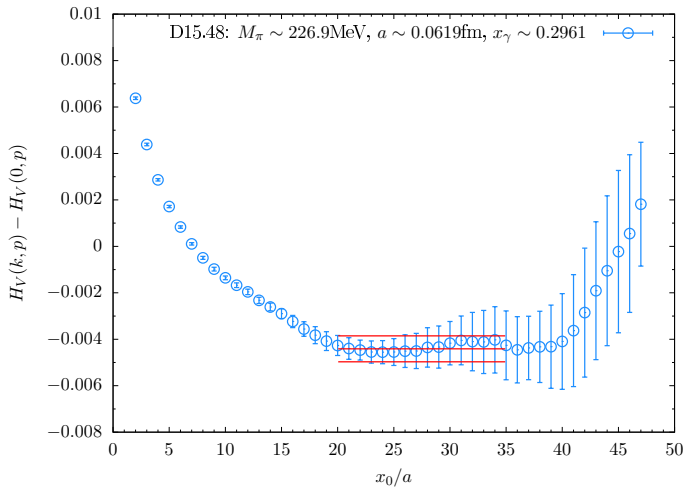
K ***D_s***

$$\lim_{t\to\infty} \overset{\alpha r}{R}(t) = H_W^{\alpha r}(p,k)$$

for the plateaus we find rather good signals (we are optimizing them)

Vectors (F_V)





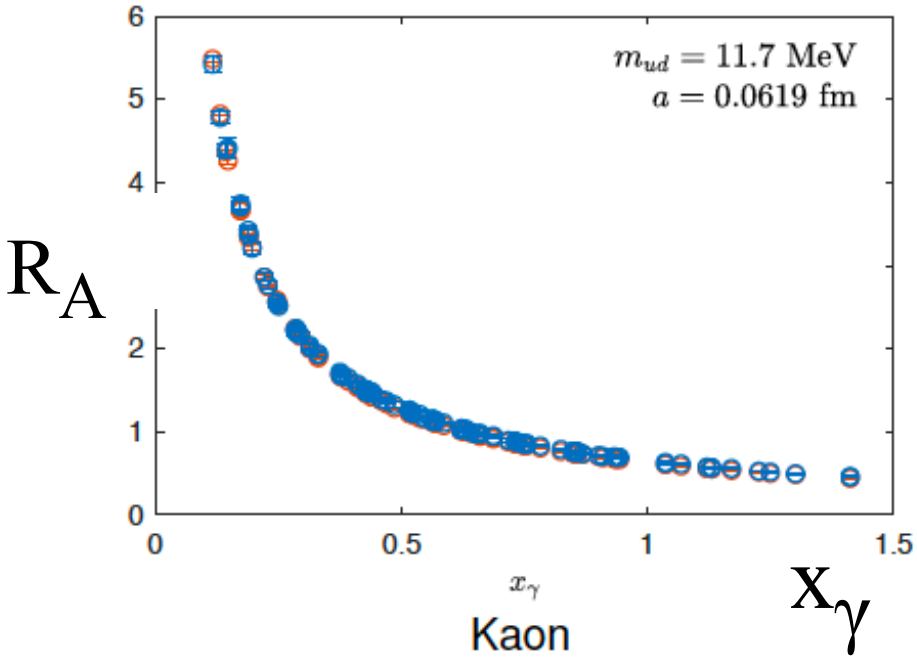
K ***D_s***

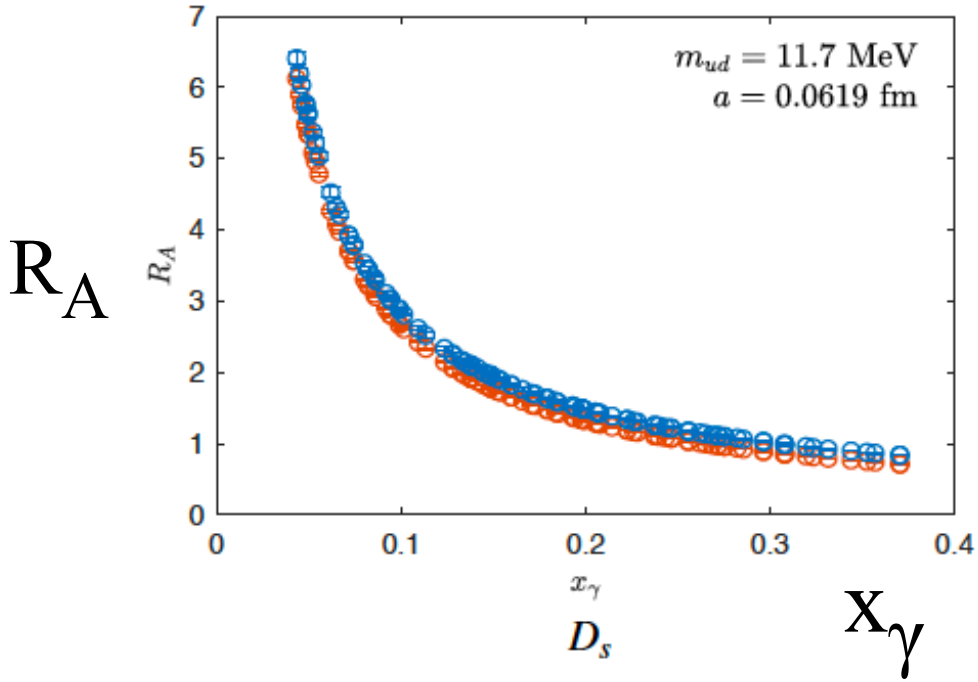
Axial-vector channel for the Kaon and D_s

m_{ud} = 11.7 MeV

a=0.0619 fm

$$R_A = \frac{F_A}{m_P} + \frac{f_P}{p \cdot k}$$





- $R_A = F_A + \frac{2f_P}{m_P x_\gamma}$; $\quad R_A^{\rm pt} = \frac{2f_P}{m_P x_\gamma}$
- For light mesons the SD form-factors are small.

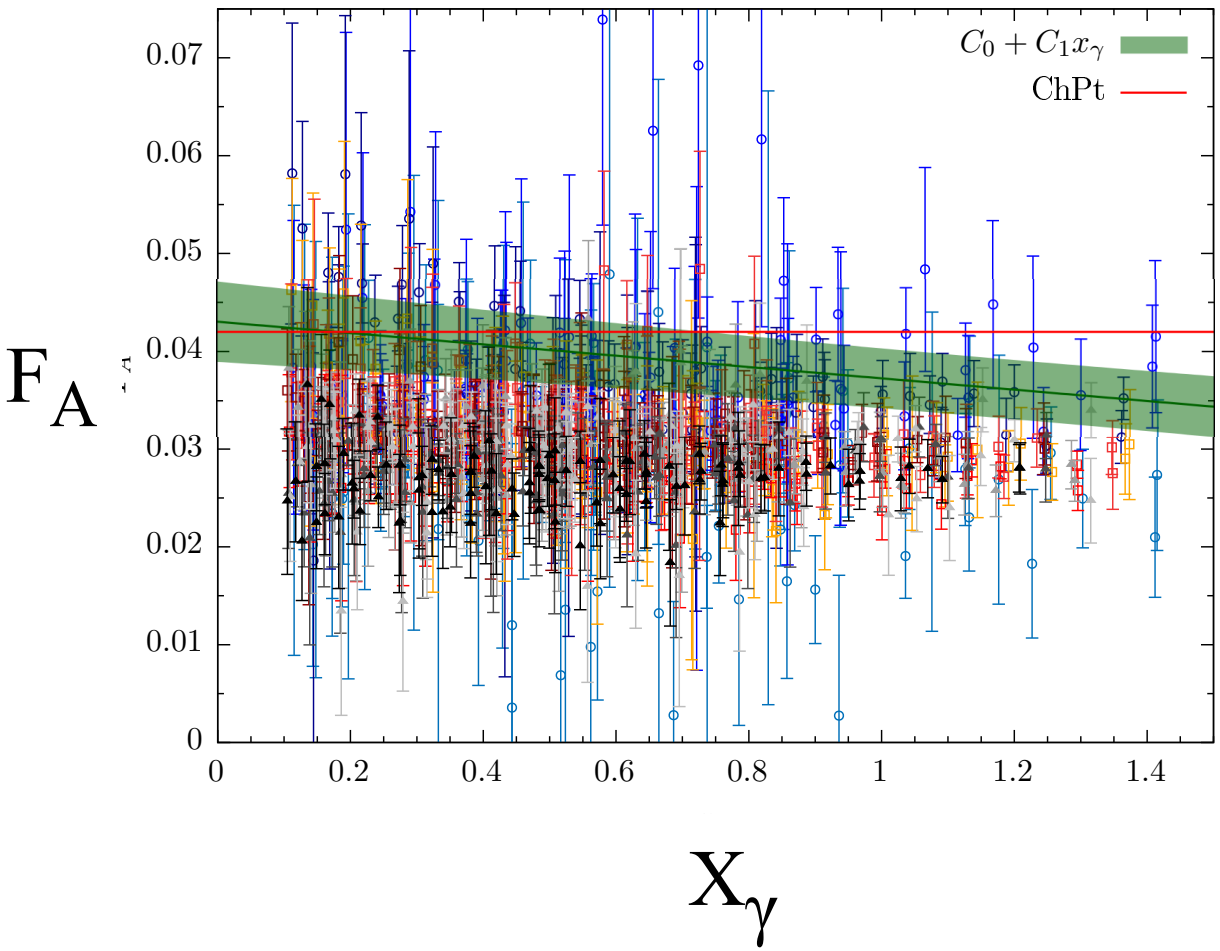
$$x_\gamma = \frac{2p \cdot k}{m_P^2}$$

we may compare with χPT that at $O(p^4)$ gives without any momentum dependence

$$F_A = \frac{8m_K}{f_\pi}\left(L_9^r + L_{10}^r\right)$$

V. Cirigliano et al. , Rev. Mod. Phys. 84 (2012) 399 [arXiv:1107.6001];

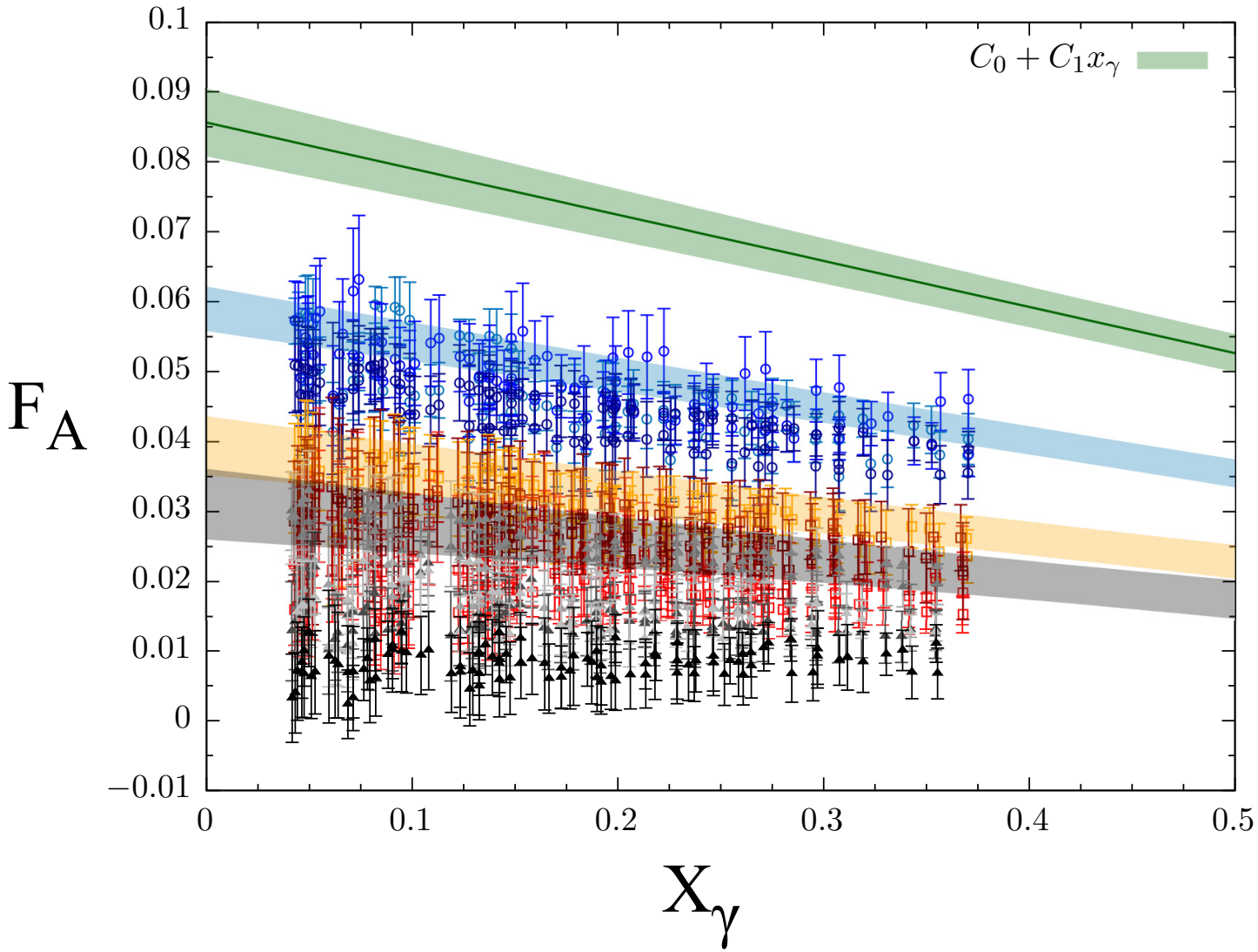
J. Bijnens and G. Ecker, Ann. Rev. Nucl. Part. Sci. 64 (2014) 149 [arXiv:1405.6488].



Kaon F_A form factor

with an improved analysis we areable to extract the momentum dependence too

we cannot compare this case with χPT



Ds

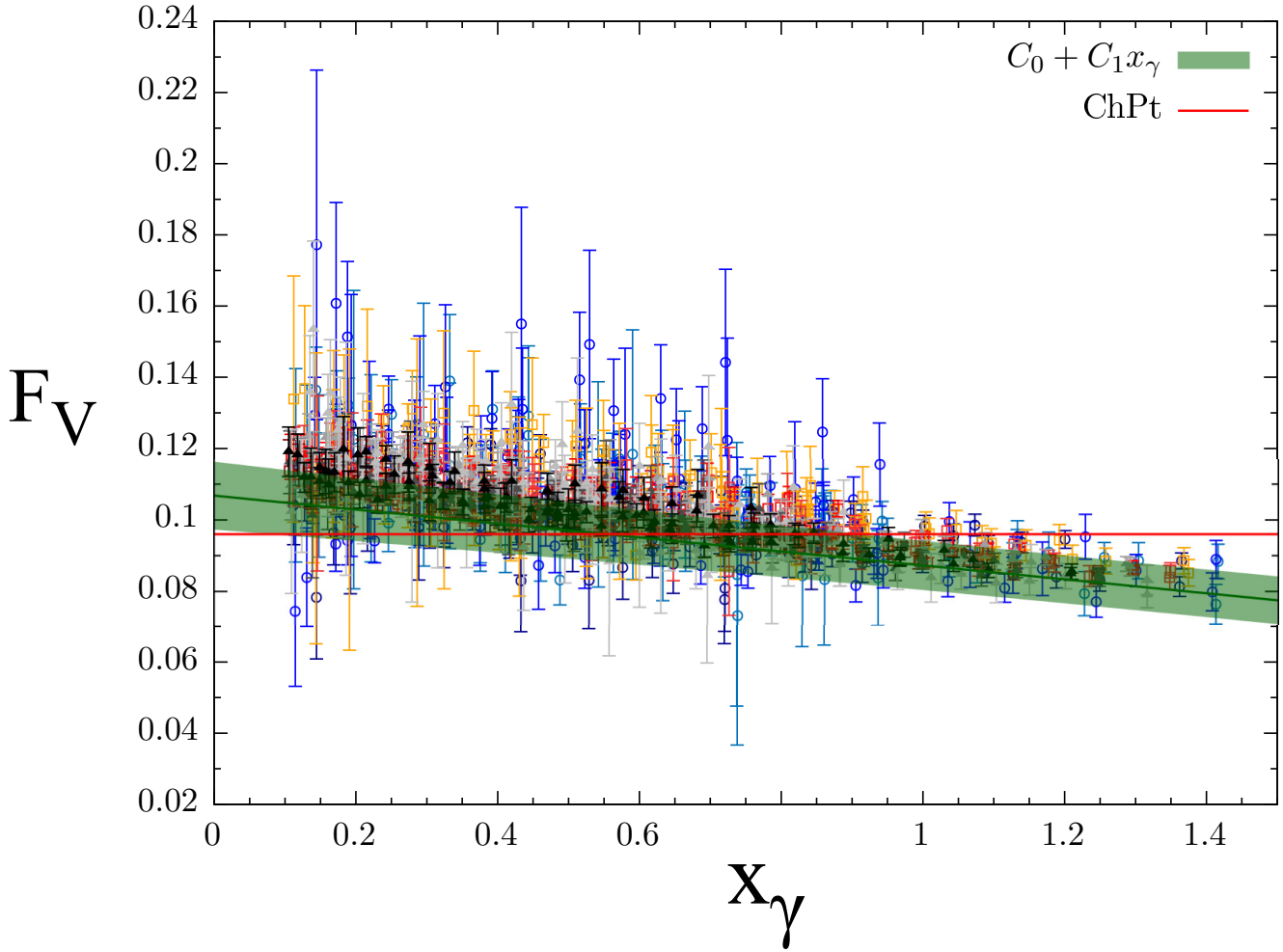
F_A form factor

large discretization effects observed but the extrapolation to the continuum is safe

we are able to extract the momentum dependence of F_A

F_V form factor for the Kaon

χPT

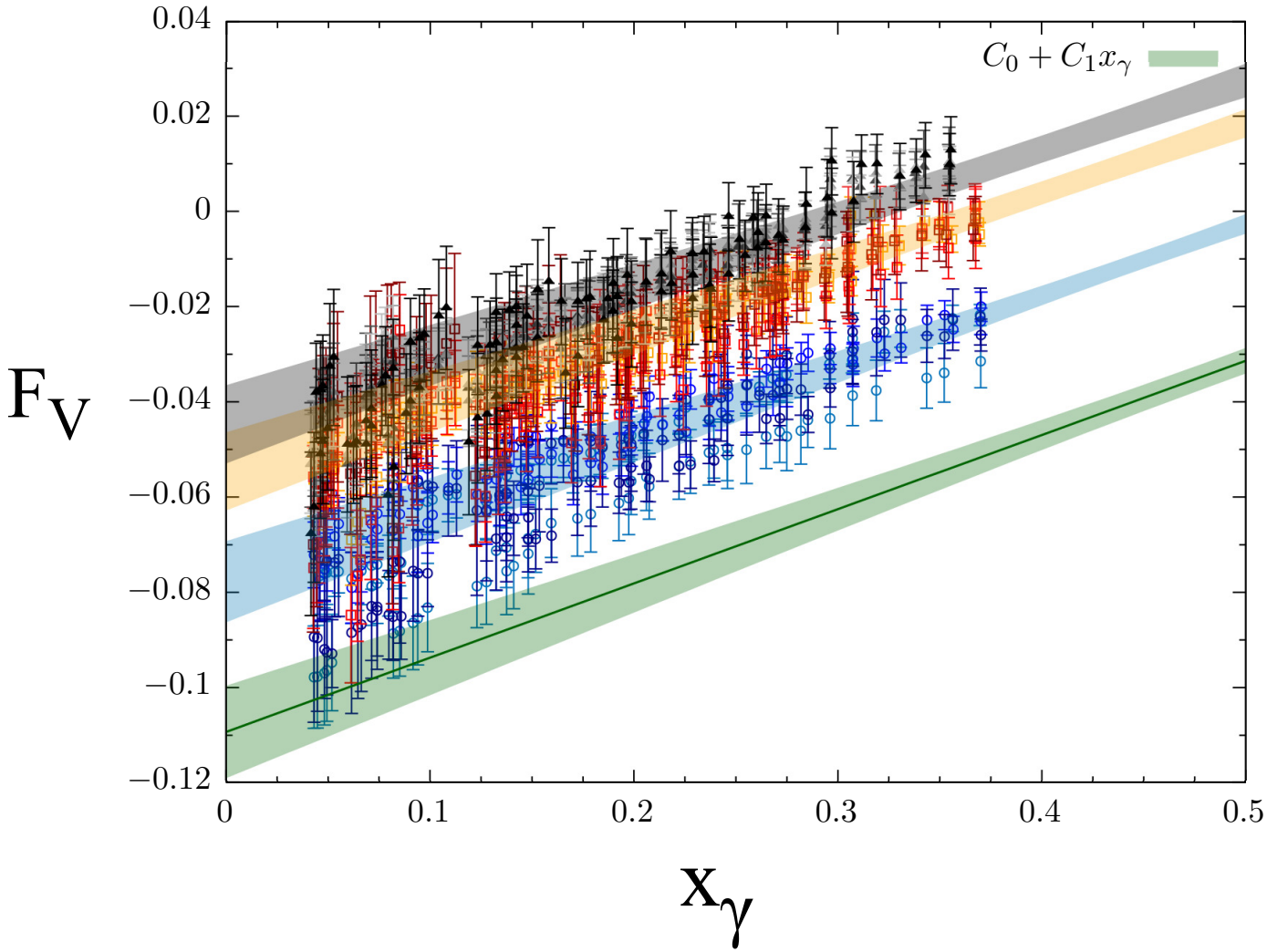


$$F_V = \frac{8\, m_K}{4\pi^2 f_\pi}$$

with an improved analysis we areable to extract the momentum dependence too

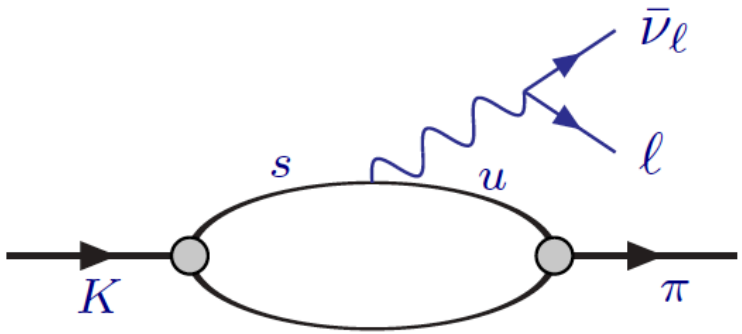
Similar results were found for the other mesons π, D and D_s

F_V form factor for the Ds



Similar results were found for the D with slightly larger uncertainties; (not all the physical range in x_γ is covered)

Semileptonic Decays: similarities and differences



Semileptonic decays are characterized by a vector and a scalar form factor, $f_+(q^2)$ and $f_0(q^2)$ respectively, corresponding to the hadronic quantity f_π (f_K , f_D ..) in the case of leptonic decays. $q=p_K-p_\pi=p_l-p_\nu$

$$\langle \pi^+(p_\pi)|\bar{u}\gamma_\mu s|\bar{K}^0(p_k)\rangle = f_+(q^2)\left[(p_k+p_\pi)_\mu - \frac{(m_K^2-m_\pi^2)}{q^2}q_\mu\right] + f_0(q^2)\frac{(m_K^2-m_\pi^2)}{q^2}q_\mu$$

For our discussion of the analytic continuation, of the infrared divergences and of finite volume effects the useful variables are q^2 and the invariant mass of the final pion-charged lepton pair $s=(p_\pi+p_l)^2$. In terms of these variables we have

$$\frac{d^2\Gamma}{dq^2ds} = G_F^2|V_{us}|^2\left(a_+(q^2,s)|f_+(q^2)|^2 + a_0(q^2,s)|f_0(q^2)|^2 + a_{+0}(q^2,s)f_+(q^2)f_0(q^2)\right)$$

where the kinematical coefficients $a_i(q^2, s)$ can be easily determined

Following the procedure used for leptonic decays we write

$$\frac{d^2\Gamma}{dq^2 ds} = \lim_{V\to\infty}\left(\frac{d^2\Gamma_0}{dq^2 ds} - \frac{d^2\Gamma_0^{pt}}{dq^2 ds}\right) + \lim_{V\to\infty}\left(\frac{d^2\Gamma_0^{pt}}{dq^2 ds} + \frac{d^2\Gamma_1^{pt}(\Delta E)}{dq^2 ds}\right)$$

where the two terms are now infrared safe. For soft photons with $\Delta E/\Lambda_{QCD} << 1$ we can compute the real emission using the eikonal approximation and the virtual one using a simple effective model with suitable defined form factors (with many caveat however).

The analogy however is not completely true: analytic continuation from Minkowsky and 1/L corrections are different

Semileptonic Decays: New problems in the continuation from Minkowski to Euclidean

Work in progress

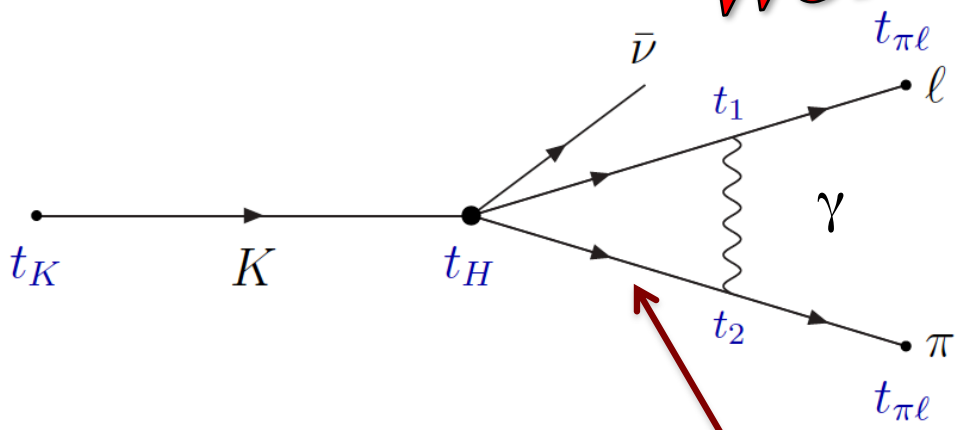


Figure 1: Diagram contributing to the $K \to \pi\ell\bar{\nu}$ decay amplitude.

A new problem may arise if there are intermediate π-l (or π-l-γ, π-γ, l-γ) states which have a smaller energy than the external π-l system.

$$G(t_H, t_{\pi\ell}) \sim e^{\Delta E\,(t_{\pi\ell} - t_H)}$$

$$\boxed{\Delta E = E^{ext}_{\pi\ell} - E^{int}_{\pi\ell}}$$

Remember that in QED_L the photon will always have a non-zero minimum energy

A close inspection, up to hadronic form factors, shows that

$$G(t_H, t_{\pi\ell}) = \frac{1}{8k\omega'_\pi\omega'_\ell}\Big[\frac{1}{\Delta E}\left(\frac{1}{\Delta\omega_\pi - k} + \frac{1}{\Delta\omega_\ell - k}\right) + \frac{1}{\Delta E}\left(\frac{1}{\Delta\omega_\pi + k} + \frac{1}{\Delta\omega_\ell + k}\right) e^{\Delta E\,(t_{\pi\ell} - t_H)}$$

$$- \frac{1}{(\Delta\omega_\pi - k)(\Delta\omega_\ell + k)}\, e^{(\Delta\omega_\pi - k)\,(t_{\pi\ell} - t_H)} - \frac{1}{(\Delta\omega_\pi + k)(\Delta\omega_\ell - k)} e^{(\Delta\omega_\ell - k)\,(t_{\pi\ell} - t_H)}$$

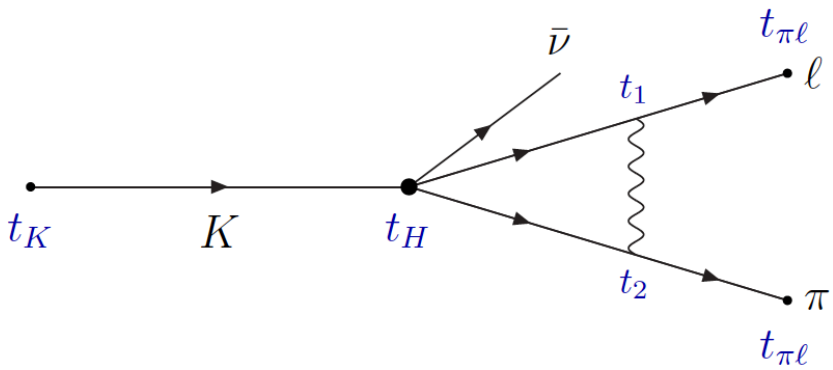
physical term

$$\times(\text{hadronic form factor})$$

$$\omega_{\pi,\ell} = \sqrt{m^2_{\pi,\ell} + \vec{p}^{\,2}_{\pi,\ell}} \quad \omega'_{\pi,\ell} = \sqrt{m^2_{\pi,\ell} + (\vec{p}_{\pi,\ell} \pm \vec{k})^2} \qquad \Delta\omega_{\pi,\ell} = \omega_{\pi,\ell} - \omega'_{\pi,\ell}$$

note that the last two terms are always well behaved at large $t_{\pi l}$-t_H since, as in the leptonic case, $\Delta\omega_l$-k < 0 and $\Delta\omega_\pi$-k <0 when we remove photon zero mode in the finite volume (necessity of a mass gap).

The dangerous term is instead the term proportional to ΔE



In general, depending on the volume and the pion-lepton invariant mass s, there are unphysical contributions from lighter intermediate π l (γ) states, which grow exponentially with the temporal integration region and must be subtracted (as in the Lellouch-Luscher formula for K$\pi\pi$ decays). In fact this is a general feature in the calculation of long distance effects.

For semileptonic decays of heavy mesons however, for much of phase space there are too many lighter intermediate states to handle (and above the inelastic threshold). This is analogous to the fact that e.g. B $\pi\pi$ and B πK decays amplitudes cannot be calculated whereas K $\pi\pi$ amplitudes can.

We also need to study the FV corrections due to the electromagnetic rescattering.

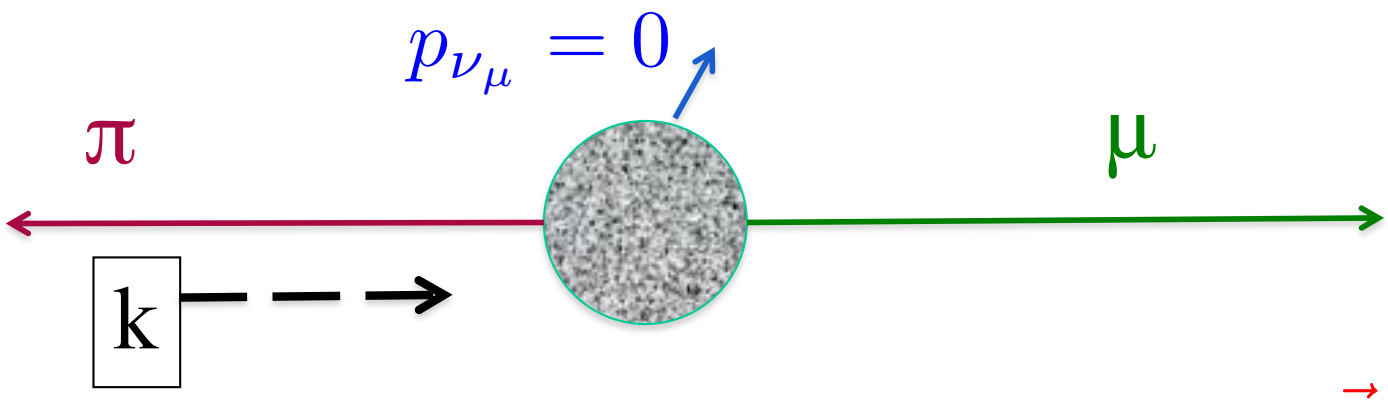
Analytic continuation continued

The relevant parameter is the invariant mass of the π-l pair

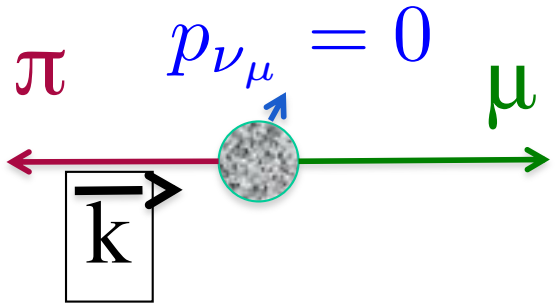
$$(m_\mu + m_\pi)^2 \leq M^2_{\pi\ell} \leq m_K^2$$

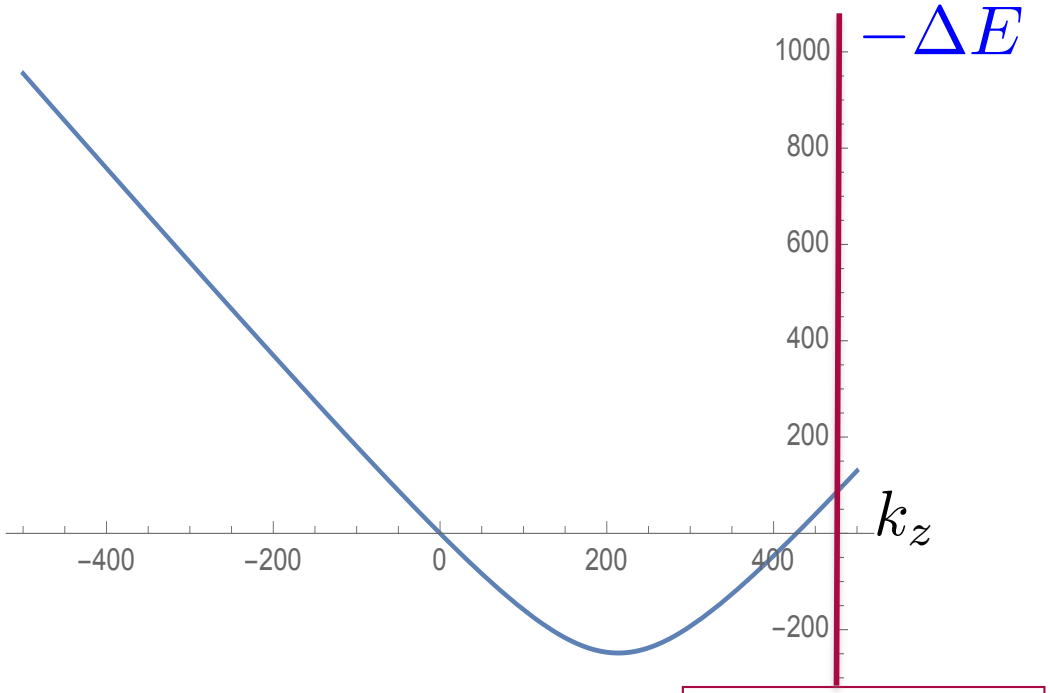
which, in turn, is determined by the neutrino momentum [or $q^2 = (p_\pi - p_l)^2$]. When the invariant mass $M_{\pi l}$ is large ΔE become positive.

$$- \Delta E = \sqrt{m_\pi^2 + (\vec{p}_\pi + \vec{k})^2} + \sqrt{m_\ell^2 + (\vec{p}_\ell - \vec{k})^2} - \sqrt{m_\pi^2 + \vec{p}_\pi^2} - \sqrt{m_\ell^2 + \vec{p}_\ell^2}$$



for large $M_{\pi l}$ max ΔE corresponds to $\vec{k} = -\vec{p}_\pi$


π
$p_{\nu_\mu} = 0$
μ
$\vec{k}$


$-\Delta E$
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600
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200
−200
−400
−200
0
200
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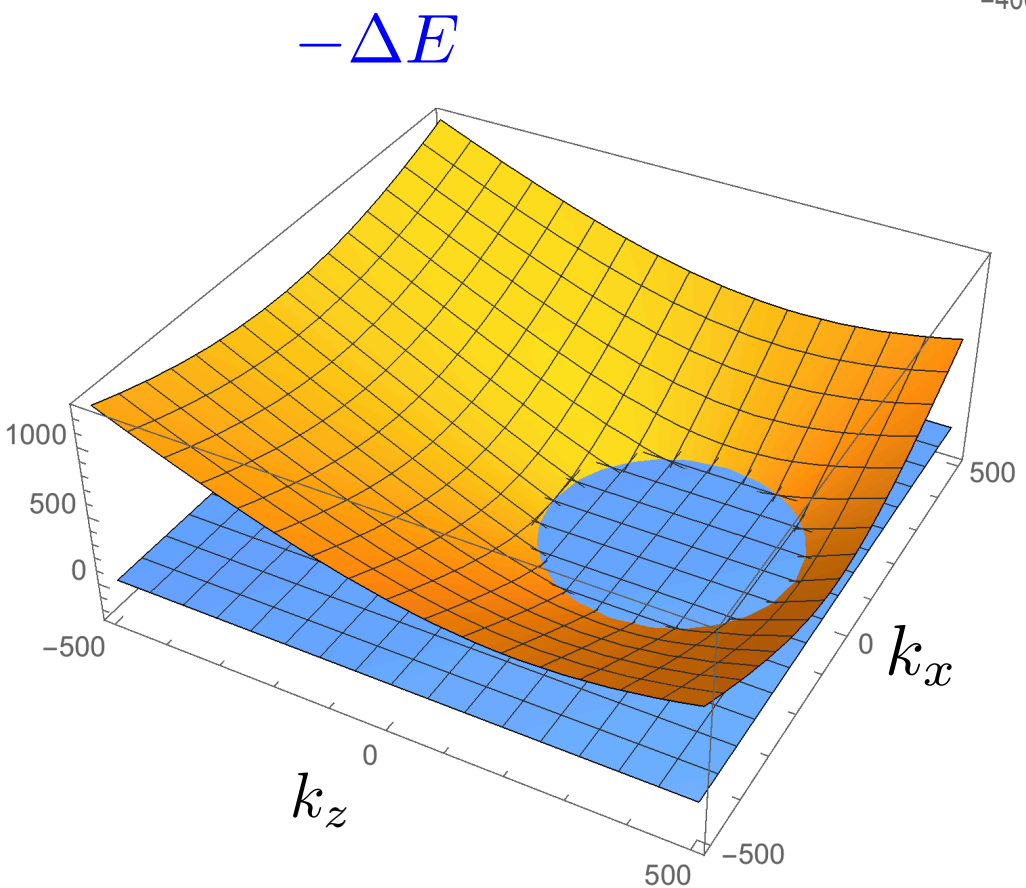
k_z

a=0.0815
& L=32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k_z
500
0
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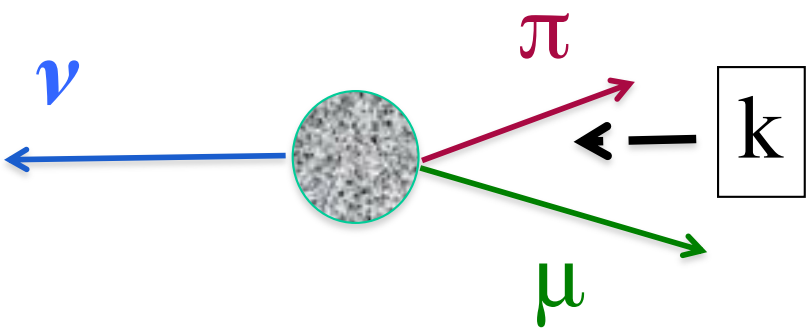
k_x


ν
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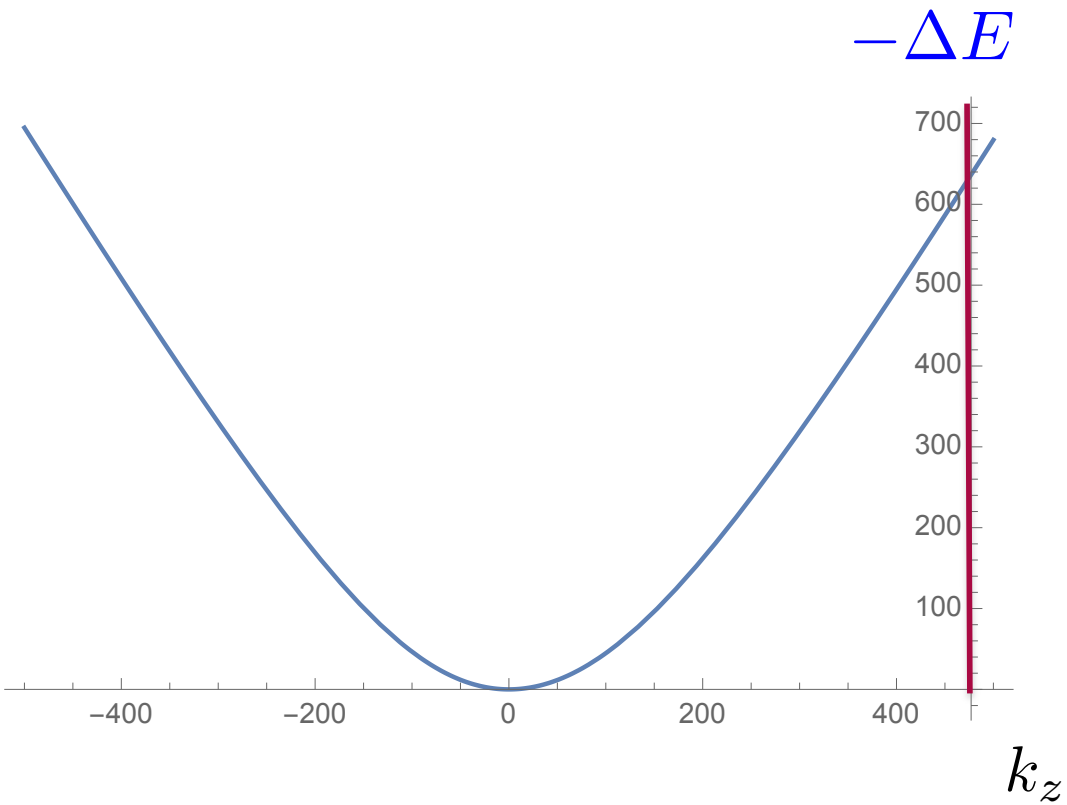
$-\Delta E$
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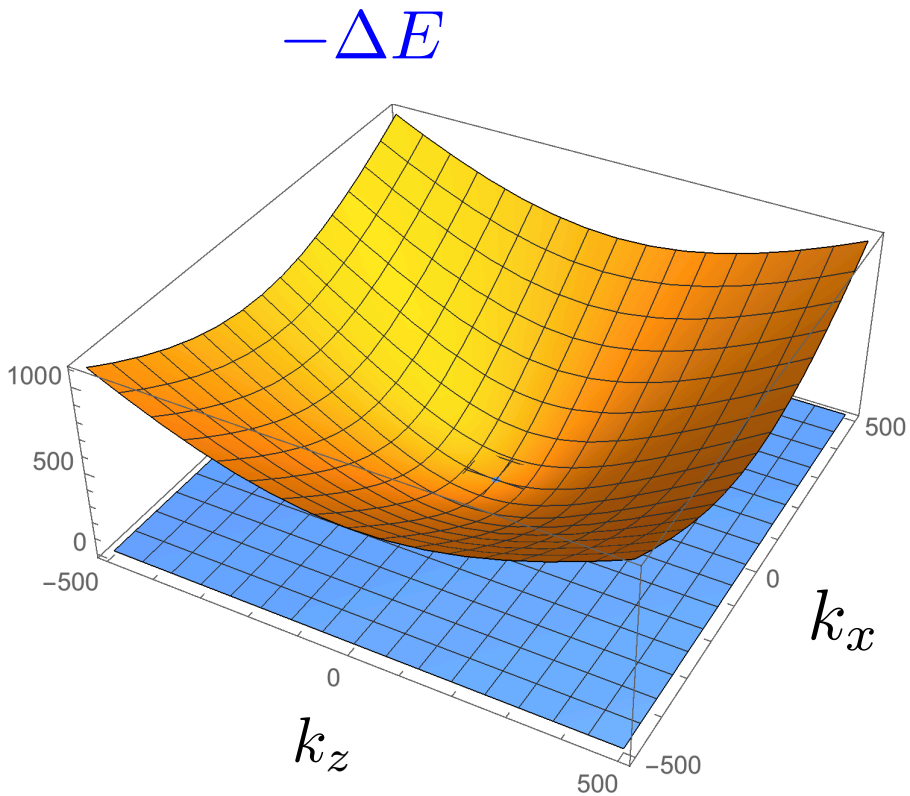
$-\Delta E$
1000
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k_z
500
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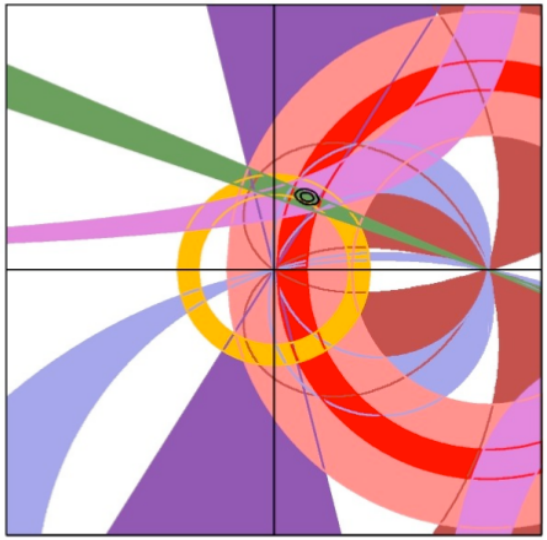
k_x

Is the presence of intermediate states with a lower energy really a problem?

Certainly is a complication

But it is not new…..



The energy non-conserving matrix elements with initial or final states having energy E^{int} can also be calculated to aid in the subtraction of the exponentially growing terms.

- IV reconstruction method to be developed for the SL decays.

- D and B decays: the large number of such terms which need to be subtracted in most of phase space, makes it *very difficult* to implement the method.
- No such exponentially growing terms are present for leptonic decays.

Γ_0^{pt} *Universality of the logarithmically divergent term and of the 1/L correction*

$$\Gamma_0^{\rm pt}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\mathrm{Log}\left[m_P L\right] + \frac{C_1(r_\ell)}{m_P L} + \ldots$$

Depends on the IR regularization. The regularization dependent part does not depend on the internal structure of the hadron

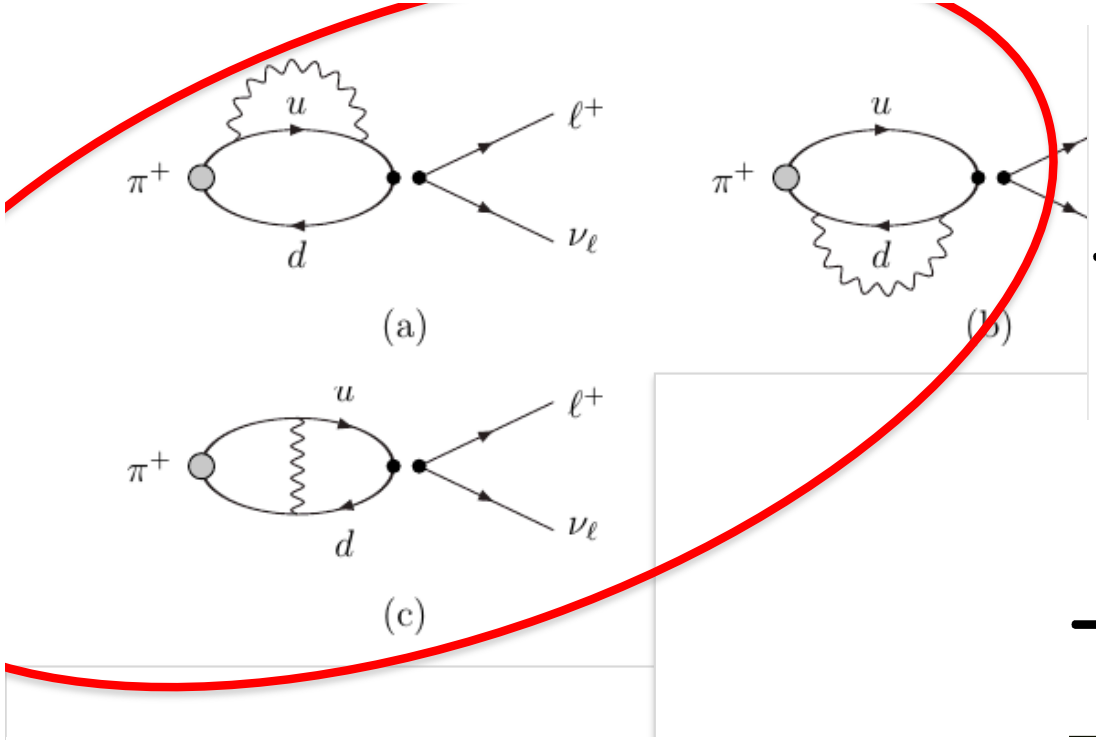
Does NOT depend on the ir regularization or on the internal structure of the hadron

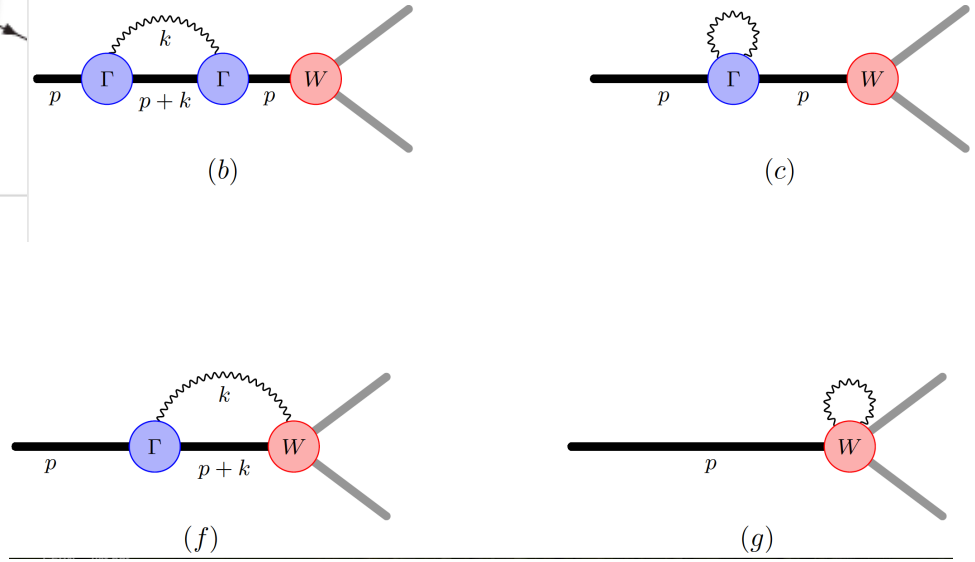
BMW, Science 347 (2015) 1452
B. Lucini et al., JHEP 1602 (2016)

Thus $\Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt}$ = Infrared finite, independent of the regularization up to $O(1/L^2)$

Universality demonstrated via skeleton expansion or effective theory

$$\Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt}$$





NON-UNIVERSAL TERMS ENTER WITH HIGHER POWERS OF THE PHOTON MOMENTUM i.e. LESS SINGULAR TERMS AND CORRESPOND TO HIGHER POWERS IN 1/L A practical rule summarising the relation between the power of the finite-volume corrections and the leading singularity of the integrand at k=0 is

$$\Delta\text{Loop} = \int \frac{dk_0}{2\pi} \left(\frac{1}{L^3} \sum_{\vec{k}\neq 0} - \int \frac{d^3k}{(2\pi)^3} \right) \frac{1}{(k^2)^{n/2}} = O\left(\frac{1}{L^{4-n}} \right)$$

Using only gauge invariance (Ward ids) one has

$$\Gamma^\mu(p,k) = (2p+k)^\mu + 4z_1 p^\mu\, p\cdot k + 4z_1\epsilon^2 p^\mu + O(k^2,\epsilon^4,\epsilon^2 k)$$

z_1 is structure dependent

$$\Gamma^{\mu\nu}(p,k,-k) = -2\delta^{\mu\nu} - 8z_1 p^\mu p^\nu + O(k,\epsilon^2)$$

$$\Delta(p+k) = \frac{1 - 2z_1 p\cdot k - \epsilon^2 z_1 + O(k^2,\epsilon^4,\epsilon^2 k)}{\epsilon^2 + 2p\cdot k + k^2}\,.$$

from $\Delta_\gamma(k^2)\left\{\Gamma^\mu(p,k)\Delta(p+k)\Gamma^\mu(p+k,-k) + \frac{1}{2}\Gamma^{\mu\mu}(p,k,-k)\right\}$

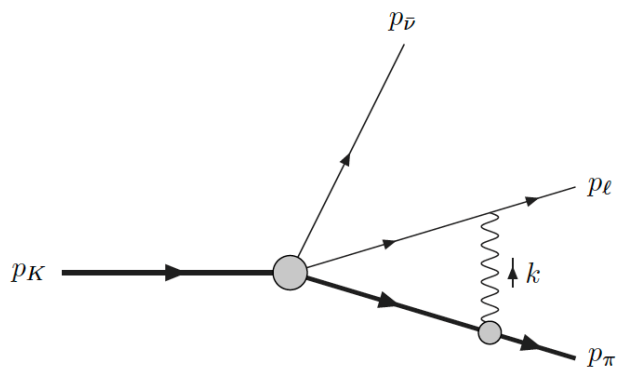
we find the following contribution to the amplitude

$$\Delta_\gamma(k^2)\left\{\frac{4m_P^2 + O(k^2)}{(2p\cdot k + k^2)^2} - \frac{8z_1 m_P^2 + O(k)}{2p\cdot k + k^2}\right\} + O\left(\frac{1}{L^2}\right)$$

infrared logarithmic divergent term

non-universal 1/L term

Finite-volume corrections - Semileptonic Decays (Cont.)



- We have seen that, as a result of the Ward Identity, we do not need the derivatives of the pion form-factors to obtain the $O(1/L)$ corrections.
- However, we also need to expand the weak-vertex which, in QCD without QED, is a linear combination of two form-factors $f^{\pm}(q^2)$.

- Off-shell, the $K\pi\ell\bar{\nu}$ weak vertex is a linear combination of two functions $F^{\pm}(p_\pi^2, p_K^2, 2p_K \cdot p_\pi)$ (which reduce on-shell to the form-factors $f^{\pm}(q^2)$).
- The WI relates the $K\pi\ell\bar{\nu}$ and $K\pi\ell\bar{\nu}\gamma$ vertices and does lead to a partial, but not complete, cancelation of $O(\frac{1}{L})$ terms.
- The $O(\frac{1}{L})$ corrections are found to depend on $\frac{df^{\pm}(q^2)}{dq^2}$, as well as on $f^{\pm}(q^2)$.
  - Such derivative terms are generic (a consequence of the Low theorem); absent only in particularly simple cases, such as leptonic decays.
- These corrections are "universal" in the sense that the coefficients are physical and can be computed in lattice simulations.
  - There are no corrections of the form $\frac{df^{\pm}}{dm_\pi^2}$ or $\frac{df^{\pm}}{dm_K^2}$, which would not be physical.

Perturbative calculation

- The starting point for our study is the relation:

$$\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}} = \lim_{V\to\infty}\left(\frac{d^2\Gamma_0}{dq^2 ds_{\pi\ell}} - \frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}}\right) + \lim_{V\to\infty}\left(\frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}} + \frac{d^2\Gamma_1(\Delta E)}{dq^2 ds_{\pi\ell}}\right)$$

  and the second term on the r.h.s. needs to be calculated in perturbation theory.
  - This has not been fully done (yet?).
- The second term has been calculated in the *soft-photon* approximation in which all terms proportional to k^n $(n \geq 1)$ are dropped in the numerator.

  De Boer, Kitahari, Nišandžić, arXiv:1803.05881
  - This work was motivated by the $R(D)$ and $R(D^*)$ anomalies in semileptonic B-decays and the suggestion that radiative corrections not present in *photos* may be the explanation. Appears to be not true.
- The soft-photon approximation is sufficient to make both terms on the r.h.s. infrared finite, but not to eliminate the $O(\frac{1}{L})$ corrections in the first term.

$$\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}} = \lim_{V\to\infty}\left(\frac{d^2\Gamma_0}{dq^2 ds_{\pi\ell}} - \frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}}\right) + \lim_{V\to\infty}\left(\frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}} + \frac{d^2\Gamma_1(\Delta E)}{dq^2 ds_{\pi\ell}}\right)$$

- The calculation of the second term on the r.h.s. has also been performed at lowest non-trivial order in ChPT. Cirigliano, Giannotti, Neufeld, arXiv:0807.4507

CONCLUSION and OUTLOOK

Present:

Full lattice calculations of radiative corrections to leptonic decays are possible

The form factors for real emissions are accessible from Euclidean correlators

Our preliminary results are very encouraging, still more work is needed and the analysis is ongoing

Future:

B-mesons are also very interesting, we expect a dynamical enhancement, but they are more difficult on current lattices. Neutron lifetime is also a fundamental piece of information in flavor physics

THANKS FOR YOUR ATTENTION

Extended Twisted Mass Collaboration

UTfit

INFN Istituto Nazionale di Fisica Nucleare