$b \to X_s \gamma \ @ N^2\text{LO}^{(*)}$ and feasibility of $b \to X_c \ell \bar{\nu} \ @ N^3\text{LO}$

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(*) In collaboration with Abdur Rehman and Matthias Steinhauser [arXiv:2002.01548],
as well as Mateusz Czaja, Tobias Huber and Go Mishima

1. Introduction

2. The radiative decay

   (i) $\mathcal{O}(\alpha_s^2)$ contributions to $\hat{G}_{17}$ and $\hat{G}_{27}$

   (ii) Non-perturbative effects in $\bar{B} \to X_s \gamma$

   (iii) Updated SM predictions for $B_s \gamma$ and $R_\gamma$

3. The semileptonic decay

   (i) Motivation for $\mathcal{O}(\alpha_s^3)$

   (ii) Challenges

4. Summary
$R(D)$ and $R(D^*)$ “anomalies” [https://hflav.web.cern.ch] \( (3.1\sigma) \)

\[
R(D^*) = \frac{\mathcal{B}(B \to D^{(*)}\tau\bar{\nu})}{\mathcal{B}(B \to D^{(*)}\mu\bar{\nu})}
\]

\[b \to s\ell^+\ell^-\] “anomalies” \( (> 5\sigma)\)  
[see, e.g., J. Aebischer et al., arXiv:1903.10434]

\[Q^\ell_9 = \begin{pmatrix} \gamma^\ell \\ s_L \\ b_L \end{pmatrix}
\]

\[Q^\ell_{10} = \begin{pmatrix} \gamma^\ell \gamma_5 \\ s_L \\ b_L \end{pmatrix}
\]

\[\ell = e \text{ or } \mu\]

$C_7$, the Wilson coefficient of $Q_7 = \begin{pmatrix} b_R \\ s_L \end{pmatrix}$ is an important input in the fits.
Sample Leading-Order (LO) contributions to $C_7$ in the SM and beyond:

\[ M_{H^\pm} > \sim 800 \text{ GeV} \]

in the 2HDM-II
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\[
\begin{align*}
\gamma & \quad \gamma \quad \gamma \\
\uparrow & \quad \uparrow & \quad \uparrow \\
\text{u, c, t} & \quad \text{u, c, t} & \quad \text{u, c, t} \\
\text{\(W^\pm\)} & \quad \text{\(W^\pm\)} & \quad \text{\(W^\pm\)} \\
b & \quad \text{s} & \quad \text{s} \\
\end{align*}
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\]

\[
\begin{align*}
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\uparrow & \quad \uparrow \\
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\chi^\pm & \quad \chi^\pm \\
b & \quad \text{s} \\
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\]

\[
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The strongest experimental constraint on $C_7$ comes from $\mathcal{B}_{s\gamma}$ —— the CP- and isospin-averaged BR of $\bar{B} \rightarrow X_s \gamma$ and $B \rightarrow X_s \bar{s} \gamma$. 

$(\bar{B}^0, B^-)$ 

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&\gamma_{t, t} \rightarrow t_{\chi^\pm_{b, s}} \\
&\gamma_{b, s} \rightarrow LQ_{b, s}
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HFLAV, arXiv:1909.12524: $\mathcal{B}_{s\gamma}^{\exp} = (3.32 \pm 0.15) \times 10^{-4}$ for $E_\gamma > E_0 = 1.6 \text{ GeV} \simeq \frac{m_b}{3}$, $(\pm4.5\%)$

averaging CLEO, BELLE and BABAR with $E_0 \in [1.7, 2.0] \text{ GeV},$ and then extrapolating to $E_0 = 1.6 \text{ GeV}.$

**TH requirement:** $E_0$ should be large ($\sim \frac{m_b}{2}$) but not too close to the endpoint ($m_b - 2E_0 \gg \Lambda_{\text{QCD}}$).
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With the full BELLE-II dataset, a $\pm 2.6\%$ uncertainty in the world average for $B_{s\gamma}^{\text{exp}}$ is expected.

SM calculations must be improved to reach a similar precision.
Determination of $\mathcal{B}(\bar{B} \to X_s \gamma)$ in the SM:

$$\mathcal{B}(\bar{B} \to X_s \gamma)_{E\gamma > E_0} = \mathcal{B}(\bar{B} \to X_c e\bar{\nu})_{\exp} \left| \frac{V_{ts} V_{tb}^*}{V_{cb}} \right|^2 \frac{6\alpha_{em}}{\pi} C \left[ P(E_0) + N(E_0) \right]$$

$$\Gamma^{pert.} \approx 96\%$$

$$\Gamma^{non-pert.} \approx 4\%$$

$$\frac{\Gamma[b \to X_s^p \gamma]_{E\gamma > E_0}}{|V_{cb}/V_{ub}|^2 \Gamma[b \to X_u^p e\bar{\nu}]} = \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{em}}{\pi} P(E_0)$$

$$C = \left| \frac{V_{ub}}{V_{cb}} \right|^2 \frac{\Gamma[\bar{B} \to X_c e\bar{\nu}]}{\Gamma[\bar{B} \to X_u e\bar{\nu}]}$$

semileptonic phase-space factor
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$$\Gamma[\bar{B} \to X^p_s \gamma]_{E_\gamma > E_0} = \left| \frac{V_{ts} V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{\text{em}}}{\pi} P(E_0)$$

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The effective Lagrangian: $L_{\text{weak}} \sim \sum_i C_i Q_i$

Eight operators $Q_i$ matter for $\mathcal{B}_{s\gamma}^{\text{SM}}$ when the NLO EW and/or CKM-suppressed effects are neglected:
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$$\frac{\Gamma[b \rightarrow X_s^P \gamma]_{E\gamma>E_0}}{|V_{cb}/V_{ub}|^2 \Gamma[b \rightarrow X_u e\bar{\nu}]} = \left| \frac{V_{ts}V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{\text{em}}}{\pi} P(E_0)$$

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$$\Gamma(b \rightarrow X_s^p \gamma) = \frac{G_F^2 m_{b,\text{pole}}^5 \alpha_{\text{em}}}{32\pi^4} \left| V_{ts}V_{tb} \right|^2 \sum_{i,j=1}^{8} C_i(\mu_b) C_j(\mu_b) \hat{G}_{ij}, \quad (\hat{G}_{ij}=\hat{G}_{ji})$$
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$$\frac{\Gamma[b \to X_s^p \gamma]_{E_\gamma > E_0}}{|V_{cb}/V_{ub}|^2 \Gamma[b \to X^p_u e \bar{\nu}]} = \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \frac{6 \alpha_{em}}{\pi} \frac{P(E_0)}{C}$$

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NLO ($\mathcal{O}(\alpha_s)$) – last missing pieces being evaluated by Tobias Huber and Lars-Thorben Moos

Most important @ NNLO ($\mathcal{O}(\alpha_s^2)$): \(\hat{G}_{77}, \hat{G}_{17}, \hat{G}_{27}\)

[arXiv:1912.07916]

between the $m_c \gg m_b$ and $m_c = 0$ limits [arXiv:1503.01791]

\[\Rightarrow \pm 3\%\] uncertainty in $\mathcal{B}_{s \gamma}^{\text{SM}}$
Sample diagrams contributing to $\hat{G}_{27}$ @ NNLO:
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3. Extending the set of master integrals \( M_k \) so that it closes under differentiation with respect to \( z = m_c^2/m_b^2 \). This way one obtains a system of differential equations

\[
\frac{d}{dz} M_k(z, \epsilon) = \sum_l R_{kl}(z, \epsilon) M_l(z, \epsilon), \quad (*)
\]

where \( R_{nk} \) are rational functions of their arguments.
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And from diagrams with closed loops of massive fermions

UV renormalization has been carried out using the results from arXiv:1702.07674.
Non-perturbative contribution from gluon-to-photon conversion in the QCD medium.

It was first considered by Lee, Neubert & Paz in hep-ph/0609224. It originates from hard gluon scattering on the valence quark or a “sea” quark that produces an energetic photon. The quark that undergoes this Compton-like scattering is assumed to remain soft in the $\bar{B}$-meson rest frame to ensure effective interference with the leading “hard” amplitude. Without interference the contribution would be negligible ($\mathcal{O}(\alpha_s^2 \Lambda^2 / m_b^2)$).

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$$\Rightarrow \frac{\delta \Gamma_c / \Gamma}{\Delta_{0-}} \simeq \frac{(B+C)(Q_u + Q_d) + 2DQ_s}{(C-B)(Q_u - Q_d)}$$

$$\equiv \frac{Q_u + Q_d}{Q_d - Q_u} \left[ 1 + 2 \frac{D-C}{C-B} \right]$$ $\equiv$ $SU(3)_F$ violation

MM, arXiv:0911.1651

$$\frac{\delta \Gamma_c}{\Gamma} \simeq -\frac{1}{3} \Delta_{0-} \left[ 1 + 2 \frac{D-C}{C-B} \right] = -\frac{1}{3} (-0.48 \pm 1.49 \pm 0.97 \pm 1.15) \% \times (1 \pm 0.3) = (0.16 \pm 0.74) \%$$

Belle, arXiv:1807.04236, $E_0 = 1.9$ GeV

Recall: $(x \pm \sigma_x)(y \pm \sigma_y) = xy \pm \sqrt{(x\sigma_y)^2 + (y\sigma_x)^2 + (\sigma_x\sigma_y)^2}$
The resolved photon contribution to the $Q_7$-$Q_{1,2}$ interference.


\[
\langle \bar{B} | \left[ \begin{array}{c}
\frac{\mu_G^2}{27m_c^2} \\
\frac{\kappa V \mu_G^2}{27m_c^2}
\end{array} \right] \langle \bar{B} \rangle 
\]

\[
\delta N(E_0) = (C_2 - \frac{1}{6}C_1)C_7 \left[ \begin{array}{c}
-\frac{\mu_G^2}{27m_c^2} \\
\frac{\Lambda_{17}}{m_b}
\end{array} \right]
\]
The resolved photon contribution to the $Q_7$-$Q_{1,2}$ interference.


\[ \langle \bar{B} \mid \bar{B} \rangle \delta N(E_0) = (C_2 - \frac{1}{6}C_1)C_7 \left[ -\frac{\mu_G^2}{27m_c^2} + \frac{\Lambda_{17}}{m_b} \right] - \frac{k_v \mu_G^2}{27m_c^2} \]

\[ \Lambda_{17} = \frac{2}{3} \text{Re} \int_{-\infty}^{\infty} \frac{d\omega_1}{\omega_1} \left[ 1 - F \left( \frac{m_c^2 - i\varepsilon}{m_b\omega_1} \right) + \frac{m_b\omega_1}{12m_c^2} \right] h_{17}(\omega_1, \mu) \]

\[ \omega_1 \leftrightarrow \text{gluon momentum}, \quad F(x) = 4x \text{arctan}^2 \left( \frac{1}{\sqrt{4x} - 1} \right) \]
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\[ \langle \bar{B} | \bar{\gamma}^\mu \nabla_\mu \gamma^\nu \nabla_\nu \gamma^\alpha \gamma^\beta (S_\bar{n}^\dagger h|0) (r \bar{n})(S_\bar{n}^\dagger h)(0)| \bar{B} \rangle \]

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The soft function $h_{17}$:

\[ h_{17}(\omega_1, \mu) = \int \frac{dr}{4\pi M_B} e^{-i\omega_1 r} \langle \bar{B} | (\bar{n}S_{\bar{n}})(0) \bar{n} \gamma^\dagger \gamma_\alpha \gamma^\beta (S_{\bar{n}}^\dagger h)(0) | \bar{B} \rangle \]

A class of models for $h_{17}$:

\[ h_{17}(\omega_1, \mu) = e^{-\frac{\omega_1^2}{2\sigma^2}} \sum_n a_{2n} H_{2n} \left( \frac{\omega_1}{\sigma \sqrt{2}} \right), \quad \sigma < 1 \text{ GeV} \]

Hermite polynomials

Constraints on moments (e.g.):

\[ \int d\omega_1 h_{17} = \frac{2}{3} \mu_G^2, \quad \int d\omega_1 \omega_1^2 h_{17} = \frac{2}{15} (5 m_5 + 3 m_6 - 2 m_9). \]
The resolved photon contribution to the $Q_7$-$Q_{1,2}$ interference.


\[ \langle \bar{B} | e^{-i\omega_1} (\bar{S}_n h_S)(0) \bar{n} \gamma_\alpha \tilde{\gamma}_\beta (S_{\tilde{n}}^\dagger g G_s^{\alpha\beta} S_{\tilde{n}})(r \bar{n}) (S_{\tilde{n}}^\dagger h)(0) | \bar{B} \rangle \]

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G+P numerically:

$$\Lambda_{17} \in [-24, 5] \text{ MeV for } m_c = 1.17 \text{ GeV.}$$

Factor-of-3 improvement w.r.t. BLNP.

In our code: $\kappa_V = 1.2 \pm 0.3$.

Warning: scheme for $m_c$!
Non-perturbative contribution proportional to $|C_8|^2$

A. Ferroglia & U. Haisch [arXiv:1009.2144],
focused on the collinear logs $\ln \frac{m_b}{m_s}$ in the corresponding contribution to $P(E_0)$.
$\Rightarrow$ fragmentation functions $\Rightarrow$ effects below 1% in $B_{s\gamma}$. 
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Numerically, we can reproduce this range by performing a replacement

$$\ln \frac{m_b}{m_s} \rightarrow \kappa_{88} \ln 50 \quad \text{with} \quad \kappa_{88} = 1.7 \pm 1.1$$

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The $[\ln 10, \ln 50]$ range remains used in other (small) terms where collinear logs arise.
Updated SM predictions for $\mathcal{B}_{s\gamma}$ and $R_\gamma \equiv \mathcal{B}_{(s+d)\gamma}/\mathcal{B}_{c\ell\bar{\nu}}$ (with $E_0 = 1.6$ GeV):

\[
\mathcal{B}_{s\gamma} = (3.40 \pm 0.17) \times 10^{-4}
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compare to $(3.36 \pm 0.23) \times 10^{-4}$ in arXiv:1503.01789

\[
(\pm 5.0\%)
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R_\gamma = (3.35 \pm 0.16) \times 10^{-3}
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When the interpolation gets removed but nothing else changes:
$$\sqrt{3^2 + 2.5^2}\% = 3.9\%$$ – still somewhat behind the expected experimental ±2.6%.
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When the interpolation gets removed but nothing else changes:
$\sqrt{3^2 + 2.5^2\%} = 3.9\%$ – still somewhat behind the expected experimental $\pm 2.6\%$.

Shifts in uncertainties related to $\frac{\delta \Gamma_c}{\Gamma}$, $\kappa_V$ and $\kappa_{88}$:
formerly: $1.25\% + 2.85\% + 1.10\% = 5.20\%$ (in quadrature: 3.30\%)
at present: $0.74\% + 0.88\% + 0.92\% = 2.54\%$ (in quadrature: 1.48\%)  $\sqrt{1.48^2 + 2.01^2\%} = 2.49\% \simeq 2.5\%$
Summary for the radiative decay

- Perturbative NNLO calculations of $\Gamma[b \rightarrow X_s^p \gamma]$ that aim at removing the $m_c$-interpolation have been finalized for diagrams involving closed fermion loops on the gluon lines. We confirm several published results, and supplement them with a previously unknown (tiny) piece.
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Determination of $|V_{cb}|$ from the inclusive $\bar{B} \to X_c \ell \nu$ rate and spectra

$|V_{cb}| = (42.00 \pm 0.64) \times 10^{-3}$  

$1.5\%$  

[1.5%] roughly:  

$\sqrt{(1.0\%)^2 + (1.1\%)^2} \simeq 1.5\%$

perturbative $\mathcal{O}(\alpha_s^3)$  

other
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$|V_{cb}|^2$ other


Impact on the uncertainty in the SM prediction for $\epsilon_K$:

$$\sqrt{(5.3\%)^2 + (6.4\%)^2} \approx 8.3\%$$  \hspace{1cm} (roughly)

$|V_{cb}|^4$ other

using Eq. (17) of [ J. Brod, M. Gorbahn and E. Stamou, arXiv:1911.06822 ].
Inclusive Decays

- Optical Theorem
- OPE – Heavy Quark Expansion (HQE): \( p_b = m_b \nu_B + k \)

Observables can be written as:

\[
d\Gamma = d\Gamma_0 + d\Gamma_{\mu_\pi} \frac{\mu_\pi^2}{m_b^2} + d\Gamma_{\mu_G} \frac{\mu_G^2}{m_b^2} + d\Gamma_{\rho_D} \frac{\rho_D^3}{m_b^3} + d\Gamma_{\rho_{LS}} \frac{\rho_{LS}^3}{m_b^3} + \ldots
\]

- \( d\Gamma_i \) are computed in perturbative QCD
- The non-perturbative dynamics is enclosed into the HQE parameters: \( \mu_\pi, \mu_G, \rho_D, \rho_{LS} \sim \langle B | \bar{b}_\nu iD^\mu \ldots iD^\nu \Gamma_{\mu...\nu} b_\nu | B \rangle \)
- HQE parameters are extracted from data.

Reviews:
Dingfelder, Mannel, Rev.Mod.Phys. 88 (2016) 035008.
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Feasibility of $b \to X_c \ell \bar{\nu}$ @ N$^3$LO

contribution to $\Gamma$

contribution to $d\Gamma/dq^2$ for $q^2 = M^2$
Feasibility of $b \to X_c \ell \bar{\nu} @ N^3$LO

Let us consider $q^2 = m_c^2$:

Real boundary condition for the differential equations at $m_c \gg m_b$
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Let us consider $q^2 = m_c^2$:

Real boundary condition for the differential equations at $m_c \gg m_b$

Possible IBP outsourcing: **Fraunhofer Institute for Industrial Mathematics**

[D. Bendle et al., arXiv:1908.04301]
BACKUP SLIDES
The “hard” contribution to $\bar{B} \to X_s \gamma$

Goal: calculate the inclusive sum

$$\sum_{X_s} \left| C_7(\mu_b) \langle X_s \gamma | O_7 | \bar{B} \rangle + C_2(\mu_b) \langle X_s \gamma | O_2 | \bar{B} \rangle + \ldots \right|^2$$

The “$77$” term in this sum is “hard”. It is related via the optical theorem to the imaginary part of the elastic forward scattering amplitude $\bar{B}(\vec{p} = 0)\gamma(\vec{q}) \to \bar{B}(\vec{p} = 0)\gamma(\vec{q})$:

$$\text{Im} \left\{ q \quad \bar{B} \quad 7 \quad \gamma \quad \bar{B} \quad 7 \quad q \right\} \equiv \text{Im} A$$

When the photons are soft enough, $m_{X_s}^2 = |m_B(m_B - 2E_\gamma)| \gg \Lambda^2 \Rightarrow \text{Short-distance dominance} \Rightarrow \text{OPE}$.

However, the $\bar{B} \to X_s \gamma$ photon spectrum is dominated by hard photons $E_\gamma \sim m_b/2$.

Once $A(E_\gamma)$ is considered as a function of arbitrary complex $E_\gamma$, Im$A$ turns out to be proportional to the discontinuity of $A$ at the physical cut. Consequently,

$$\int_{1 \text{ GeV}}^{E_{\gamma}^{\text{max}}} dE_\gamma \text{ Im} A(E_\gamma) \sim \int_{\text{circle}} dE_\gamma A(E_\gamma).$$

Since the condition $|m_B(m_B - 2E_\gamma)| \gg \Lambda^2$ is fulfilled along the circle, the OPE coefficients can be calculated perturbatively, which gives

$$A(E_\gamma)|_{\text{circle}} \sim \sum_j \left[ \frac{F^{(j)}_{\text{polynomial}}(2E_\gamma/m_b)}{m_b^{n_j}(1 - 2E_\gamma/m_b)^{k_j}} + \mathcal{O}(\alpha_s(\mu_{\text{hard}})) \right] \langle \bar{B}(\vec{p} = 0) | Q_{\text{local operator}}^{(j)} | \bar{B}(\vec{p} = 0) \rangle.$$ 

Thus, contributions from higher-dimensional operators are suppressed by powers of $\Lambda/m_b$.

At $(\Lambda/m_b)^0$: $\langle \bar{B}(\vec{p}) | \bar{b}_\gamma \gamma^\mu b \bar{B}(\vec{p}) \rangle = 2p^\mu \Rightarrow \Gamma(\bar{B} \to X_s \gamma) = \Gamma(b \to X_s^{\text{parton}} \gamma) + \mathcal{O}(\Lambda/m_b)$.

At $(\Lambda/m_b)^1$: Nothing! All the possible operators vanish by the equations of motion.

At $(\Lambda/m_b)^2$: $\langle \bar{B}(\vec{p}) | \bar{b}_v D^\mu D_\mu b_v | \bar{B}(\vec{p}) \rangle \sim m_B \mu_\pi^2$, $\langle \bar{B}(\vec{p}) | \bar{b}_v g_s G_{\mu\nu} \sigma^{\mu\nu} b_v | \bar{B}(\vec{p}) \rangle \sim m_B \mu_G^2$.

The HQET heavy-quark field: $b_v(x) = \frac{1}{2}(1 + \gamma) b(x) \exp(i m_b v \cdot x)$ with $v = p/m_B$. 

The same method has been applied to the 3-loop counterterm diagrams

[MM, A. Rehman, M. Steinhauser, PLB 770 (2017) 431]

Master integrals:

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Results for the bare NLO contributions up to $\mathcal{O}(\epsilon)$:

$$\hat{G}_{27}^{(1)2P} = -\frac{92}{81\epsilon} + f_0(z) + \epsilon f_1(z) \quad \overset{z \to 0}{\rightarrow} \quad -\frac{92}{81\epsilon} - \frac{1942}{243\epsilon} + \epsilon \left( -\frac{26231}{729} + \frac{259}{243\pi^2} \right)$$

Dots: solutions to the differential equations and/or the exact $z \to 0$ limit.
Lines: large- and small-$z$ asymptotic expansions

Small-$z$ expansions of $\hat{G}_{27}^{(1)2P}$:

- $f_0$ from C. Greub, T. Hurth, D. Wyler, hep-ph/9602281, hep-ph/9603404,

Analogous results for the 3-body final state contributions ($\delta = 1$):

$$\hat{G}^{(1)3P}_{27} = g_0(z) + \epsilon g_1(z) \quad \overset{z \to 0}{\rightarrow} \quad -\frac{4}{27} - \frac{106}{81}\epsilon$$

\[ g_0(z) = \begin{cases} 
-\frac{4}{27} - \frac{14}{9}z + \frac{8}{3}z^2 + \frac{8}{3}z(1 - 2z) s L + \frac{16}{9}z(6z^2 - 4z + 1) \left(\frac{\pi^2}{4} - L^2\right), & \text{for } z \leq \frac{1}{4}, \\
-\frac{4}{27} - \frac{14}{9}z + \frac{8}{3}z^2 + \frac{8}{3}z(1 - 2z) t A + \frac{16}{9}z(6z^2 - 4z + 1) A^2, & \text{for } z > \frac{1}{4}, 
\end{cases} \]

where $s = \sqrt{1 - 4z}$, $L = \ln(1 + s) - \frac{1}{2} \ln 4z$, $t = \sqrt{4z - 1}$, and $A = \arctan(1/t)$. 

Dots: solutions to the differential equations and/or the exact $z \to 0$ limit.
Lines: exact result for $g_0$, as well as large- and small-$z$ asymptotic expansions for $g_1$. 

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[20]