The volume of the black hole interior at late times

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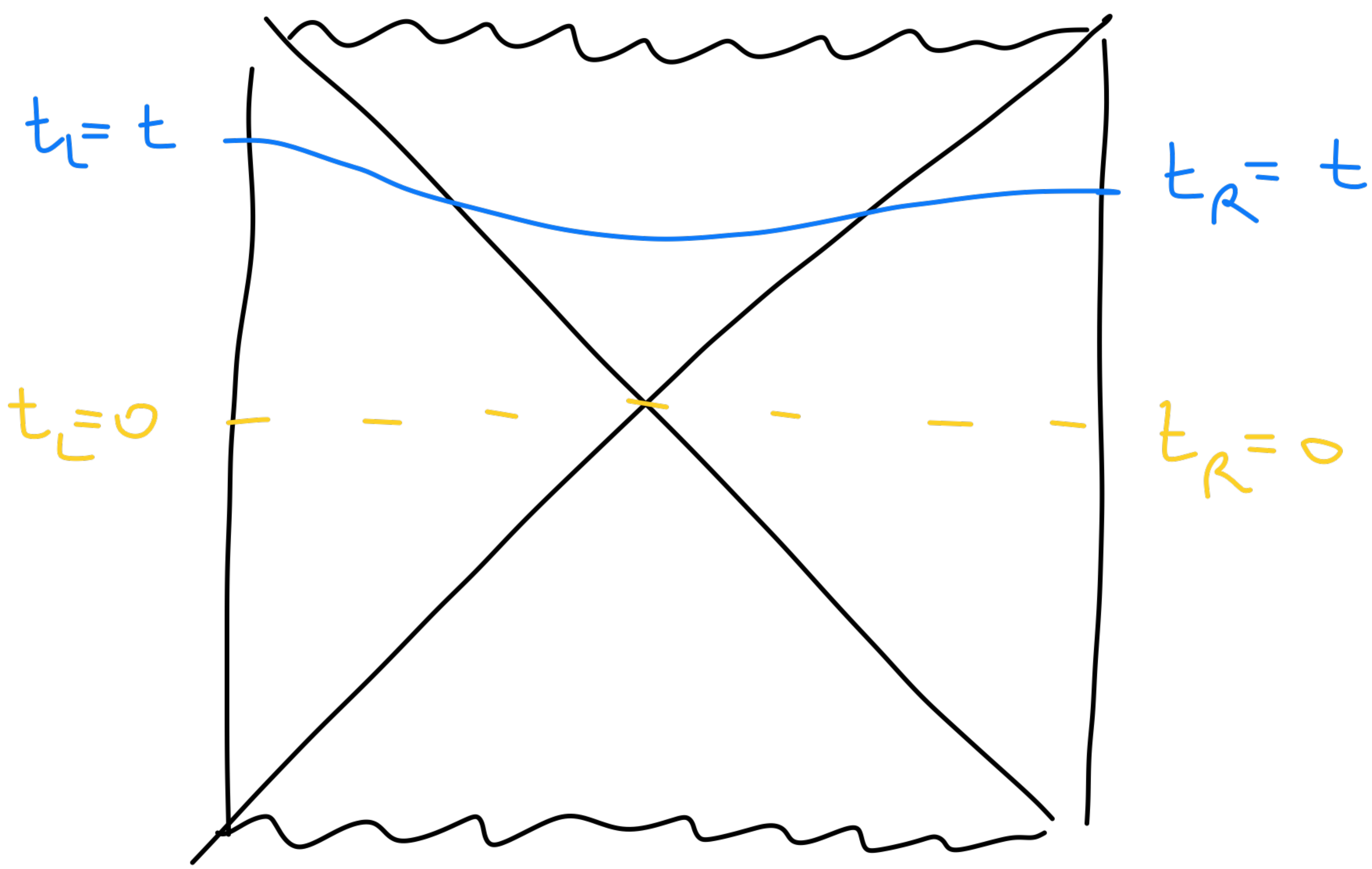
Based on:
2107.06286
with Luca Iliesiu & Mark Mezei

Plan

- Motivation
- JT gravity as a matrix integral
- The volume of the interior in JT gravity

Growth of the interior

Black hole interior: an expanding cosmology

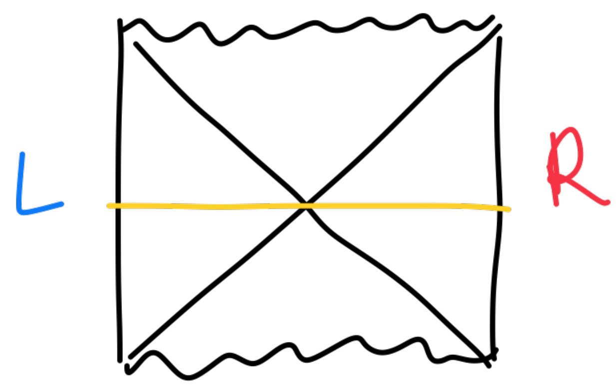


volume of maximal Cauchy slice $\propto Mt$

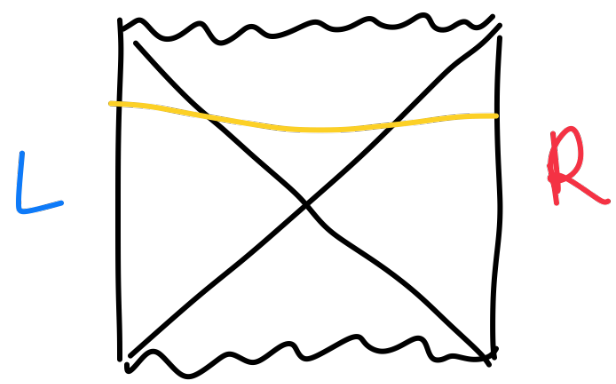
Question: what is the microscopic origin of this "creation of space"?

Growth of the interior

In AdS/CFT:



$$= \quad |TFD\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\beta E_n/2} |n\rangle_L |n\rangle_R$$



$$= \quad e^{-i(H_L+H_R)t} |TFD\rangle$$

Probes of the TFD state:

Correlation functions: $\langle TFD | O_1(t_1) \ldots O_k(t_k) | TFD \rangle$ **thermalize in** $\propto \beta$ **times** (**$\beta \log G_N^{-1}$ for OTOC**)

Entanglement entropy: **thermalize in** $\propto \beta$ **times** [Hartman-Maldacena]

Growth of the interior

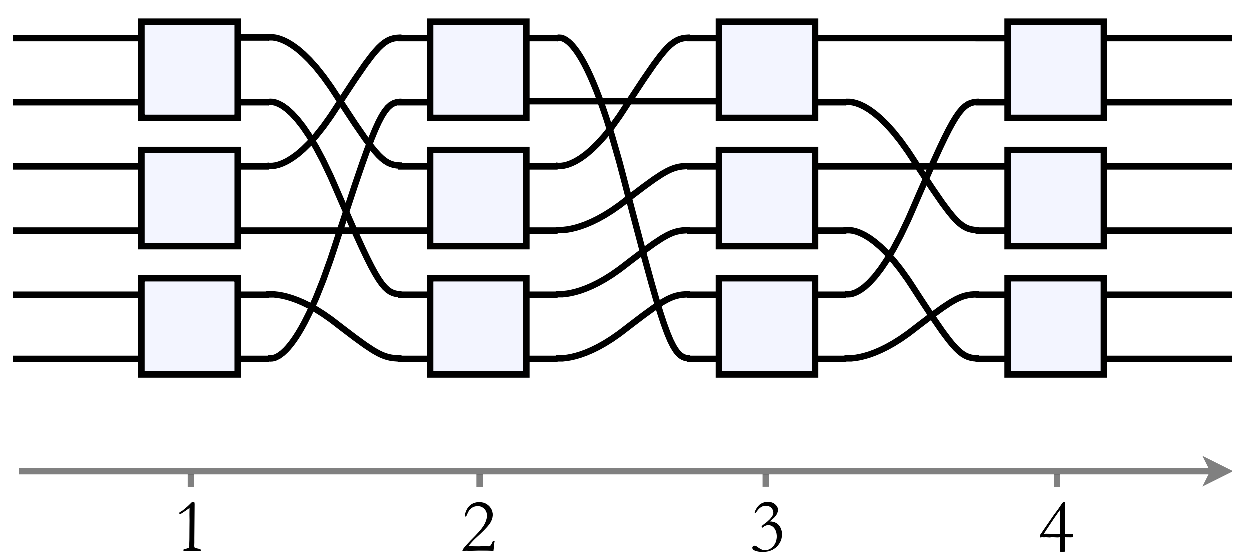
Perturbative corrections in G_N are unlikely to terminate this growth

Upshot: We need a quantity that is not thermalizing

for $\propto e^{\frac{1}{G_N}}$ times

Candidate: complexity of a state [Susskind et. al.]

$|0\rangle^{\otimes K}$

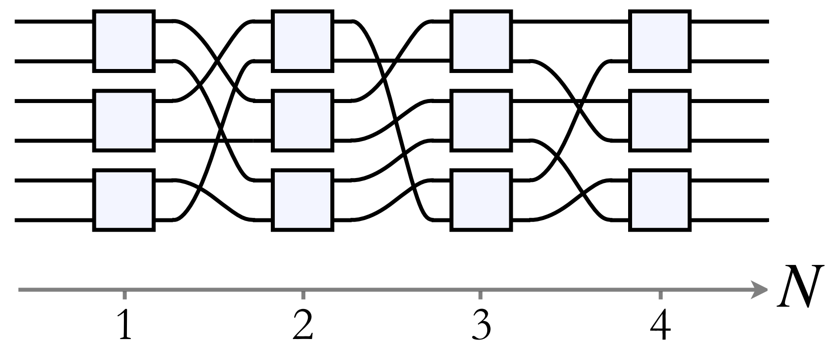


$|\psi\rangle_{\text{target}}$

Size of the minimal quantum circuit producing the target

Complexity growth

$|0\rangle^{\otimes K}$



$|\psi\rangle_{\text{target}}$

Somewhat ambiguous, but so is how we measure the size of the interior

Argument for the time dependence of complexity [Susskind,Brown-Susskind-Zhao]

$$|\psi\rangle_{\text{target}} = e^{-iHt}|TFD\rangle$$

$$\downarrow$$

$$\underbrace{\left(1 - iH\frac{t}{N}\right)}_{\text{One ``layer'' of the circuit}}{}^{N}$$

integral of local operators with derivatives

$$\textbf{\# of gates} = \frac{K}{2}N \approx St$$

Complexity growth

$$|\psi\rangle_{\text{target}} = e^{-iHt}|TFD\rangle$$

# of gates $= \frac{K}{2}N \approx St$

At later times, there could be a shortcut in getting to $|\psi\rangle_{\text{target}}$

[Brown-Susskind-Zhao]

All possible unitaries on K **qubits:** $SU(2^K)$

$$\left.\begin{array}{r} \mathrm{Vol}[SU(2^K)] \propto 2^{\frac{K}{2}4^K} \\ \text{\# of possible circuits of depth } N \propto (\#\text{gates})^{KN} \end{array}\right\} \Rightarrow$$

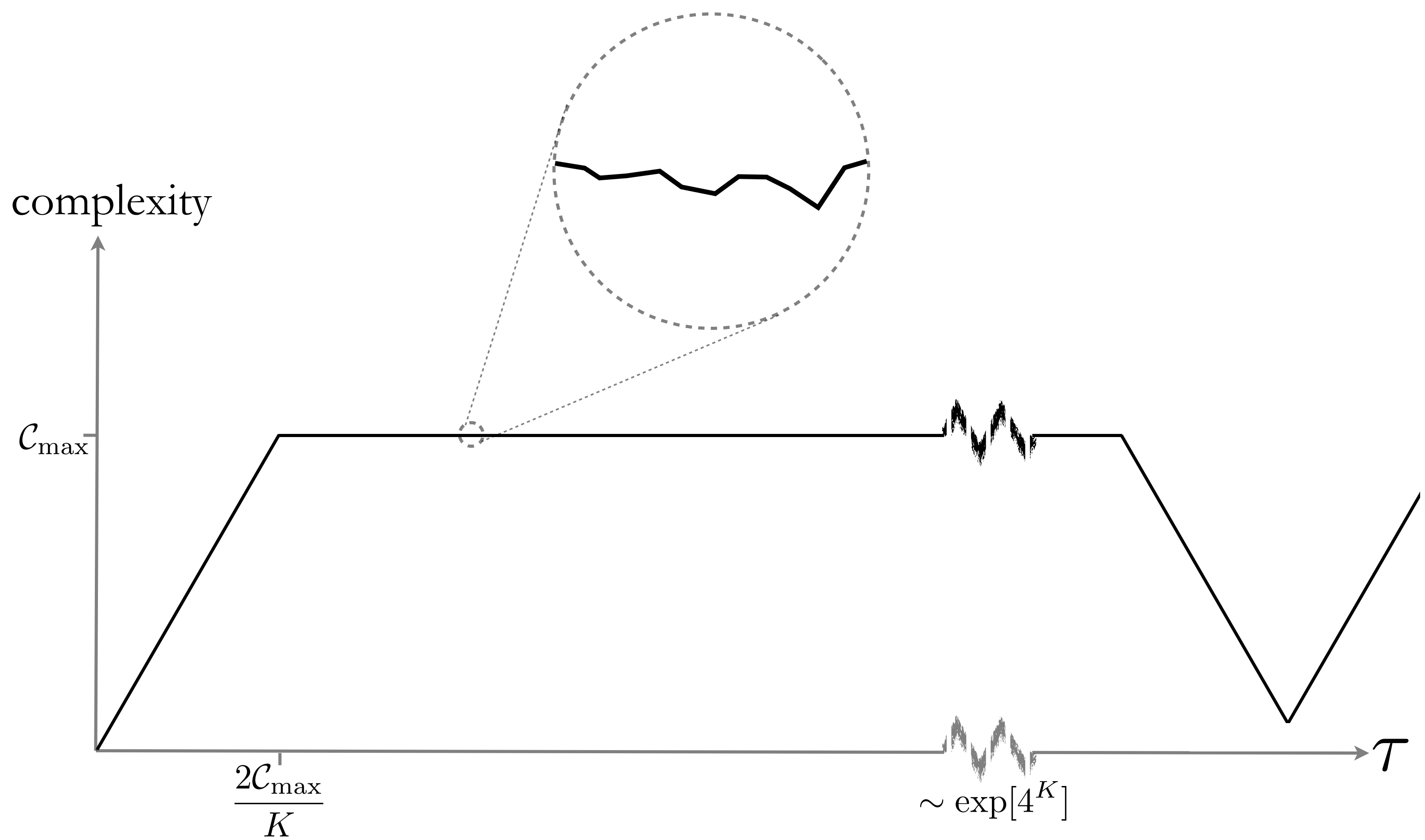
max complexity $\propto 4^K \propto e^S$

Time required to reach this with $\propto St$ **growth:** $t_{\text{sat}} \propto e^S$

Complexity growth

Conjectured time dependence of complexity in chaotic systems

[Susskind,Brown-Susskind-Zhao], see [Balasubramanian-DeCross-Kar-Parrikar] for progress towards proving it



semi-classical gravity

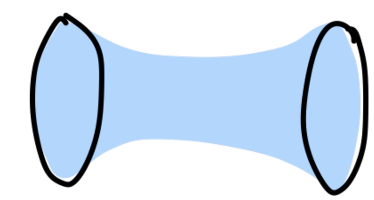
Today, we try to see this in gravity

Complexity growth

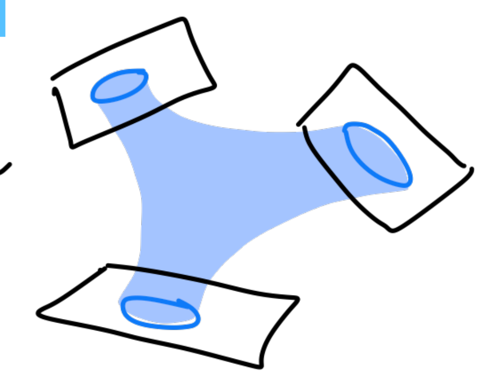
Why do we have a chance of seeing something like this in gravity?

Because Euclidean gravity seems to have an unreasonably large regime of validity, e.g.

Universal energy level repulsion of chaotic systems
[Saad-Shenker-Stanford,Cotler-Jensen,...]

$Z(\beta)^2 \supset$ 

Entropy Page curve of an evaporating black hole
[Penington-Shenker-Stanford-Yang,Amheiri-Hartman-Maldacena-Shaghoulian-Tajdini]

$\mathrm{tr}\, \rho_{rad}^{n} \sim$ 

Both are about corrections coming from summing topologies

Today: how do these effects affect the volume of the interior?

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Jackiw-Teitelboim (JT) gravity

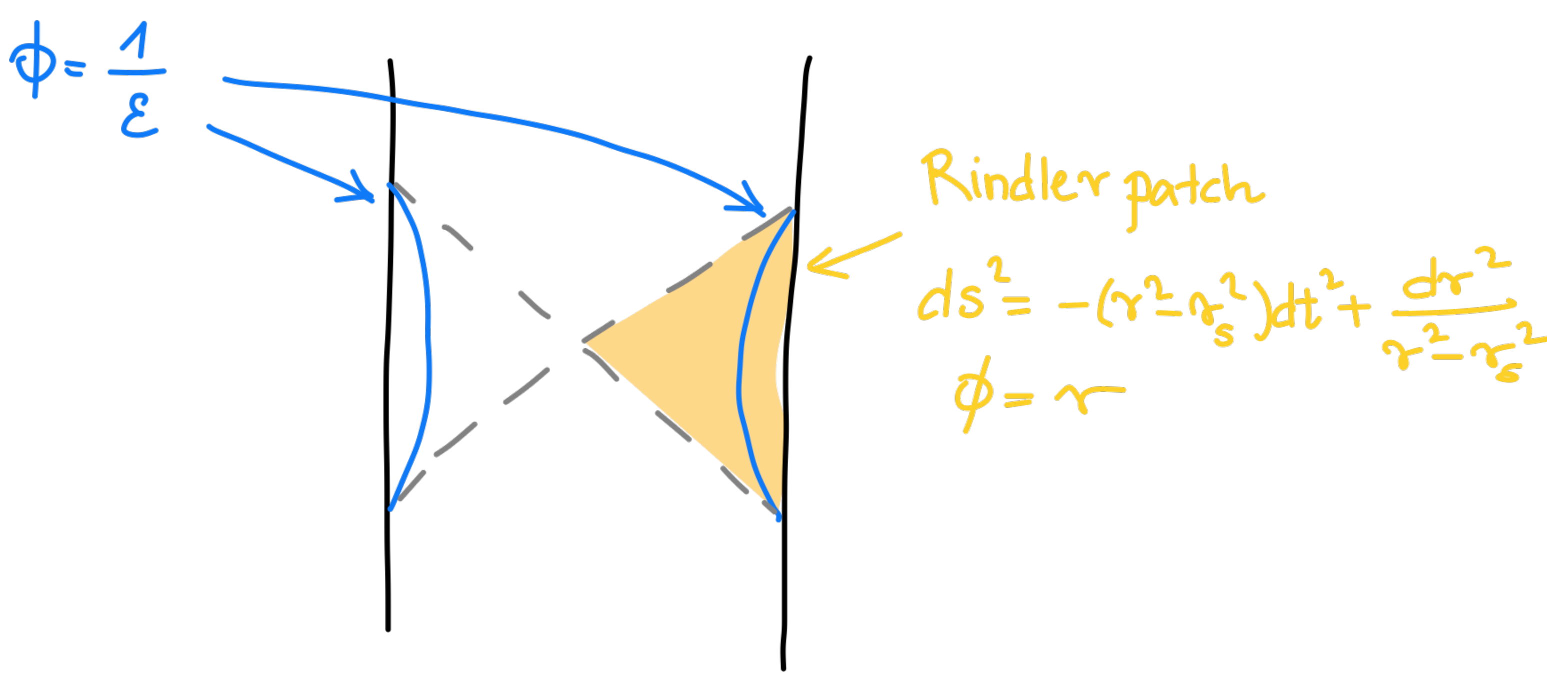
Dilaton-gravity in two dimensions:

$$I_{JT} = -S_0\chi(\mathscr{M}) - \frac{1}{2}\int\sqrt{g}\phi(R+2) - \int_{\partial\mathscr{M}}\sqrt{h}\phi(K-1)$$

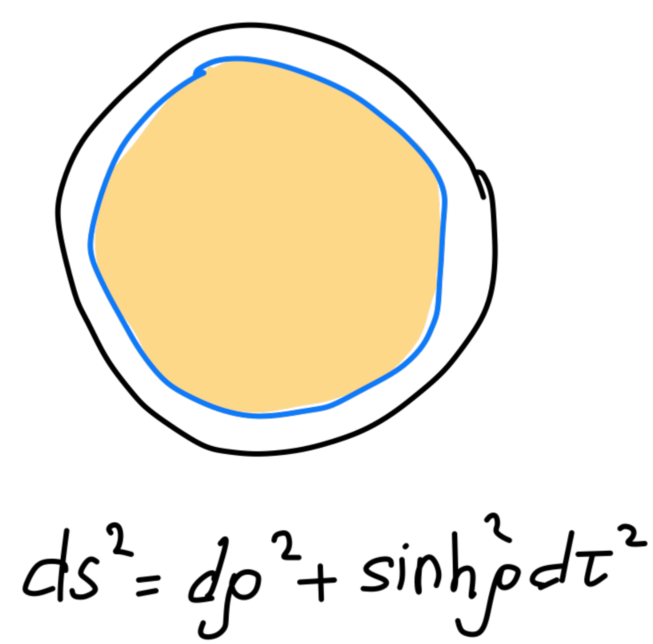
Arises by dimensional reduction
of near-horizon region of near-extremal black holes
[Maldacena-Stanford-Yang,Sarosi,Nayak-Shukla-Soni-Trivedi]

Black hole solution: Global AdS_2 with a cutoff at large constant ϕ

Lorentzian



Euclidean

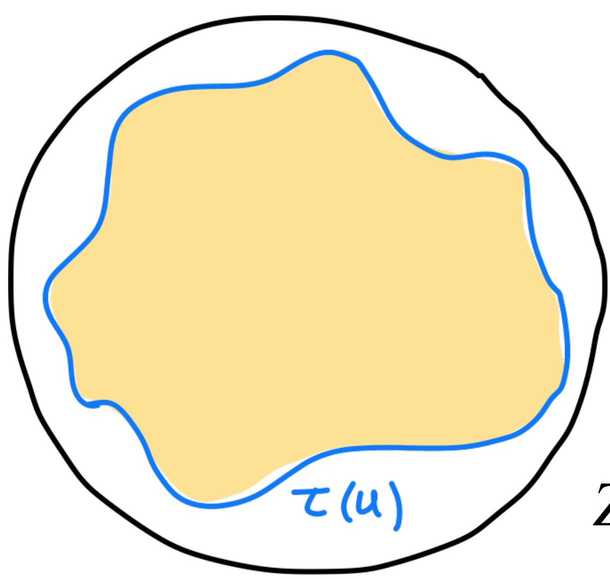


Jackiw-Teitelboim (JT) gravity

Dilaton-gravity in two dimensions:

$$I_{JT} = -S_0 \chi(\mathscr{M}) - \frac{1}{2}\int \sqrt{g}\phi(R+2) - \int_{\partial \mathscr{M}} \sqrt{h}\phi(K-1)$$

Quantum theory: path integrating out ϕ gives $\delta(R+2)$
which in turn fixes all off-shell geometries
in the path integral to be pieces of the hyperbolic disk
[Maldacena-Stanford-Yang]



Off-shell degrees of freedom: boundary "wiggles"

$$\int_{\partial \mathscr{M}} \sqrt{h}\phi(K-1) \rightarrow \int du \left\{ \tan \frac{\tau(u)}{2}, u \right\}$$

$$Z_{\text{disk}} = \int \frac{d\mu(\tau)}{SL(2,\mathbb{R})} \exp\left[-\gamma \int du \left\{ \tan \frac{\tau(u)}{2}, u \right\}\right] = \left[\frac{\gamma}{\beta}\right]^{3/2} e^{\frac{2\pi^2 \gamma}{\beta}}$$

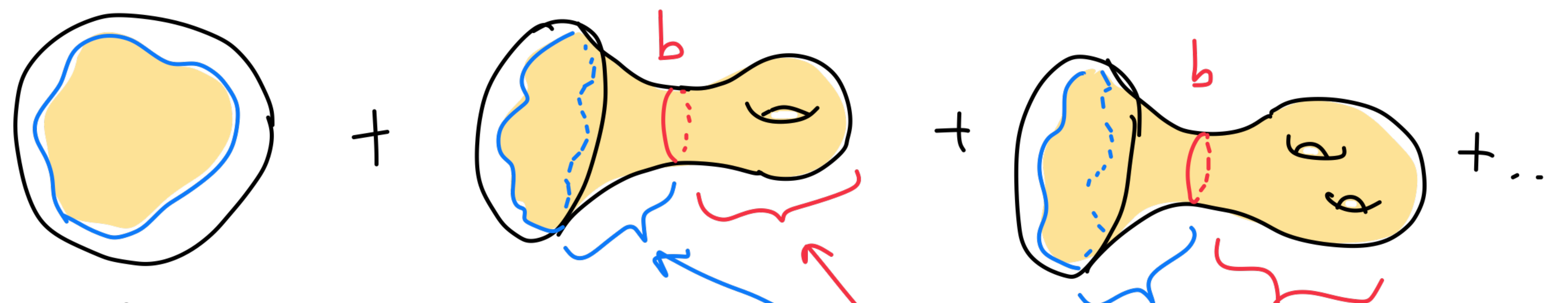
One loop exact [Stanford-Witten]

Jackiw-Teitelboim (JT) gravity

Higher topologies also contribute, weighted by $\chi(\mathscr{M}) = 2g + n - 2$

[Saad-Shenker-Stanford]

$Z(\beta) =$



$$e^{S_0}\left[\frac{\gamma}{\beta}\right]^{3/2} e^{\frac{2\pi^2\gamma}{\beta}} \quad + \quad \sum_g e^{-S_0(2g-1)} \int_0^\infty b\,db\,\underbrace{V_{g,1}(b)}\,\underbrace{Z^{\text{trumpet}}(\beta, b)}$$

$$\left[\frac{\gamma}{\beta}\right]^{1/2} e^{-\frac{\gamma b^2}{2\beta}}$$

integral over wiggles
one loop exact

Volume of moduli space of bordered Riemann surfaces

Jackiw-Teitelboim (JT) gravity

Works similarly for n boundaries $Z(\beta_1)\ldots Z(\beta_n) \to V_{g,n}(b_1,\ldots,b_n)$

$V_{g,n}(b_1,\ldots,b_n)$ **satisfy a recursion relation** [Mirzakhani]

Related to genus expansion in matrix integrals [Eynard-Orantin]

JT gravity is a matrix integral [Saad-Shenker-Stanford]

$$[Z(\beta)]_{\text{JT gravity}} \simeq \int dH e^{-V(H)} \mathrm{Tr}[e^{-\beta H}]$$

In the genus expansion
and in the double scaling limit
(large matrix, zoomed to
the bottom of the spectrum)

Explains factorization puzzle

$$[Z^2]_{\text{gravity}} \neq [Z]^2_{\text{gravity}}$$

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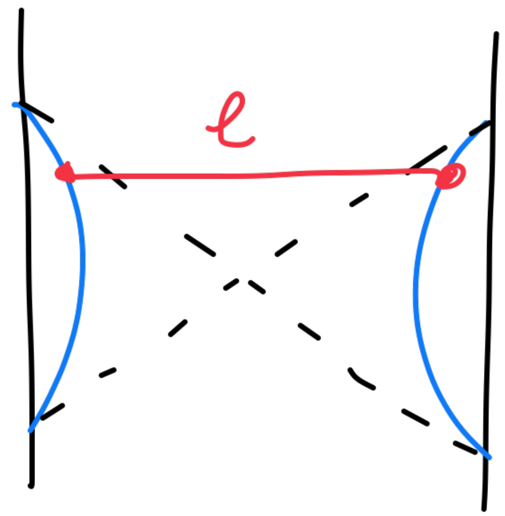
Volume of the interior in JT gravity

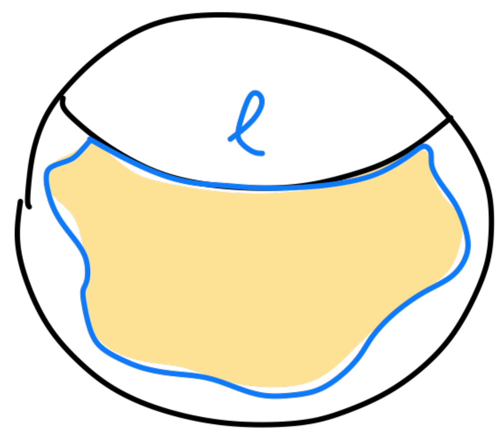
Classical volume



$$\ell_{\text{ren}} = 2\log\left(2\cosh\left[\frac{2\pi}{\beta}t\right]\right) \approx \frac{4\pi}{\beta}t$$

Quantum volume (perturbative): using HH wave function [Yang,Harlow-Jafferis]

$$\psi^{\text{Disk}}_{\beta/2}(\ell) =$$



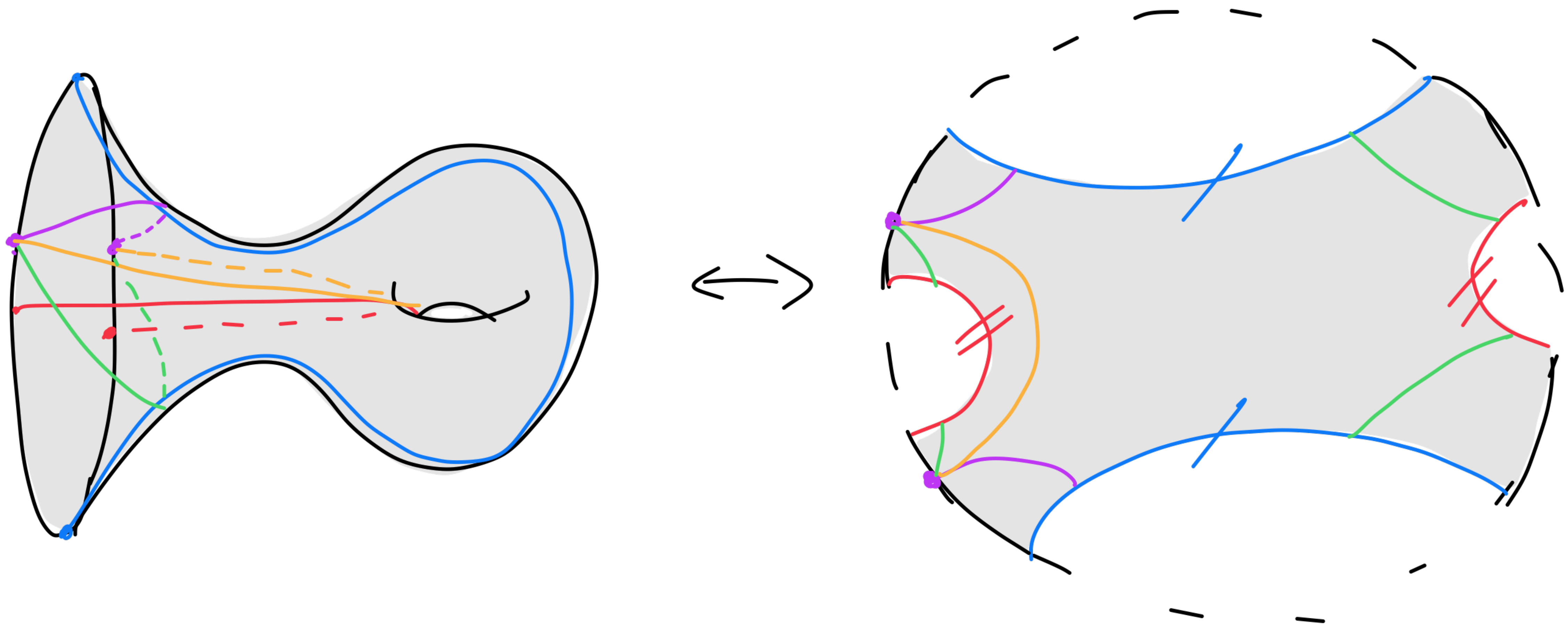
$$= \int_0^\infty dE\rho_0(E)e^{-\beta E/2}[4e^{-\ell/2}K_{i\sqrt{8E}}(4e^{-\ell/2})]$$

$$\langle \ell \rangle = \frac{e^{-S_0}}{Z_{\text{disk}}}\int e^{\ell}d\ell\,|\psi^{\text{Disk}}_{\frac{\beta}{2}+it}(\ell)|^2\,\ell$$

Volume of the interior in JT gravity

Non-perturbative quantum volume:

Challenge: infinite number of extremal geodesics on higher genus surfaces



Taking minimal geodesic on each surface is not an option:
we want to continue to Lorentzian!
Euclidean minimal geodesic changes abruptly:
leads to non-analiticity

Volume of the interior in JT gravity

Non-perturbative quantum volume:

Challenge: infinite number of extremal geodesics on higher genus surfaces

Instead: average over a well defined set of extremal geodesics

$$\langle \ell \rangle \equiv \sum_g e^{S_0(1-2g)} \sum_\gamma \langle \ell_\gamma \rangle_{\text{wiggles \& moduli space}}$$

Divergent!

Natural regularization:

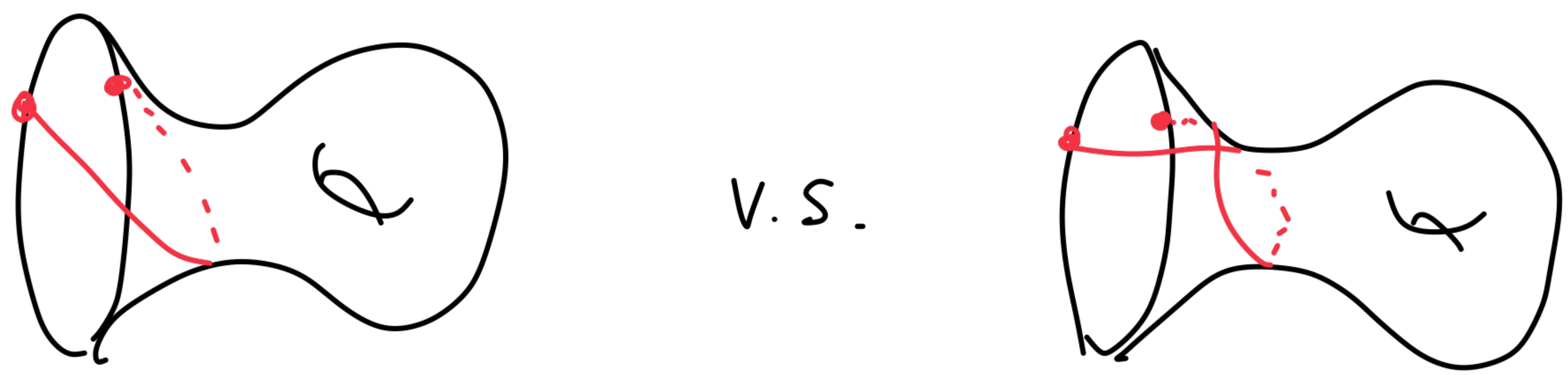
$$\langle \ell \rangle \equiv -\lim_{\Delta\to 0} \frac{d}{d\Delta} \sum_g e^{S_0(1-2g)} \sum_\gamma \langle e^{-\Delta \ell_\gamma} \rangle_{\text{wiggles \& moduli space}}$$

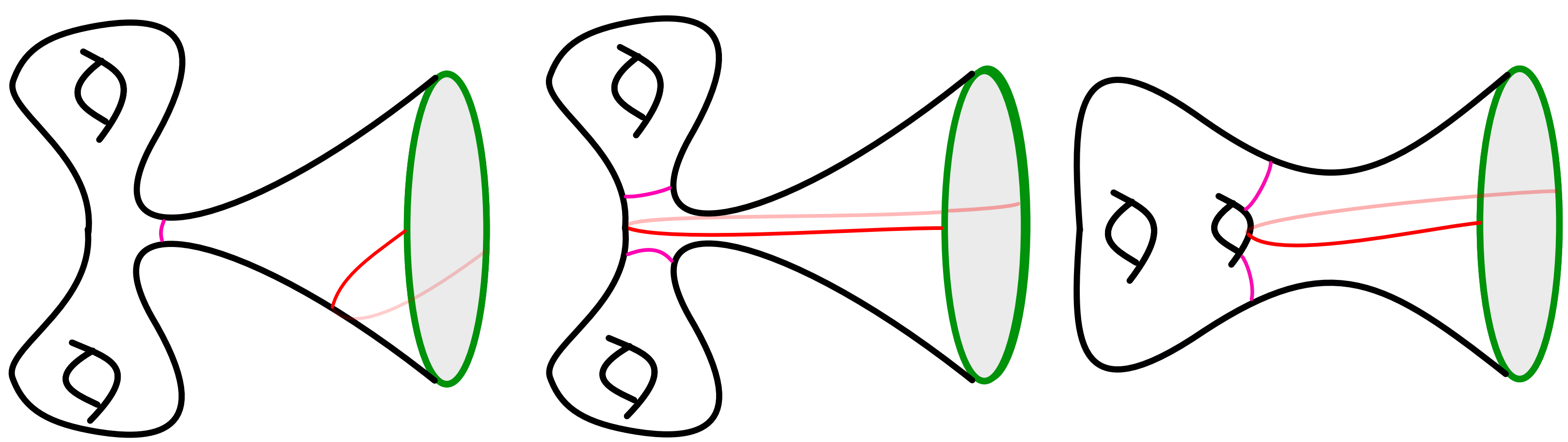
If all geodesics are summed: two point function [Saad]

Volume of the interior in JT gravity

Non-perturbative quantum volume:

We choose to sum only non-self intersecting geodesics, so that a state can live on the slice defined by the geodesic





Also technically simpler: cutting along the geodesic we can evaluate $\langle e^{-\Delta \ell} \rangle$ by integrating against the respective HH wave functions, just like for the disk

Volume of the interior in JT gravity

Result:

$$\langle \ell(t) \rangle = -\frac{e^{-S_0}}{4\pi^2 Z_{\text{disk}}(\beta)} \int_0^\infty dE_1 dE_2 \langle \rho(E_1)\rho(E_2)\rangle M(E_1, E_2) e^{-\frac{1}{2}\beta\left(E_1+E_2\right) - i\left(E_1-E_2\right)t}$$

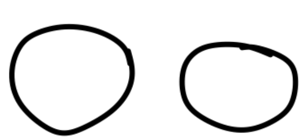
$$M(E_1, E_2) = \frac{8\pi^4 \operatorname{csch}\left(\sqrt{2}\pi\left(\sqrt{E_1}-\sqrt{E_2}\right)\right) \operatorname{csch}\left(\sqrt{2}\pi\left(\sqrt{E_1}+\sqrt{E_2}\right)\right)}{E_1 - E_2}$$

$$\propto \frac{1}{(E_1-E_2)^2}$$

In matrix integrals, universally:

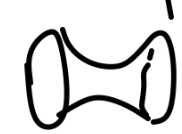
$$\langle \rho(E_1)\rho(E_2)\rangle = \langle \rho(E_1)\rangle\langle\rho(E_2)\rangle + \langle\rho(E_1)\rangle\delta(E_1-E_2) - \frac{\sin^2\left[\pi\langle\rho(E_2)\rangle(E_1-E_2)\right]}{\pi^2(E_1-E_2)^2}$$

disk



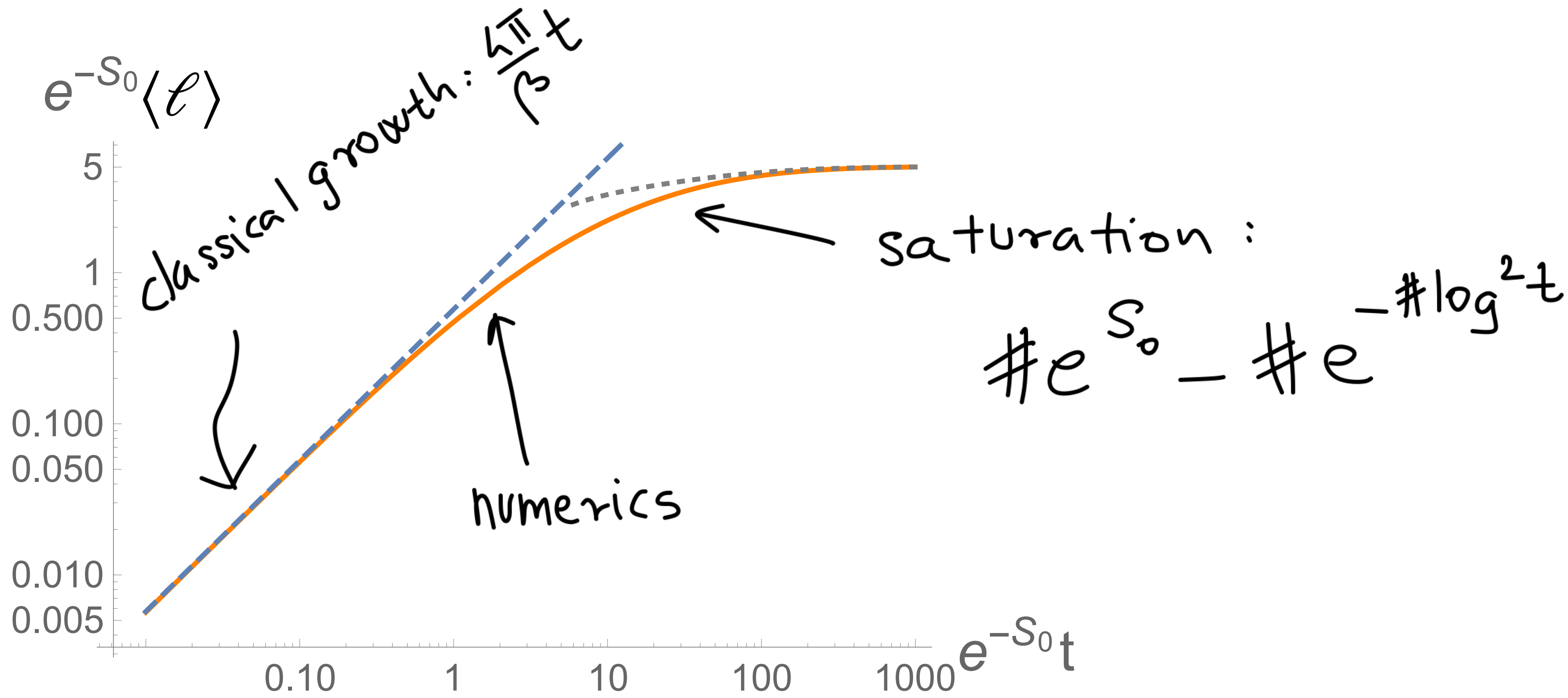
time indep divergence (Δ regulated)

trumpet

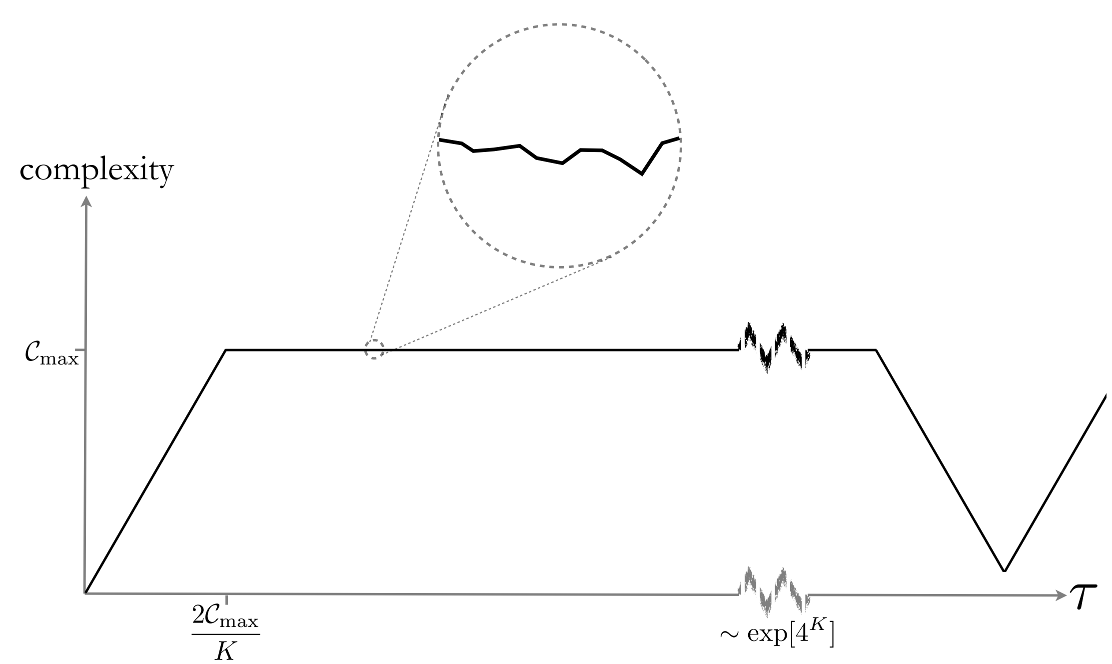


doubly non-perturbative

Volume of the interior in JT gravity



Reminder:

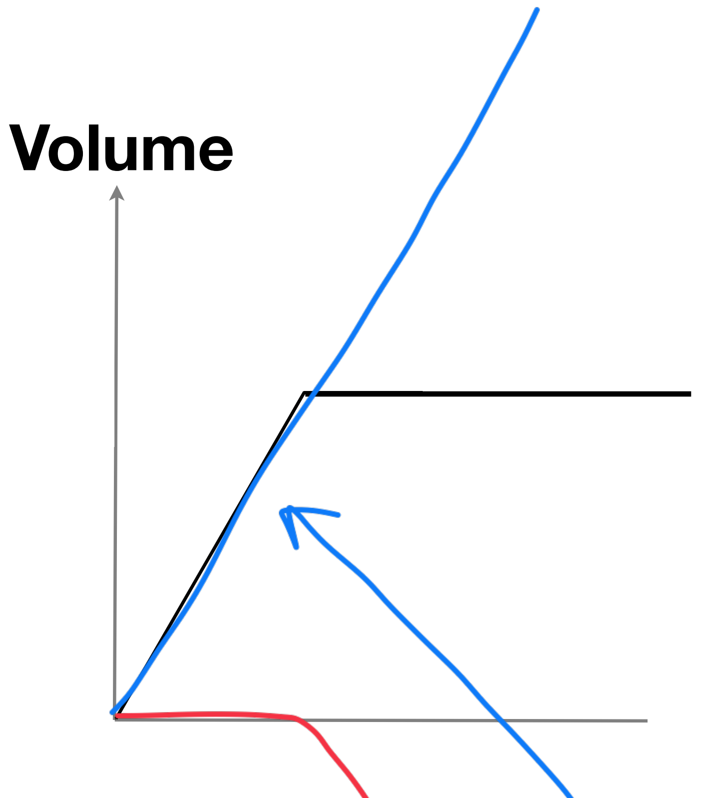


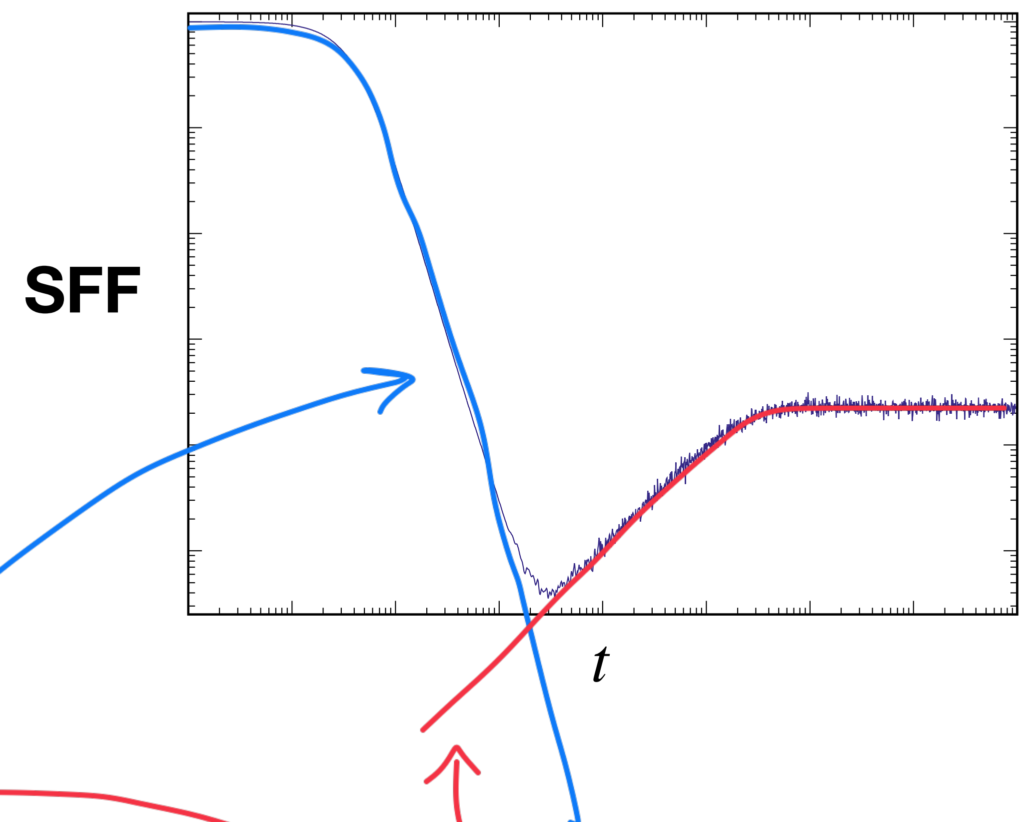
Comments

Volume of the interior in JT gravity

Very similar quantity: spectral form factor $\overline{Z(\beta - it)Z(\beta + it)} \equiv \textbf{SFF}(t)$

$$\int_0^\infty dE_1 dE_2 \langle\rho(E_1)\rho(E_2)\rangle e^{-\frac{1}{2}\beta(E_1+E_2)-i(E_1-E_2)t} \times \begin{cases} M(E_1,E_2) & \textbf{for } \langle \ell(t)\rangle \\ 1 & \textbf{for SFF}(t) \end{cases}$$



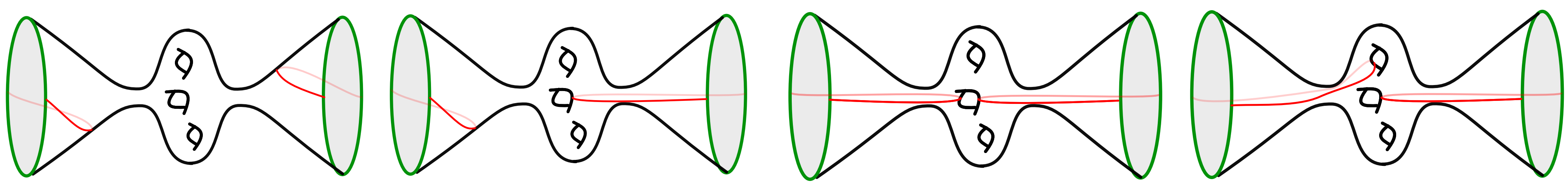


$$\langle\rho(E_1)\rho(E_2)\rangle = \langle\rho(E_1)\rangle\langle\rho(E_2)\rangle + \langle\rho(E_1)\rangle\delta(E_1-E_2) - \frac{\sin^2\left[\pi\langle\rho(E_2)\rangle(E_1-E_2)\right]}{\pi^2(E_1-E_2)^2}$$

Volume of the interior in JT gravity

Noise on volume plateau: absent because JT gravity is an ensemble average

Need to calculate $\overline{\langle \ell(t) \rangle^2}$ **to quantify noise**

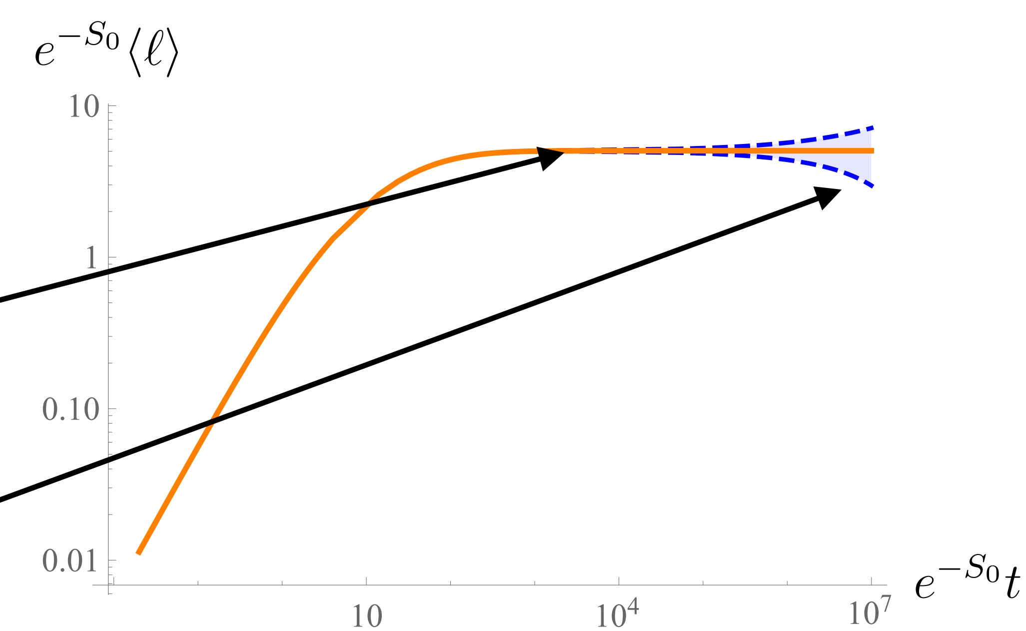


Geodesic cuts will introduce at most four boundaries: depends on $\langle \rho\rho\rho\rho \rangle$

Result: $\sigma_\ell \equiv \sqrt{\overline{[\ell - \overline{\ell}]^2}} \propto \sqrt{t}$ **for** $t \gg e^{S_0}$

$\frac{\sigma_\ell}{\ell} \propto e^{-S_0/2}$ **for** $t \propto e^{S_0}$

$\frac{\sigma_\ell}{\ell} \propto O(1)$ **for** $t \propto e^{2S_0}$

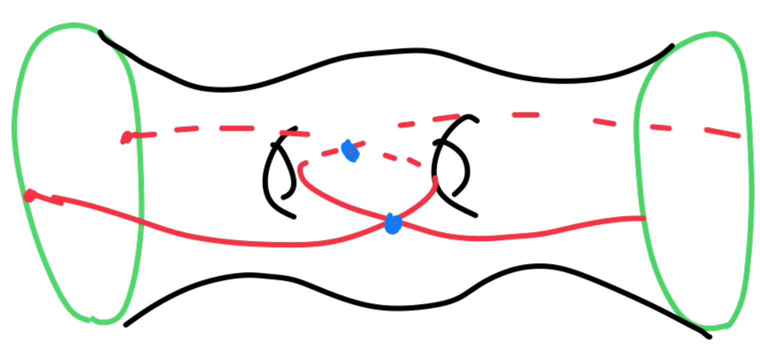


Volume of the interior in JT gravity

$\sigma_\ell \propto \sqrt{t}$ **for** $t \gg e^{S_0}$

Comments:

- **For complexity, one would expect the noise not to change on the plateau**
- **The calculation suggests the noise is highly non-Gaussian, contrary to the spectral form factor**
- **Are we calculating the variance right? E.g. we omit intersecting geodesics**



Summary

Defined regularized non-perturbative volume in JT gravity

Showed that it saturates at $t \propto e^{S_0}$, which is expected from complexity

Result is similar to SFF, but the origin of "ramp" and plateau are different

Questions

Doubly non-perturbative effects are needed for volume saturation
Further: truncating genus expansion at any finite order gives wrong approximation — How to understand the geometric origin of the saturation?

The final formula makes sense in any matrix integral, can also be derived for more general dilaton potential (with methods of [Maxfield-Turiaci,Witten])
call it spectral complexity — can it match some definition of complexity?

Late time variance larger than expected, and not equilibrating

Higher dimensions: is the spectral density 2pt function still in control?