Complexity and Conformal Transformations

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Gauge/Gravity Duality 2021, 27.07.2021

Based on 1806.08376 with Nina Miekley,
1902.06499,
and 2005.02415 and 2007.11555 with Michal P. Heller
Outline

- Part I: What is complexity?
- Part II: Holographic complexity
- Part III: Complexity on the Virasoro group
- Part IV: Relation to wave equations
- Part V: Summary
What is complexity?
What is the "complexity" of an abstract operation or a state?

Abstract operation:

Input state

Operation 1

Intermediate state 1

Operation 2

Intermediate state 2

... Operation n

Output state

Complexity = n

Example: Rubik's cube

Example: Quantum computation

Example: Nielsen's idea of geometric complexity

Complexity = geometric measure of distance between input and output state

Operator complexity $C(U)$ is the minimal number of gates $\mu_i$, that have to be applied to implement $U$.

The complexity of the state $|\psi\rangle$, $C(\psi, \mathcal{R})$, is $\min(C(U))$ for all $U$ such that $|\psi\rangle = U |\mathcal{R}\rangle$ holds for a reference state $|\mathcal{R}\rangle$. 
Idea [Nielsen 2005]: Define a distance measure on $SU(N)$ and calculate geodesic curve

$$U(s) = \mathcal{P} e^{-i \int_0^s Q(\gamma) d\gamma}$$

from $1 = U(0)$ to $U = U(s_f)$. The complexity is

$$C(U) \equiv d(U, 1) = \int_0^{s_f} F[U(s), \dot{U}(s)] ds,$$

with a cost function $F[U(s), \dot{U}(s)]$. Finsler geometry:

- $F$ should be smooth
- Positivity: $F[U, V] \geq 0$ with equality iff $V = 0$.
- Positive homogeneity: $F[U, \sigma V] = \sigma F[U, V]$ for any $\sigma > 0$.
Holographic complexity
Holographic complexity

There are two proposals how to compute complexity $C$ holographically:

- **Volume proposal (CV):** $C \propto \frac{V}{L G_N}$ with the extremal surface volume $V$ [Susskind 2016].

- **Action proposal (CA):** $C = \frac{A}{\pi \hbar}$ where $A$ is the action of the bulk gravity integrated over the *Wheeler de-Witt patch* $\mathcal{W}$ [Brown et al. 2016].
How can we test this?

In AdS$_3$/CFT$_2$, we can study *local conformal transformations* both in the bulk and on the boundary.

In field theory terms, we know how such transformations are generated by the energy momentum tensor (see below.)

In the bulk, we know the corresponding *Bañados geometries* \cite{Banados:1999wn}.

Idea: Calculate complexity change under *infinitesimal* local conformal transformations

\[
x^\pm = G_\pm (\tilde{x}^\pm) = \tilde{x}^\pm + \sigma g_\pm (\tilde{x}^\pm), \quad \sigma \ll 1,
\]

in AdS$_3$/CFT$_2$ according to CV and CA proposal in order to compare CA and CV to each other and to field theory complexity.
CV proposal

For the CV proposal, we found \[MF\text{ and Miekley}\ 2019\]

\[V = V^{\sigma=0} + \sigma V^{(1)} + \sigma^2 V^{(2)} + \mathcal{O}(\sigma^3).\]

The subleading term can be expressed in terms of the Fourier transforms of \(g_{\pm}\):

\[V^{(2)} \propto \int_{-\infty}^{\infty} d\xi \ |\xi|^{3} \left| \hat{g}_{+}(-\xi)e^{2i\pi\xi t_{0}} + \hat{g}_{-}(\xi)e^{-2i\pi\xi t_{0}} \right|^{2}.\]

Features:

- Invariant under \(g_{\pm} \rightarrow -g_{\pm}: C_{(2)}(U_{+}U_{-}|0\rangle) = C_{(2)}(U_{-}^{-1}U_{+}^{-1}|0\rangle),\)
- Non negative \(V^{(2)} \geq 0,\)
- Agreement with field theory proposal of \([\text{Belin et al.}]^{2018}\).
CA proposal

As ultimately worked out in detail in [Lehner et al. 2016], the action is

$$\mathcal{A} = \frac{1}{2} \int_\mathcal{W} (R - 2\Lambda) \sqrt{-g} d^3x + \sum_{\mathcal{T}_i} \int_{\mathcal{T}_i} K \sqrt{-\gamma} d^2x + \sum_{\mathcal{S}_i} \int_{\mathcal{S}_i} K \sqrt{-\gamma} d^2x$$

$$\quad + \sum_{\mathcal{N}_i} \int_{\mathcal{N}_i} \kappa d\lambda \sqrt{\rho} dx + \sum_{\mathcal{J}_i} \int_{\mathcal{J}_i} \eta \mathcal{J}_i \sqrt{\rho} dx$$

$$\quad + \sum_{\mathcal{N}_i} \int_{\mathcal{N}_i} \theta \log(|\theta \ell_c|) d\lambda \sqrt{\rho} dx,$$

with terms for bulk, time- and spacelike boundaries, null boundaries, joints and so called counter terms.

In order to investigate the action proposal, it is necessary to compute the Wheeler De-Witt patch $\mathcal{W}$.

As local conformal transformations break translation invariance, it’s null-boundaries will only be piecewise smooth: They have caustics and creases [Akers et al. 2018].
Caustics and Null-Null joints

As shown in [MF2019]:

\[
A_{\text{crease}} = \int \left( \text{sign}(k, k') \log \left( \frac{1}{2} |k \cdot k'| \right) - \log(|\theta(k)\theta(k')\ell_c^2|) \right) \sqrt{\rho} dy \\
\sim \mathcal{O}(\sigma, \sigma \log(\sigma))
\]

because \( \sqrt{\rho} \sim \mathcal{O}(\sigma) \), \( k \cdot k' \sim \mathcal{O}(\sigma^0) \), \( \theta \sim \mathcal{O}(1/\sigma) \). Why? Global AdS-picture:

caucstic: \( \theta \) diverges

crease: length \( \sim \sigma \), \( \theta \sim 1/\sigma \)
Possible field theory duals

We make the following assumptions ($\sigma \ll 1$):

1. The conformal transformation is generated by an operator
   $U(\sigma) = 1 + \sigma V + \mathcal{O}(\sigma^2)$ with $\mathcal{C}(U(\sigma)) = \sigma \mathcal{K}' + \mathcal{O}(\sigma^2)$ (positive homogeneity).

2. The change of complexity of the state $\psi$ caused by applying the operator $U$ has to be less than the complexity of $U$:
   $\mathcal{C}(U(\sigma)) \geq |\delta \mathcal{C}(\psi)|$ (Triangle ineq.).

3. $|\delta \mathcal{C}(\psi)| = \mathcal{K}|\sigma \log(\sigma)|$ because of CA proposal.

$\Rightarrow \sigma \mathcal{K}' \geq \mathcal{K}|\sigma \log(\sigma)|$ as $\sigma \to 0$ for constants $0 < \mathcal{K}, \mathcal{K}' < \infty$. Contradiction!

Any definition of field-theory complexity that satisfies assumptions 1 and 2 can not be exactly dual to the CA proposal (as defined so far) in AdS$_3$/CFT$_2$ (which implies 3) [MF$_{2019}$].
Complexity on the Virasoro group
Complexity on the Virasoro group

Let us now focus on the \textit{field-theory side} of AdS$_3$/CFT$_2$ and transformations within the \textit{Virasoro group} (e.g. \cite{CaputaMagan2018}, \cite{Oblak2016}).

Orientation preserving diffeomorphisms on the circle:

\[ f : \sigma \to f(\sigma), \quad f(\sigma + 2\pi) \sim f(\sigma) + 2\pi, \quad f'(\sigma) > 0. \]

Group operations:

\[(f_1 \cdot f_2)(\sigma) \equiv f_1 \circ f_2(\sigma) = f_1(f_2(\sigma)), \quad \text{identity element: } f(\sigma) = \sigma.\]

The Virasoro group is obtained as the \textit{central extension} of this group:

\[(f_1(\sigma), \beta) \cdot (f_2(\sigma), \alpha) = (f_1 \circ f_2, \alpha + \beta + C(f_1, f_2)).\]

Herein, \( C(f_1, f_2) \) is the Bott-cocycle.
In CFT$_2$, for *circuits* (= paths on the group manifold parametrized by $\tau$) of the form $f(\tau, \sigma)$ we write unitary operators

$$U(\tau) = \hat{P} e^{-i \int_0^\tau Q(\gamma) d\gamma},$$

with the *generator*

$$Q(\tau) = \int_0^{2\pi} \frac{d\sigma}{2\pi} T(\sigma) \epsilon(\tau, \sigma),$$

$$\epsilon(\tau, f(\tau, \sigma)) \equiv \dot{f}(\tau, \sigma),$$

where $T$ is a component of the CFT *energy-momentum tensor*.

We ignore the central extension for the moment, and as in [Caputa and Magan 2018], [Erdmenger et al. 2020], we assume that the circuit $U(\tau)$ acts on a reference state $|h\rangle$ such that

$$|\psi(\tau)\rangle = U(\tau)|h\rangle.$$
Introducing

\[ \tilde{Q}(\tau) = U(\tau)^{-1} Q(\tau) U(\tau) = \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} \left( T(\sigma) - \frac{c}{12} \{f(\sigma),\sigma\} \right) , \]

we can write

\[ \langle h| \tilde{Q}^2(\tau) | h \rangle = \langle \psi(\tau)| Q^2(\tau) | \psi(\tau) \rangle , \]

and similarly for the one-point function.

We can now define

\[ |\langle \psi(\tau)| \psi(\tau + d\tau) \rangle| \approx 1 - G_{\tau\tau}(\tau)d\tau^2 + \mathcal{O}(d\tau^3) \]

where \( G_{\tau\tau} \) is the \textit{fidelity susceptibility} or \textit{Fubini-Study-metric}.

\[ [\text{Caputa and Magan 2018}] : G_{\tau\tau}(\tau) \sim \langle h| \tilde{Q}^2(\tau) | h \rangle - \langle h| \tilde{Q}(\tau) | h \rangle^2 . \]
The Fubini-Study cost

The Fubini-Study cost associated with the circuit $f(\tau, \sigma)$ (from $\tau = 0$ to $\tau = 1$) takes the form

$$L_{FS} = \int_0^1 d\tau \sqrt{G_{\tau\tau}}$$

$$= \int_0^1 \frac{d\tau}{2\pi} \sqrt{\int \int_0^{2\pi} d\sigma d\kappa \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} \frac{\dot{f}(\tau, \kappa)}{f'(\tau, \kappa)} \Pi(\sigma - \kappa)}$$

with

$$\Pi(\sigma - \kappa) = \frac{c}{32 \sin(\frac{\sigma - \kappa}{2})^4} - \frac{h}{2 \sin(\frac{\sigma - \kappa}{2})^2}.$$

We will treat $\Pi$ as a general function as much as possible.

This is what we took as our definition of complexity in [MF and Heller], [MF and Heller], motivated also by [Chapman et al.], [Belin et al.].
Equations of motion

This allows us to calculate the \textit{equations of motion for affine parametrisation}:

\[
0 \equiv \int_0^{2\pi} d\sigma \left[ -\Pi(\sigma - \kappa) \frac{d}{d\tau} \left( \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)f'(\tau, \kappa)} \right) \right.
\]
\[
+ \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} \partial_\kappa \left( \Pi(\sigma - \kappa) \frac{\dot{f}(\tau, \kappa)}{f'(\tau, \kappa)^2} \right) \left] \right.
\]

Instead of solving the equation of motion for examples of specific boundary conditions, we can gain an understanding of the \textit{qualitative features} of the geometry imposed by our metric by calculating \textit{sectional curvatures}.

\begin{itemize}
\item \textbf{Sectional curvatures mostly} $< 0$ (although \textbf{some} $> 0$ should always exist).
\item This is tied to a sensitivity of the geodesic problem to initial conditions ($\sim$ chaos) \cite{Arnold2014}. See \cite{Brown2017} for qualitative importance of $K < 0$ in computational complexity models.
\end{itemize}
Relation to wave equations
Relation to wave equations

PDEs on the Virasoro group (or $\text{Diff}(S^1)$) can be derived as \textit{Euler-Arnold type equations} from a right- (or left-) invariant inner product on the Virasoro algebra. For example \cite{Khesin and Wendt}:

$$\left\langle (v, b^{(v)}), (w, b^{(w)}) \right\rangle_{\text{Vir}} = \left\langle v, w \right\rangle_{\text{Diff}} + \left\langle b^{(v)}, b^{(w)} \right\rangle_{\mathbb{R}}$$

$$\equiv \int (\alpha v w + \beta v' w') \, d\sigma + b^{(v)} \cdot b^{(w)}$$

leads to EOMs

$$\alpha (\dot{v} + 3v v') - \beta (\dot{v}'' + 2v' v'' + v v''') - b^{(v)} v''' = 0, \quad \dot{b}^{(v)} = 0,$$

Here $v$ corresponds to $\epsilon(\tau, \sigma)$ defined earlier:

$$\epsilon(\tau, f(\tau, \sigma)) = \dot{f}(\tau, \sigma) \Rightarrow \epsilon(\tau, \sigma) = -\frac{\dot{F}(\tau, \sigma)}{F'(\tau, \sigma)},$$

where $F$ is the inverse of $f$, i.e. $F(\tau, f(\tau, \sigma)) = \sigma$. 

This fits (up to \( f \leftrightarrow F \)) together with our general approach when writing

\[
\langle v, w \rangle_{\text{Diff}^+} = \int \int d\sigma\, d\kappa\, \Pi(\sigma - \kappa)\, v(\sigma)\, w(\kappa),
\]

\[
\Pi(\sigma - \kappa) = \alpha \delta(\sigma - \kappa) + \beta \delta''(\sigma - \kappa), \quad (\ast)
\]

and yields several well-known \textit{integrable wave equations}:

- \( \alpha = \beta = 1 \): Camassa-Holm equation
- \( \alpha = 0, \beta = 1 \): Hunter-Saxton equation
- \( \alpha = 1, \beta = 0 \): \textit{Korteweg-de Vries equation}, discussed as complexity model in \cite{CaputaMagan2018, ErdmengerEtAl2020}.

Questions:

- What kind of complexity is defined by such equations?
- When does \((\ast)\) arise as a 2-point function? Which EOMs are ”best”?
- What results from the math literature can be applied to Virasoro complexity, concerning \textit{solutions, curvature, conjugate points, geodesic completeness} etc.?
This fits (up to $f \leftrightarrow F$) together with our general approach when writing

$$
\langle v, w \rangle_{\text{Diff}^+} = \int \int d\sigma \, d\kappa \, \Pi(\sigma - \kappa) \, v(\sigma) \, w(\kappa),
$$

$$
\Pi(\sigma - \kappa) = \alpha \delta(\sigma - \kappa) + \beta \delta''(\sigma - \kappa), \tag{*}
$$

and yields several well-known *integrable wave equations*:

- $\alpha = \beta = 1$: Camassa-Holm equation
- $\alpha = 0, \beta = 1$: Hunter-Saxton equation
- $\alpha = 1, \beta = 0$: *Korteweg-de Vries equation, discussed as complexity model in* [Caputa and Magan 2018], [Erdmenger et al. 2020].

E.g. Hunter-Saxton equation [Lenells 2007]:

- Constant *positive curvature*, geometry of an *open subset of a sphere*.
- No conjugate points, but space of *invertible* maps is geodesically incomplete.
- This is related to *wave breaking*. 
Summary

Idea: Calculate complexity change under *infinitesimal local conformal transformation* in AdS$_3$/CFT$_2$ according to CV and CA proposal in order to compare CA and CV to each other and to field theory complexity.

- For the *CV proposal*, bulk calculations [MF and Miekley 2019] to leading order agree exactly with field theory proposal of [Belin et al. 2018].

- For the *CA proposal* proposal, it is important to carefully determine the WdW-patch. The leading order is $\delta A \propto \sigma \log(\sigma)$, causing a contradiction between the CA proposal and Nielsen complexity. Solution in [Mounim and M"uck 2021]?

- See *first law of complexity* [Bernamonti et al. 2019], [Bernamonti et al. 2020], [Hashemi et al. 2019] for a similar programme.

*Question*: Can we understand the field theory side better? Can we extend this beyond leading order calculations?
Summary

► In [MF and Heller 2020b, MF and Heller 2020a], we studied one natural way of assigning complexity to Virasoro circuits based on the Fubini-Study distance.

► Related papers are [Caputa and Magan 2018, Erdmenger et al. 2020] and recent [Chagnet et al. 2021] in higher dimensions.

► Euler-Arnold equations appear under the same conceptional umbrella. See also [Caputa and Magan 2018, Balasubramanian et al. 2019, Erdmenger et al. 2020, Auzzi et al. 2021, Balasubramanian et al. 2021].

► Upcoming paper with Erdmenger, Gerbershagen, Heller and Weigel!
Thank you very much for your attention
Back up slides...
The geodesic analogy

This is analogous to the *geodesic problem* in finite dimensions, but we use the continuous coordinates \( \sigma, \kappa \in [0, 2\pi] \) instead of discrete indices:

\[
\sum_\sigma \leftrightarrow \int d\sigma,
\]

\[
X^\sigma(\tau) \leftrightarrow f(\tau, \sigma),
\]

\[
g_{\sigma\kappa} \leftrightarrow \frac{\Pi(\sigma - \kappa)}{f'(\tau, \sigma)f'(\tau, \kappa)}.
\]

\[
\int_0^1 d\tau \sqrt{g_{\sigma\kappa} \ddot{X}^\sigma(\tau) \ddot{X}^\kappa(\tau)} \leftrightarrow \\
\int_0^1 \frac{d\tau}{2\pi} \sqrt{\int_0^{2\pi} d\sigma d\kappa \frac{\dot{f}(\tau, \sigma) \dot{f}(\tau, \kappa)}{f'(\tau, \sigma)f'(\tau, \kappa)} \Pi(\sigma - \kappa)}
\]
Differential regularisation

In order to make sense of our expressions, we need to tread the poles of \( \Pi(\sigma - \kappa) \). We use **differential regularisation**, see [Freedman et al. 1992], [Latorre et al. 1994], [Erdmenger and Osborn 1997]:

\[
\int \int_{0}^{2\pi} d\sigma d\kappa \frac{\dot{f}(\tau, \sigma) \dot{f}(\tau, \kappa)}{f'(\tau, \sigma) f'(\tau, \kappa)} \left( \frac{c}{32 \sin \left( \frac{\sigma - \kappa}{2} \right)^4} - \frac{h}{2 \sin \left( \frac{\sigma - \kappa}{2} \right)^2} \right) \Pi(\sigma - \kappa)
\]

\[
= \int \int_{0}^{2\pi} d\sigma d\kappa \log \left( \sin \left( \frac{\sigma - \kappa}{2} \right)^2 \right)
\]

\[
\times \left[ - \frac{c}{24} \frac{\partial^2 \dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} \frac{\partial^2 \dot{f}(\tau, \kappa)}{f'(\tau, \kappa)} + \left( \frac{c}{24} - h \right) \frac{\partial \dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} \frac{\partial \dot{f}(\tau, \kappa)}{f'(\tau, \kappa)} \right].
\]
Degeneracy of the metric

\[
0 = \int\int_0^{2\pi} d\sigma d\kappa \log \left( \sin \left( \frac{\sigma - \kappa}{2} \right)^2 \right) \times \left[ -\frac{c}{24} \partial_{\sigma}^2 \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} \partial_{\kappa}^2 \frac{\dot{f}(\tau, \kappa)}{f'(\tau, \kappa)} + \left( \frac{c}{24} - h \right) \partial_{\sigma} \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} \partial_{\kappa} \frac{\dot{f}(\tau, \kappa)}{f'(\tau, \kappa)} \right].
\]

- for \( \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} = \text{const.} \Rightarrow \partial_{\sigma} \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} = 0 \) for any \( c, h \).

- for \( \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} = \sin(\sigma + \delta) \) for \( h = 0 \).

This is related to the Virasoro sub-groups generated

- by \( L_0 \) for general \( c, h \)

- and \( \{L_0, L_{\pm 1}\} \) for \( h = 0 \)

which transform the state \( |h\rangle \) only by a complex phase when acting on it.

Our notion of complexity does not assign complexity cost to transformations that only generate a phase (unlike Erdmenger et al. 2020).
**$U(1)$ (gauge) symmetry**

Our action is invariant under *shifts of the coordinate* $\sigma$

$$f(\tau, \sigma) \rightarrow f(\tau, \sigma + \delta \sigma) \Rightarrow Q' = 2 \int\int_0^{2\pi} d\sigma \, d\kappa \, \Pi(\sigma - \kappa) \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)}.$$

- When $\Pi(\sigma - \kappa) = \partial_\sigma \partial_\kappa \tilde{\Pi}(\sigma - \kappa) \Rightarrow Q' = 0$.

- In this case, if $f(\tau, \sigma)$ is a solution to the equations of motion, then so will be $f(\tau, \sigma + \alpha(\tau))$ for any function $\alpha(\tau)$: $U(1)$ gauge symmetry.
$U(1)$ (gauge) symmetry

Our action is invariant under *shifts of the coordinate $\sigma$*

\[
f(\tau, \sigma) \to f(\tau, \sigma + \delta \sigma) \Rightarrow Q' = 2 \int \int_0^{2\pi} d\sigma \, d\kappa \, \Pi(\sigma - \kappa) \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)}.
\]

▶ When $\Pi(\sigma - \kappa) = \partial_\sigma \partial_\kappa \tilde{\Pi}(\sigma - \kappa)$ \Rightarrow $Q' = 0$.

▶ In this case, if $f(\tau, \sigma)$ is a solution to the equations of motion, then so will be $f(\tau, \sigma + \alpha(\tau))$ for any function $\alpha(\tau)$: *$U(1)$ gauge symmetry*.

▶ *Fixing the gauge* addresses the degeneracy of the metric and allows for unique solutions to the equations of motion.

▶ For $h = 0$, this is enlarged to a $PSL(2, \mathbb{R})$ gauge invariance.
Our action is invariant under **affine shifts**

\[ \tau \rightarrow \tau + \delta \tau, \]

hence the Lagrangian is conserved:

\[
Q = \int_0^{2\pi} d\sigma d\kappa \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} \frac{\dot{f}(\tau, \kappa)}{f'(\tau, \kappa)} \Pi(\sigma - \kappa),
\]

\[
\frac{d}{d\tau} Q = 0.
\]

This, will ensure affineness on all solutions.
Symmetries - conformal symmetry

Our action is invariant under \textit{conformal transformations}

\[ f \rightarrow F(f) \equiv f + \delta g(f). \]

Note that this is a symmetry \textit{for any choice of} \( \Pi(\sigma - \kappa) \). The conserved charge is

\[ Q_{\delta g} = 2 \int_0^{2\pi} d\sigma \, d\kappa \, \Pi(\sigma - \kappa) \left( \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma) f'(\tau, \kappa)} \right) \delta g(f(\tau, \kappa)). \]

for \textit{any} \( \delta g \). Because of this arbitrariness of \( \delta g \), the conservation of \( Q_{\delta g} \) actually implies the EOMs:

\[ EOMs = 0 \iff \frac{d}{d\tau} Q_{\delta g} = 0 \quad \forall \delta g. \]

But (to the best of my knowledge), the EOMs are not integrable for general \( \Pi(\sigma - \kappa) \) or our specific choice of \( \Pi(\sigma - \kappa) \).
Geodesics and Curvature

Suppose we want to calculate the *geodesic circuit* $f(\tau, \sigma)$ from $f(0, \sigma) = \sigma$ to e.g.

$$f(1, \sigma) = \sigma + \varepsilon \sin(\sigma) \text{ for } \varepsilon \ll 1.$$  

This can be done perturbatively in $\varepsilon$, and we find

$$f(\tau, \sigma) \approx \sigma + \varepsilon \tau \sin(\sigma) + \varepsilon^2 \frac{c \tau^2 - c \tau + 20 h \tau^2 - 20 h \tau}{4 (c + 8 h)} \sin(2 \sigma) + \ldots.$$  

For this example, the squared distance is

$$L_{sq} \propto \int_0^{2\pi} d\sigma d\kappa \frac{\dot{f}(\tau, \sigma) \dot{f}(\tau, \kappa)}{f'(\tau, \sigma) f'(\tau, \kappa)} \Pi(\sigma - \kappa)$$

$$= 2\pi^2 h \varepsilon^2 + \frac{\pi^2 (3 c^2 + 56 c h + 112 h^2)}{96 (c + 8 h)} \varepsilon^4 + \ldots.$$
Curvature

Instead of solving the equation of motion for examples of specific boundary conditions, we can gain an understanding of the **qualitative features** of the geometry imposed by our metric by calculating **sectional curvatures**.

A computation for $\dot{f}(0, \sigma) \equiv u = \sin(m\sigma)$ and $\dot{f}(0, \sigma) \equiv v = \sin(n\sigma)$ gives

$$K(u, v) \propto \frac{(2m + n)(m + 2n)}{24h + c(m + n - 1)(m + n + 1)} - \frac{(2m - n)(m + n)^2}{m(24h + c(m^2 - 1))}$$

for $m > n$. Example $c = 1, h = 0$:

<table>
<thead>
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<th></th>
<th>$m = 1$</th>
<th>$m = 2$</th>
<th>$m = 3$</th>
<th>$m = 4$</th>
<th>$m = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>X</td>
<td>$-\frac{2}{\pi^2}$</td>
<td>$-\frac{3}{4\pi^2}$</td>
<td>$-\frac{2}{5\pi^2}$</td>
<td>$-\frac{1}{4\pi^2}$</td>
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<td>X</td>
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<tr>
<td>3</td>
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<td>-0.11</td>
<td>X</td>
<td>$-\frac{43}{56\pi^2}$</td>
<td>$-\frac{461}{840\pi^2}$</td>
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<tr>
<td>4</td>
<td>-0.041</td>
<td>-0.067</td>
<td>-0.078</td>
<td>X</td>
<td>$-\frac{71}{120\pi^2}$</td>
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<tr>
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<td>-0.044</td>
<td>-0.056</td>
<td>-0.060</td>
<td>X</td>
</tr>
</tbody>
</table>
Curvature

Instead of solving the equation of motion for examples of specific boundary conditions, we can gain an understanding of the qualitative features of the geometry imposed by our metric by calculating sectional curvatures.

Lessons:

- *Sectional curvatures mostly* $< 0$ (although *some* $> 0$ should always exist).

- This is tied to a sensitivity of the geodesic problem to initial conditions ($\sim$ chaos) \[\text{Arnold}^{2014}\].

- See \[\text{Brown et al.}^{2017}\] for qualitative importance of $K < 0$ in computational complexity models.
Conformal transformations in $\text{AdS}_3/\text{CFT}_2$

We consider the groundstate of a $\text{CFT}_2$, given holographically by Poincaré-$\text{AdS}_3$ with metric

$$ds^2 = \frac{1}{4\lambda^2} d\lambda^2 - \lambda dx^+ dx^-.$$

A local conformal transformation on the boundary is described by a bulk solution generating diffeo (SGD) $[\text{Bañados}]^{1999}$ of the (possible) form $[\text{Mandal et al.}]^{2015}$

$$x^+ = G_+(\tilde{x}^+), \quad x^- = G_-(\tilde{x}^-), \quad \lambda = \frac{\tilde{\lambda}}{G'_+(\tilde{x}^+)G'_-(\tilde{x}^-)}.$$

We look at infinitesimal $(\sigma \ll 1)$ diffeos of the form

$$x^\pm = G_\pm(\tilde{x}^\pm) = \tilde{x}^\pm + \sigma g_\pm(\tilde{x}^\pm).$$
Conformal transformations in AdS$_3$/CFT$_2$

From the point of view of the coordinates $(\tilde{\lambda}, t, x)$, the SGD changes two things:

- **different cutoff surface**
  \[
  \tilde{\lambda} = \frac{1}{\epsilon^2} \gg 1
  \]
  \[
  \lambda = \frac{\tilde{\lambda}}{G'_+(\tilde{x}^+) G'_-(\tilde{x}^-)}
  \]
  → different regularization
  → metric and EM-Tensor change appropriately
  → similar to AdS$_2$ holography
  [Maldacena et al. 2016]

- **different equal time slice**, 
  \[
  \tilde{t} = \text{const.} \Leftrightarrow t = t^{\text{bdy}}(x),
  \]
  → different boundary conditions
Counter terms

Via integration by parts, the so called counter terms can be reformulated as joint-terms:

\[ A_{\text{counter}} = \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} \int \theta \log(|\theta \ell_c|) \, d\lambda \sqrt{\rho} \, dy \]

\[ = \int [\sqrt{\rho} \log(|\theta \ell_c|)] \bigg|_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} \, dy - \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} \int \frac{\partial \lambda \theta}{\theta} \, d\lambda \sqrt{\rho} \, dy. \]

\[ = \int [\sqrt{\rho} \log(|\theta \ell_c|)] \bigg|_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} \, dy + \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} \int \partial \lambda \sqrt{\rho} \, d\lambda \, dy \]

\[ = \int [\sqrt{\rho} \log(|\theta \ell_c'|)] \bigg|_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} \, dy, \]

with \( \theta = \frac{1}{\sqrt{\rho}} \partial_\lambda \sqrt{\rho} \) and the Null-Raychaudhuri equation \( \frac{\partial \lambda \theta}{\theta} = -\theta \) in 3d vacuum backgrounds.
Wheeler de-Witt patch

In order to investigate the action proposal, it is necessary to compute the Wheeler De-Witt patch \( \mathcal{W} \).

In Poincaré -patch coordinates, due to the wrapping of the cutoff surface, \( t^{\text{bdy}}(x) \) depends on the chosen conformal transformation \( g_{\pm} \):

\[
t^{\text{bdy}}(x) = \sigma g_{\pm}(x) + \mathcal{O}(\sigma^3) \quad \text{for} \quad t_0 = 0, \quad g_{-}(-x) = g_{+}(+x).
\]
Wheeler de-Witt patch

In order to investigate the action proposal, it is necessary to compute the Wheeler De-Witt patch $\mathcal{W}$.

$\mathcal{W}$ is the set of bulk points which are not in causal contact with the chosen boundary slice $t = t^{bdy}(x)$. I.e. the points in $\mathcal{W}$ are outside all lightcones starting on the boundary slice. $\mathcal{W}$ is bounded by *lightsheets*, generated by lightrays.
Caustics and Null-Null joints

For general $t^{bdy}(x)$, these lightsheets are only *piecewise smooth*: They have *caustics* and *creases* [Akers et al. 2018]. The *focussing theorem* implies that these features will occur at $z$-coordinates of order $\mathcal{O}(1/\sigma)$. 
Caustics and Null-Null joints

$\mathbf{t}^{\text{bdy}}(x) = \frac{0.01}{1+x^2}$

- Caustic, - - crease/null-null-joint, • "hyperbolic point" at boundary, corresponding to caustic in bulk.
Similar situation in black holes

[Carmi et al. 2017] looked at time-evolution of (CA) complexity for black hole backgrounds. Two regimes separated by critical time $t_c$: $t < t_c$ (left) and $t_L + t_R > t_c$ (right) ($t = t_L + t_R$).

For $t < t_c$: $\mathcal{C}_A(t) = \text{const.}$

For $1 \gg t - t_c > 0$: $\dot{\mathcal{C}}_A \sim \log(t - t_c)$, $\mathcal{C}_A$ finite.
Results of \cite{Carmi:2017} seem to suggest that applying the operator \( U(\delta t) = e^{-iH\delta t} \) for infinitesimal \( \delta t \) on the state \(|TFD(t_c)\rangle\) leads to a complexity change of the state exceeding \( C(U(\delta t)) \). Contradiction?

- See also \cite{Couch:2018} for a similar divergence in \( \dot{C}(t) \).

- This problem in \cite{Carmi:2017} (also) appears in higher dimensions.

- This problem appears \textit{with and without} the counterterms
  \( \sim \int \theta \log(|\theta \ell_c|) d\lambda \sqrt{\rho} dx \) \cite{Carmi:2017}.

- \cite{Carmi:2017} proposed as solution that \( C(t) \) should be \textit{averaged over timescales} \( \sim 1/\text{Temp} \sim t_c \).
References


