Complexity and Conformal Transformations

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Gauge/Gravity Duality 2021, 27.07.2021

Based on *1806.08376* with Nina Miekley,
1902.06499,
and *2005.02415* and *2007.11555* with Michał P. Heller

Outline

- Part I: What is complexity?
- Part II: Holographic complexity
- Part III: Complexity on the Virasoro group
- Part IV: Relation to wave equations
- Part V: Summary

What is complexity?

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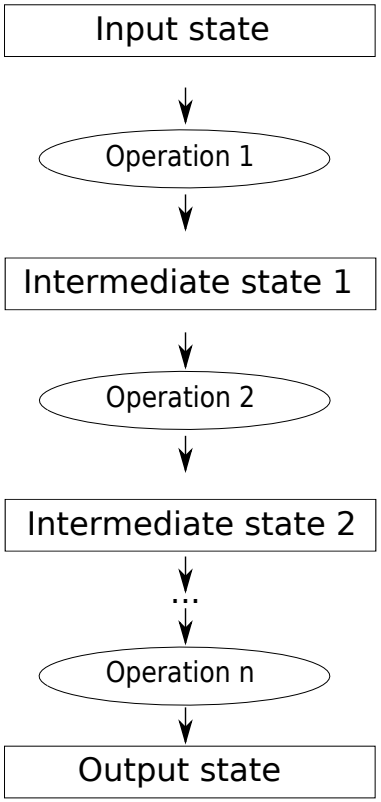
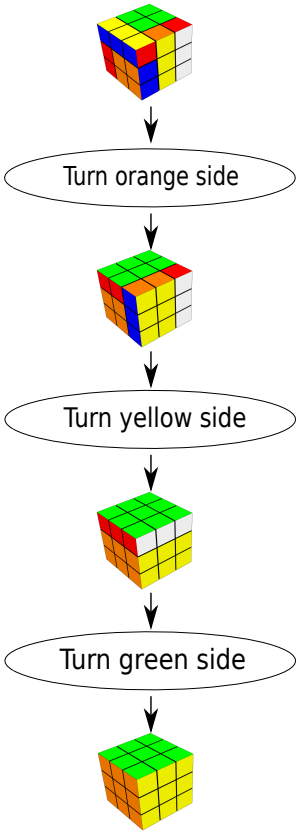
What is the "complexity" of an abstract operation or a state?

Abstract operation:



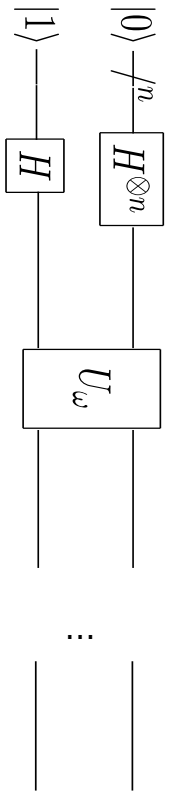
Complexity = n

Example:
Rubik's cube



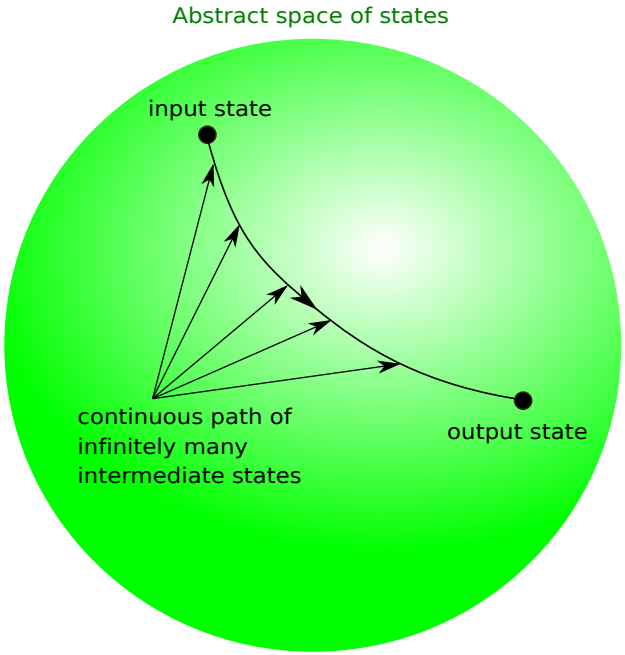
Complexity = 3

Example:
Quantum computation



Complexity = ?

Example:
Nielsen's idea of geometric complexity



Complexity = geometric measure of distance between input and output state

- *Operator complexity* $\mathcal{C}(U)$ is the minimal number of gates μ_i, that have to be applied to implement U.
- The *complexity of the state* $|\psi\rangle$, $\mathcal{C}(\psi, \mathcal{R})$, is $min(\mathcal{C}(U))$ for all U such that $|\psi\rangle = U\,|\mathcal{R}\rangle$ holds for a reference state $|\mathcal{R}\rangle$.

Idea $\left[\begin{smallmatrix}\textit{Nielsen}\\ \textit{2005}\end{smallmatrix}\right]$*:* Define a distance measure on $SU(N)$ and calculate *geodesic curve*

$$U(s) = \overleftarrow{\mathcal{P}}\, e^{-i\int_0^s Q(\gamma)\,d\gamma}$$

from $\mathbb{1} = U(0)$ to $U = U(s_f)$. The complexity is

$$\boxed{\mathcal{C}(U) \equiv d(U, \mathbb{1}) = \int_0^{s_f} F[U(s), \dot{U}(s)]\,ds,}$$

with a *cost function* $F[U(s), \dot{U}(s)]$. *Finsler geometry:*

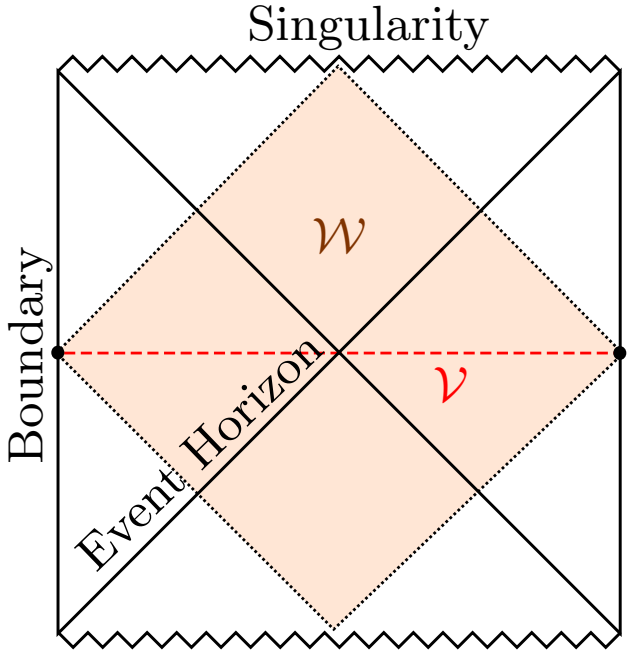
- F should be smooth
- Positivity: $F[U, V] \geq 0$ with equality iff $V = 0$.
- Positive homogeneity: $F[U, \sigma V] = \sigma F[U, V]$ for any $\sigma > 0$.
- Triangle inequality: $F[U, V + V'] \leq F[U, V] + F[U, V']$.

Holographic complexity

Holographic complexity

There are two proposals how to compute complexity $\mathcal{C}$ holographically:

- *Volume proposal (CV):* $\mathcal{C} \propto \frac{\mathcal{V}}{LG_N}$ with the extremal surface volume $\mathcal{V}$ [Susskind 2016].
- *Action proposal (CA):* $\mathcal{C} = \frac{\mathcal{A}}{\pi\hbar}$ where $\mathcal{A}$ is the action of the bulk gravity integrated over the *Wheeler de-Witt patch* $\mathcal{W}$ [Brown et al. 2016].



How can we test this?

In AdS_3/CFT_2, we can study *local conformal transformations* both in the bulk and on the boundary.

In field theory terms, we know how such transformations are generated by the energy momentum tensor (see below.)

In the bulk, we know the corresponding *Bañados geometries* [Bañados 1999].

Idea: Calculate complexity change under *infinitesimal* local conformal transformations

$$x^{\pm} = G_{\pm}(\tilde{x}^{\pm}) = \tilde{x}^{\pm} + \sigma g_{\pm}(\tilde{x}^{\pm}), \ \sigma \ll 1,$$

in AdS_3/CFT_2 according to CV and CA proposal in order to compare CA and CV to each other and to field theory complexity.

CV proposal

For the CV proposal, we found [MF and Miekley 2019]

$$\mathcal{V} = \mathcal{V}\big|_{\sigma=0} + \sigma \cancel{\mathcal{V}_{(1)}} + \sigma^2\, \mathcal{V}_{(2)} + \mathcal{O}(\sigma^3).$$

The subleading term can be expressed in terms of the Fourier transforms of $g_\pm$:

$$\mathcal{V}_{(2)} \propto \int_{-\infty}^{\infty} d\xi\, |\xi|^3 \left| \hat{g}_+(-\xi) e^{2i\pi\xi t_0} + \hat{g}_-(\xi) e^{-2i\pi\xi t_0} \right|^2 .$$

Features:

- Invariant under $g_\pm \to -g_\pm$: $\mathcal{C}_{(2)}(U_+ U_- |0\rangle) = \mathcal{C}_{(2)}(U_+^{-1} U_-^{-1} |0\rangle)$,
- Non negative $\mathcal{V}_{(2)} \geq 0$,
- *Agreement* with field theory proporsal of [Belin et al. 2018].

CA proposal

As ultimately worked out in detail in [Lehner et al. 2016], the action is

$$\mathcal{A} = \frac{1}{2}\int_{\mathcal{W}} (R - 2\Lambda)\sqrt{-g}\,d^3x + \sum_{\mathcal{T}_i}\int_{\mathcal{T}_i} K\sqrt{-\gamma}\,d^2x + \sum_{\mathcal{S}_i}\int_{\mathcal{S}_i} K\sqrt{\gamma}\,d^2x$$
$$+ \sum_{\mathcal{N}_i}\int_{\mathcal{N}_i} \kappa\, d\lambda\sqrt{\rho}\,dx + \sum_{\mathcal{J}_i}\int_{\mathcal{J}_i} \eta_{\mathcal{J}_i}\sqrt{\rho}\,dx$$
$$+ \sum_{\mathcal{N}_i}\int_{\mathcal{N}_i} \theta\log(|\theta\ell_c|)\,d\lambda\sqrt{\rho}\,dx,$$

with terms for bulk, time- and spacelike boundaries, null boundaries, joints and so called counter terms.

In order to investigate the action proposal, it is necessary to compute the *Wheeler De-Witt* patch $\mathcal{W}$.

As local conformal transformations break translation invariance, it's null-boundaries will only be *piecewise smooth*: They have *caustics* and *creases* [Akers et al. 2018].

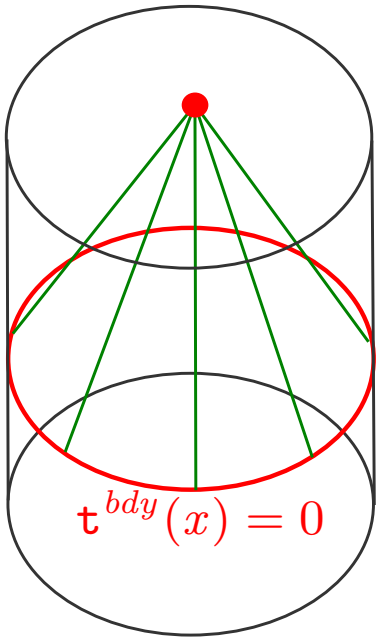
Caustics and Null-Null joints

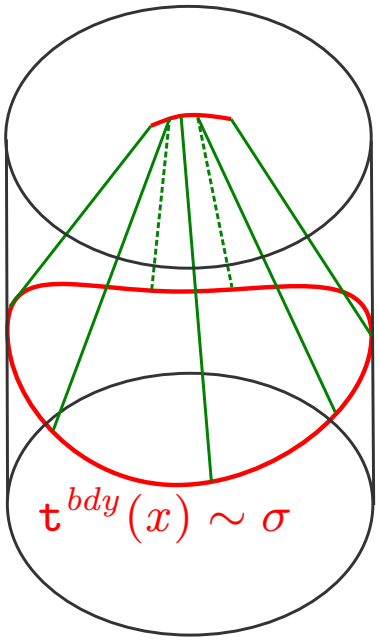
As shown in $\left[\begin{smallmatrix}\text{MF}\\2019\end{smallmatrix}\right]$:

$$\mathcal{A}_{crease} = \int \left(sign(k,k') \log\left(\frac{1}{2}|k\cdot k'|\right) - \log(|\theta(k)\theta(k')\ell_c'^{\,2}|)\right)\sqrt{\rho}dy$$
$$\sim \mathcal{O}(\sigma, \sigma\log(\sigma))$$

because $\sqrt{\rho} \sim \mathcal{O}(\sigma)$, $k\cdot k' \sim \mathcal{O}(\sigma^0)$, $\theta \sim \mathcal{O}(1/\sigma)$. Why? *Global AdS-picture:*

caustic:
θ diverges





crease:
length $\sim \sigma$, $\theta \sim 1/\sigma$

Possible field theory duals

We make the following assumptions ($\sigma \ll 1$):

1. The conformal transformation is generated by an operator $U(\sigma) = \mathbb{1} + \sigma V + \mathcal{O}(\sigma^2)$ with $\mathcal{C}(U(\sigma)) = \sigma\mathcal{K}' + \mathcal{O}(\sigma^2)$ (*positive homogeneity*).

2. The change of complexity of the state ψ caused by applying the operator U has to be less than the complexity of U: $\mathcal{C}(U(\sigma)) \geq |\delta\mathcal{C}(\psi)|$ (*Triangle ineq.*).

3. $|\delta\mathcal{C}(\psi)| = \mathcal{K}|\sigma \log(\sigma)|$ because of *CA proposal*.

$\Rightarrow \sigma\mathcal{K}' \geq \mathcal{K}|\sigma \log(\sigma)|$ as $\sigma \to 0$ for constants $0 < \mathcal{K}, \mathcal{K}' < \infty$. Contradiction! ↯

Any definition of field-theory complexity that satisfies assumptions **1** and **2** can not be exactly dual to the CA proposal (as defined so far) in $\mathrm{AdS}_3/\mathrm{CFT}_2$ (which implies **3**) [MF 2019].

Complexity on the Virasoro group

By Jan Matejko - Public Domain

Complexity on the Virasoro group

Let us now focus on the *field-theory side* of $\mathrm{AdS}_3/\mathrm{CFT}_2$ and transformations within the *Virasoro group* (e.g. [Caputa and Magan 2018], [Oblak 2016]).

Orientation preserving diffeomorphisms on the circle:

$$f : \sigma \to f(\sigma), \quad f(\sigma + 2\pi) \sim f(\sigma) + 2\pi, \quad f'(\sigma) > 0.$$

Group operations:

$$(f_1 \cdot f_2)(\sigma) \equiv f_1 \circ f_2(\sigma) = f_1(f_2(\sigma)), \quad \text{identity element: } f(\sigma) = \sigma.$$

The Virasoro group is obtained as the *central extension* of this group:

$$(f_1(\sigma), \beta) \cdot (f_2(\sigma), \alpha) = (f_1 \circ f_2, \alpha + \beta + \mathfrak{C}(f_1, f_2)).$$

Herein, $\mathfrak{C}(f_1, f_2)$ is the Bott-cocycle.

In CFT_2, for *circuits* (= paths on the group manifold parametrized by τ) of the form $f(\tau, \sigma)$ we write unitary operators

$$U(\tau) = \overleftarrow{\mathcal{P}}\, e^{-i \int_0^\tau Q(\gamma)\, d\gamma},$$

with the *generator*

$$Q(\tau) = \int_0^{2\pi} \frac{d\sigma}{2\pi}\, T(\sigma)\, \epsilon(\tau, \sigma),$$
$$\epsilon(\tau, f(\tau, \sigma)) \equiv \dot{f}(\tau, \sigma),$$

where T is a component of the CFT *energy-momentum tensor*.

We ignore the central extension for the moment, and as in [Caputa and Magan 2018], [Erdmenger et al. 2020], we assume that the circuit $U(\tau)$ acts on a reference state $|h\rangle$ such that

$$|\psi(\tau)\rangle = U(\tau)|h\rangle.$$

Introducing

$$\tilde{Q}(\tau) = U(\tau)^{-1} Q(\tau) U(\tau) = \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} \left(T(\sigma) - \frac{c}{12}\{f(\sigma),\sigma\} \right),$$

we can write

$$\langle h| \tilde{Q}^2(\tau) |h\rangle = \left\langle \psi(\tau) | Q^2(\tau) | \psi(\tau) \right\rangle,$$

and similarly for the one-point function.

We can now define

$$|\langle \psi(\tau) | \psi(\tau + d\tau) \rangle| \approx 1 - G_{\tau\tau}(\tau) d\tau^2 + \mathcal{O}(d\tau^3)$$

where $G_{\tau\tau}$ is the *fidelity susceptibility* or *Fubini-Study-metric*.

$\left[\begin{smallmatrix}\text{Caputa and Magan}\\ 2018\end{smallmatrix}\right]$: $G_{\tau\tau}(\tau) \sim \langle h| \tilde{Q}^2(\tau) |h\rangle - \langle h| \tilde{Q}(\tau) |h\rangle^2 .$

The Fubini-Study cost

The *Fubini-Study cost* associated with the circuit $f(\tau,\sigma)$ (from $\tau=0$ to $\tau=1$) takes the form

$$\begin{aligned}\mathrm{L}_{\mathrm{FS}} &= \int_0^1 d\tau\sqrt{G_{\tau\tau}} \\ &= \int_0^1 \frac{d\tau}{2\pi}\sqrt{\iint_0^{2\pi} d\sigma\, d\kappa \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)}\frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)}\Pi(\sigma-\kappa)}\end{aligned}$$

$$\text{with} \qquad \Pi(\sigma-\kappa) = \frac{c}{32\sin(\frac{\sigma-\kappa}{2})^4} - \frac{h}{2\sin(\frac{\sigma-\kappa}{2})^2}.$$

We will treat Π as a *general function* as much as possible.

This is what we took as our definition of *complexity* in [MF and Heller 2020a], [MF and Heller 2020b], motivated also by [Chapman et al. 2018], [Belin et al. 2018].

Equations of motion

This allows us to calculate the *equations of motion for affine parametrisation*:

$$0 \equiv \int_0^{2\pi} d\sigma \Bigg[-\Pi(\sigma-\kappa)\frac{d}{d\tau}\left(\frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)f'(\tau,\kappa)}\right) + \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)}\partial_\kappa\left(\Pi(\sigma-\kappa)\frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)^2}\right)\Bigg]$$

Instead of solving the equation of motion for examples of specific boundary conditions, we can gain an understanding of the *qualitative features* of the geometry imposed by our metric by calculating *sectional curvatures.*

- *Sectional curvatures mostly* < 0 (although *some* > 0 should always exist).
- This is tied to a sensitivity of the geodesic problem to initial conditions ($\sim$ chaos) [Arnold 2014]. See [Brown et al. 2017] for qualitative importance of $K < 0$ in computational complexity models.

Relation to wave equations

By Katsushika Hokusai - Metropolitan Museum of Art, Public Domain

Relation to wave equations

PDEs on the Virasoro group (or $Diff(S^1)$) can be derived as *Euler-Arnold type equations* from a right- (or left-) invariant inner product on the Virasoro algebra. For example [Khesin and Wendt 2008]:

$$\left\langle (v, b^{(v)}), (w, b^{(w)}) \right\rangle_{Vir} = \langle v, w \rangle_{Diff^+} + \left\langle b^{(v)}, b^{(w)} \right\rangle_{\mathbb{R}}$$

$$\equiv \int \left(\alpha\, v\, w + \beta\, v'\, w' \right)\, d\sigma + b^{(v)} \cdot b^{(w)}$$

leads to EOMs

$$\alpha \left(\dot{v} + 3v\, v' \right) - \beta \left(\dot{v}'' + 2v' v'' + v\, v''' \right) - b^{(v)} v''' = 0, \quad \dot{b}^{(v)} = 0,$$

Here v corresponds to $\epsilon(\tau, \sigma)$ defined earlier:

$$\epsilon(\tau, f(\tau, \sigma)) = \dot{f}(\tau, \sigma) \Rightarrow \epsilon(\tau, \sigma) = -\frac{\dot{F}(\tau, \sigma)}{F'(\tau, \sigma)},$$

where F is the inverse of f, i.e. $F(\tau, f(\tau, \sigma)) = \sigma$.

This fits (up to $f \leftrightarrow F$) together with our general approach when writing

$$\langle v, w \rangle_{Diff^+} = \iint d\sigma \, d\kappa \, \Pi(\sigma - \kappa) \, v(\sigma) \, w(\kappa),$$
$$\Pi(\sigma - \kappa) = \alpha \, \delta(\sigma - \kappa) + \beta \, \delta''(\sigma - \kappa), \qquad (*)$$

and yields several well-known *integrable wave equations:*

- $\alpha = \beta = 1$: Camassa-Holm equation
- $\alpha = 0, \beta = 1$: Hunter-Saxton equation
- $\alpha = 1, \beta = 0$: *Korteweg-de Vries equation, discussed as complexity model in* [Caputa and Magan 2018], [Erdmenger et al. 2020].

Questions:

- What kind of complexity is defined by such equations?
- When does $(*)$ arise as a 2-point function? Which EOMs are "best"?
- What results from the math literature can be applied to Virasoro complexity, concerning *solutions, curvature, conjugate points, geodesic completeness* etc.?

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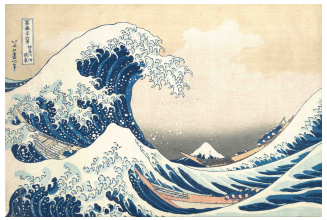
E.g. Hunter-Saxton equation [Lenells 2007]:

- Constant *positive curvature*, geometry of an *open subset of a sphere*.
- No conjugate points, but space of *invertible* maps is geodesically incomplete.
- This is related to *wave breaking*.



Summary

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Idea: Calculate complexity change under *infinitesimal local conformal transformation* in AdS_3/CFT_2 according to CV and CA proposal in order to compare CA and CV to each other and to field theory complexity.

- For the *CV proposal*, bulk calculations [MF and Miekley 2019] to leading order agree exactly with field theory proposal of [Belin et al. 2018].
- For the *CA proposal* proposal, it is important to carefully determine the WdW-patch. The leading order is $\delta\mathcal{A} \propto \sigma \log(\sigma)$, causing a contradiction between the CA proposal and Nielsen complexity. Solution in [Mounim and Mück 2021]?
- See *first law of complexity* [Bernamonti et al. 2019], [Bernamonti et al. 2020], [Hashemi et al. 2019] for a similar programme.

Question: Can we understand the field theory side better? Can we extend this beyond leading order calculations?

Summary

- In [MF and Heller 2020b], [MF and Heller 2020a], we studied *one natural way* of assigning complexity to Virasoro circuits based on the Fubini-Study distance.
- Related papers are [Caputa and Magan 2018], [Erdmenger et al. 2020] and recent [Chagnet et al. 2021] in higher dimensions.
- *Euler-Arnold equations* appear under the same conceptional umbrella. See also [Caputa and Magan 2018], [Balasubramanian et al. 2019], [Erdmenger et al. 2020], [Auzzi et al. 2021], [Balasubramanian et al. 2021].
- *Upcoming paper* with Erdmenger, Gerbershagen, Heller and Weigel!

Thank you very much for your attention

Back up slides...

The geodesic analogy

This is analogous to the *geodesic problem* in finite dimensions, but we use the continuous coordinates $\sigma, \kappa \in [0, 2\pi[$ instead of discrete indices:

$$\sum_\sigma \leftrightarrow \int d\sigma,$$

$$X^\sigma(\tau) \leftrightarrow f(\tau, \sigma),$$

$$g_{\sigma\kappa} \leftrightarrow \frac{\Pi(\sigma - \kappa)}{f'(\tau, \sigma) f'(\tau, \kappa)}.$$

$$\int_0^1 d\tau \sqrt{g_{\sigma\kappa} \dot{X}^\sigma(\tau) \dot{X}^\kappa(\tau)} \quad \leftrightarrow$$

$$\int_0^1 \frac{d\tau}{2\pi} \sqrt{\iint_0^{2\pi} d\sigma\, d\kappa \frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma)} \frac{\dot{f}(\tau, \kappa)}{f'(\tau, \kappa)} \Pi(\sigma - \kappa)}$$

Differential regularisation

In order to make sense of our expressions, we need to tread the poles of $\Pi(\sigma-\kappa)$. We use *differential regularisation*, see [Freedman et al. 1992], [Latorre et al. 1994], [Erdmenger and Osborn 1997]:

$$\iint_0^{2\pi} d\sigma\, d\kappa \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} \frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)} \underbrace{\left(\frac{c}{32\,\sin\left(\frac{\sigma-\kappa}{2}\right)^4} - \frac{h}{2\,\sin\left(\frac{\sigma-\kappa}{2}\right)^2} \right)}_{\Pi(\sigma-\kappa)}$$

$$\stackrel{!}{=} \iint_0^{2\pi} d\sigma\, d\kappa \log\left(\sin\left(\frac{\sigma-\kappa}{2}\right)^2 \right)$$

$$\times \left[-\frac{c}{24} \partial_\sigma^2 \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} \partial_\kappa^2 \frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)} + \left(\frac{c}{24} - h \right) \partial_\sigma \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} \partial_\kappa \frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)} \right].$$

Degeneracy of the metric

$$0 = \iint_0^{2\pi} d\sigma\, d\kappa \log\left(\sin\left(\frac{\sigma-\kappa}{2}\right)^2\right)$$
$$\times \left[-\frac{c}{24}\partial_\sigma^2 \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} \partial_\kappa^2 \frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)} + \left(\frac{c}{24} - h\right) \partial_\sigma \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} \partial_\kappa \frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)} \right].$$

- for $\frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} = const. \Rightarrow \partial_\sigma \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} = 0$ for *any* c, h.
- for $\frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} = \sin(\sigma+\delta)$ for $h = 0$.

This is related to the *Virasoro sub-groups* generated

- by L_0 for general c, h
- and $\{L_0, L_{\pm 1}\}$ for $h = 0$

which transform the state $|h\rangle$ only by a complex phase when acting on it.

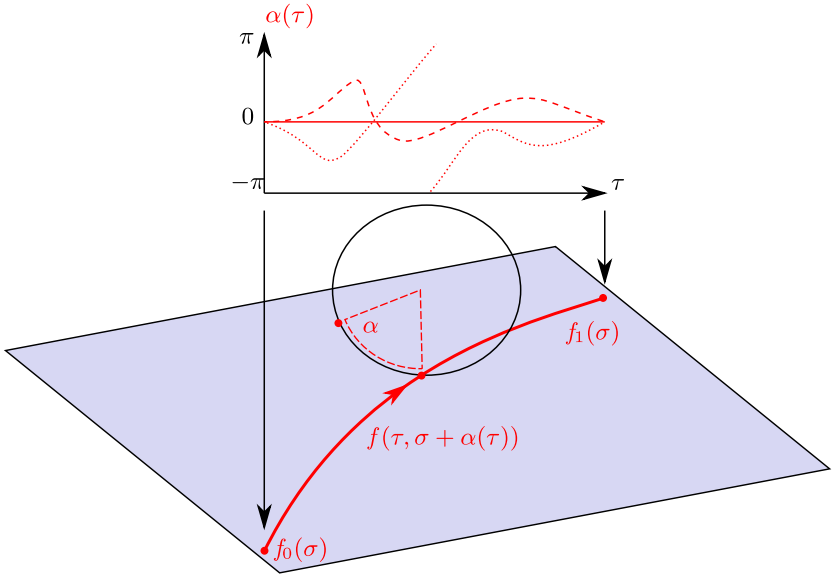
Our notion of complexity does not assign complexity cost to transformations that only generate a phase (unlike [Erdmenger et al. 2020]).

$U(1)$ (gauge) symmetry

Our action is invariant under *shifts of the coordinate σ*

$$f(\tau,\sigma) \to f(\tau,\sigma+\delta\sigma) \Rightarrow \mathcal{Q}' = 2\iint_0^{2\pi} d\sigma\, d\kappa\, \Pi(\sigma-\kappa)\, \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)}.$$

- When $\Pi(\sigma-\kappa) = \partial_\sigma \partial_\kappa \tilde{\Pi}(\sigma-\kappa) \Rightarrow \mathcal{Q}' = 0$.
- In this case, if $f(\tau,\sigma)$ is a solution to the equations of motion, then so will be $f(\tau,\sigma+\alpha(\tau))$ for any function $\alpha(\tau)$: *U(1) gauge symmetry*.



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- When $\Pi(\sigma-\kappa) = \partial_\sigma \partial_\kappa \tilde{\Pi}(\sigma-\kappa) \Rightarrow \mathcal{Q}' = 0$.
- In this case, if $f(\tau,\sigma)$ is a solution to the equations of motion, then so will be $f(\tau,\sigma+\alpha(\tau))$ for any function $\alpha(\tau)$: *U(1) gauge symmetry*.
- *Fixing the gauge* addresses the degeneracy of the metric and allows for unique solutions to the equations of motion.
- For $h=0$, this is enlarged to a $PSL(2,\mathbb{R})$ gauge invariance.

Symmetries - affine shifts

Our action is invariant under *affine shifts*

$$\tau \to \tau + \delta\tau,$$

hence the Lagrangian is conserved:

$$\mathcal{Q} = \iint_0^{2\pi} d\sigma\, d\kappa \frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)} \frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)} \Pi(\sigma-\kappa),$$

$$\frac{d}{d\tau}\mathcal{Q} = 0.$$

This, will ensure affineness on all solutions.

Symmetries - conformal symmetry

Our action is invariant under *conformal transformations*

$$f \to F(f) \equiv f + \delta g(f).$$

Note that this is a symmetry *for any choice of* $\Pi(\sigma - \kappa)$. The conserved charge is

$$\mathcal{Q}_{\delta g} = 2 \iint_0^{2\pi} d\sigma \, d\kappa \, \Pi(\sigma - \kappa) \left(\frac{\dot{f}(\tau, \sigma)}{f'(\tau, \sigma) \, f'(\tau, \kappa)} \right) \delta g(f(\tau, \kappa)).$$

for *any* of δg. Because of this arbitrariness of δg, the conservation of $\mathcal{Q}_{\delta g}$ actually implies the EOMs:

$$EOMs = 0 \Leftrightarrow \frac{d}{d\tau} \mathcal{Q}_{\delta g} = 0 \quad \forall \delta g.$$

But (to the best of my knowledge), the EOMs are not integrable for general $\Pi(\sigma - \kappa)$ or our specific choice of $\Pi(\sigma - \kappa)$.

Geodesics and Curvature

Suppose we want to calculate the *geodesic circuit* $f(\tau,\sigma)$ from $f(0,\sigma)=\sigma$ to e.g.

$$f(1,\sigma)=\sigma+\varepsilon\sin(\sigma) \text{ for } \varepsilon \ll 1.$$

This can be done perturbatively in ε, and we find

$$f(\tau,\sigma)\approx\sigma+\varepsilon\,\tau\,\sin(\sigma)+\varepsilon^2\,\frac{c\,\tau^2-c\,\tau+20\,h\,\tau^2-20\,h\,\tau}{4\,(c+8\,h)}\sin(2\,\sigma)+\ldots\,.$$

For this example, the squared distance is

$$\begin{aligned}\mathrm{L}_{sq}\propto&\iint_0^{2\pi}d\sigma\,d\kappa\,\frac{\dot{f}(\tau,\sigma)}{f'(\tau,\sigma)}\,\frac{\dot{f}(\tau,\kappa)}{f'(\tau,\kappa)}\Pi(\sigma-\kappa)\\&=2\pi^2\,h\,\varepsilon^2+\frac{\pi^2\,\left(3\,c^2+56\,c\,h+112\,h^2\right)}{96\,(c+8\,h)}\varepsilon^4+\ldots\end{aligned}$$

Curvature

Instead of solving the equation of motion for examples of specific boundary conditions, we can gain an understanding of the *qualitative features* of the geometry imposed by our metric by calculating *sectional curvatures.*

A computation for $\dot{f}(0,\sigma) \equiv u = \sin(m\sigma)$ and $\dot{f}(0,\sigma) \equiv v = \sin(n\sigma)$ gives

$$K(u,v) \propto \frac{(2\,m+n)(m+2\,n)}{24\,h + c\,(m+n-1)\,(m+n+1)} - \frac{(2\,m-n)\,(m+n)^2}{m\,\left(24\,h + c\,\left(m^2-1\right)\right)}$$

for $m > n$. Example $c = 1, h = 0$:

n	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
1	X	$-\frac{2}{\pi^2}$	$-\frac{3}{4\pi^2}$	$-\frac{2}{5\pi^2}$	$-\frac{1}{4\pi^2}$
2	-0.20	X	$-\frac{11}{10\pi^2}$	$-\frac{23}{35\pi^2}$	$-\frac{61}{140\pi^2}$
3	-0.076	-0.11	X	$-\frac{43}{56\pi^2}$	$-\frac{461}{840\pi^2}$
4	-0.041	-0.067	-0.078	X	$-\frac{71}{120\pi^2}$
5	-0.025	-0.044	-0.056	-0.060	X

Curvature

Instead of solving the equation of motion for examples of specific boundary conditions, we can gain an understanding of the *qualitative features* of the geometry imposed by our metric by calculating *sectional curvatures.*

Lessons:

- *Sectional curvatures mostly* < 0 (although *some* > 0 should always exist).
- This is tied to a sensitivity of the geodesic problem to initial conditions ($\sim$ chaos) [Arnold 2014].
- See [Brown et al. 2017] for qualitative importance of $K < 0$ in computational complexity models.

Conformal transformations in $\mathrm{AdS}_3/\mathrm{CFT}_2$

We consider the groundstate of a CFT_2, given holographically by Poincaré -AdS_3 with metric

$$ds^2 = \frac{1}{4\lambda^2} d\lambda^2 - \lambda dx^+ dx^-.$$

A local conformal transformation on the boundary is described by a bulk *solution generating diffeo (SGD)* [Bañados 1999] of the (possible) form [Mandal et al. 2015]

$$x^+ = G_+(\tilde{x}^+), \quad x^- = G_-(\tilde{x}^-), \quad \lambda = \frac{\tilde{\lambda}}{G'_+(\tilde{x}^+) G'_-(\tilde{x}^-)}.$$

We look at *infinitesimal* ($\sigma \ll 1$) diffeos of the form

$$x^\pm = G_\pm(\tilde{x}^\pm) = \tilde{x}^\pm + \sigma g_\pm(\tilde{x}^\pm).$$

Conformal transformations in AdS_3/CFT_2

From the point of view of the coordinates (λ, t, x), the SGD changes two things:

- *different cutoff surface*

$$\tilde{\lambda} = \frac{1}{\epsilon^2} \gg 1$$

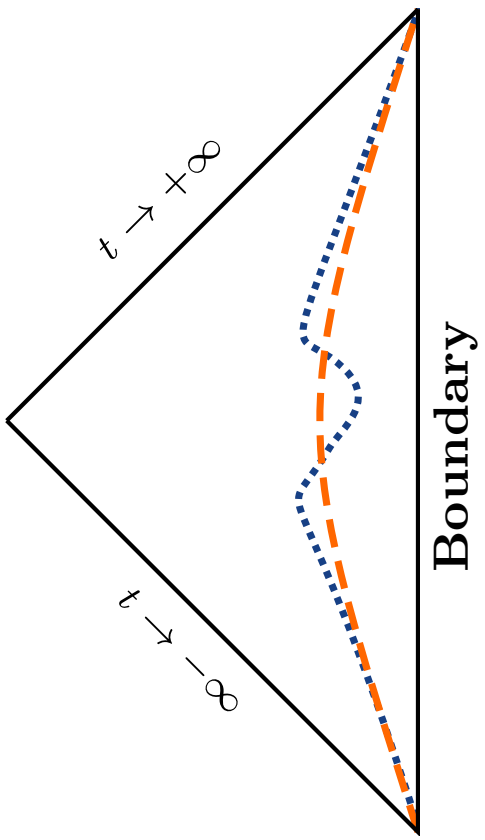
$$\lambda = \frac{\tilde{\lambda}}{G'_+(\tilde{x}^+)\, G'_-(\tilde{x}^-)}$$

$\rightarrow$ different regularization
$\rightarrow$ metric and EM-Tensor change appropriately
$\rightarrow$ similar to AdS_2 holography [Maldacena et al. 2016]

- *different equal time slice*,

$$\tilde{t} = const. \Leftrightarrow t = \mathtt{t}^{bdy}(x),$$

$\rightarrow$ different boundary conditions

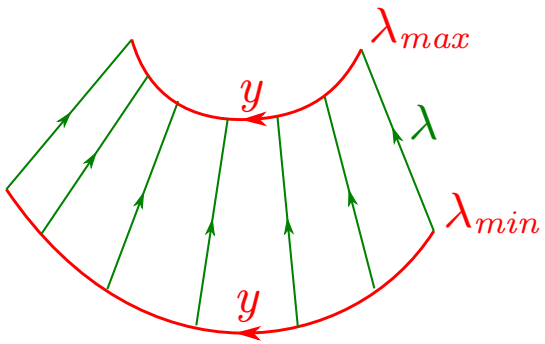


Counter terms

Via integration by parts, the so called counter terms can be reformulated as joint-terms [Reynolds and Ross 2017] [MF 2019]:

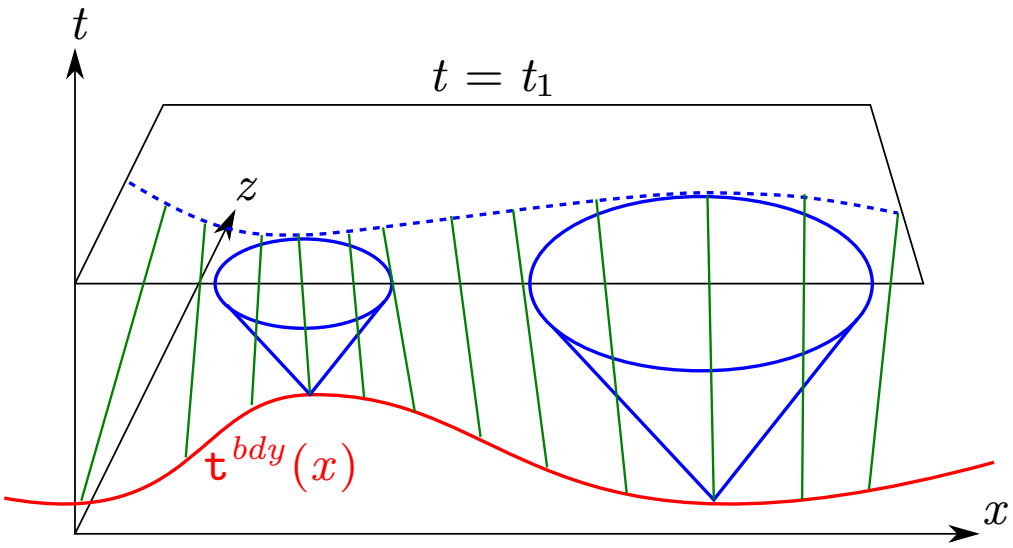
$$\begin{aligned}
\mathcal{A}_{counter} &= \int_{\lambda_{min}}^{\lambda_{max}} \int \theta \log(|\theta \ell_c|)\, d\lambda \sqrt{\rho}\, dy \\
&= \int \left[\sqrt{\rho} \log(|\theta \ell_c|)\right]\Big|_{\lambda_{min}}^{\lambda_{max}} dy - \int_{\lambda_{min}}^{\lambda_{max}} \int \frac{\partial_\lambda \theta}{\theta}\, d\lambda \sqrt{\rho}\, dy. \\
&= \int \left[\sqrt{\rho} \log(|\theta \ell_c|)\right]\Big|_{\lambda_{min}}^{\lambda_{max}} dy + \int_{\lambda_{min}}^{\lambda_{max}} \int \partial_\lambda \sqrt{\rho}\, d\lambda\, dy \\
&= \int \left[\sqrt{\rho} \log(|\theta \ell'_c|)\right]\Big|_{\lambda_{min}}^{\lambda_{max}} dy,
\end{aligned}$$

with $\theta = \frac{1}{\sqrt{\rho}} \partial_\lambda \sqrt{\rho}$ and the *Null-Raychaudhuri equation* $\frac{\partial_\lambda \theta}{\theta} = -\theta$ in 3d vacuum backgrounds.



Wheeler de-Witt patch

In order to investigate the action proposal, it is necessary to compute the *Wheeler De-Witt* patch $\mathcal{W}$.

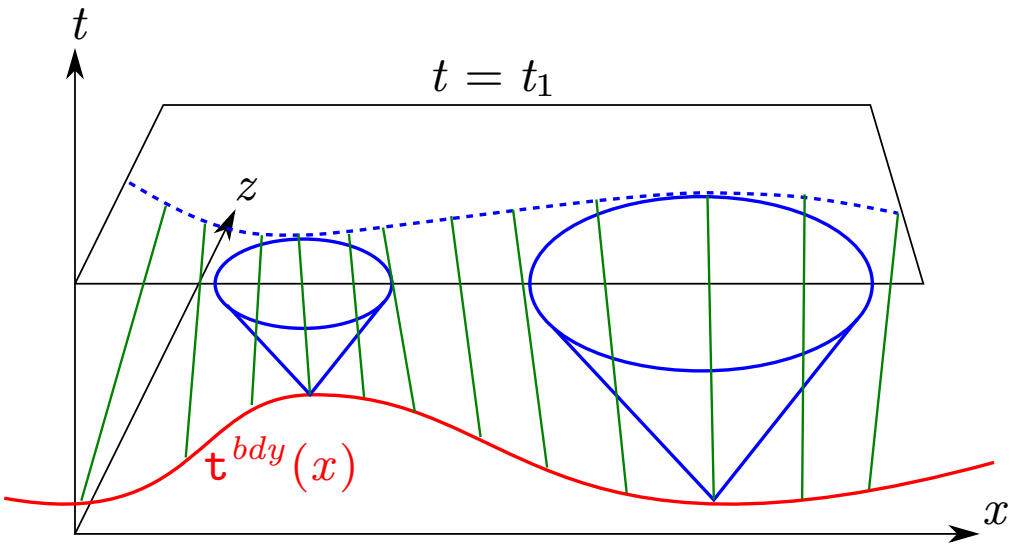


In Poincaré -patch coordinates, due to the wrapping of the cutoff surface, $\mathtt{t}^{bdy}(x)$ depends on the chosen conformal transformation $g_\pm$:

$$\mathtt{t}^{bdy}(x) = \sigma g_+(x) + \mathcal{O}(\sigma^3) \text{ for } t_0 = 0, g_-(-x) = g_+(+x).$$

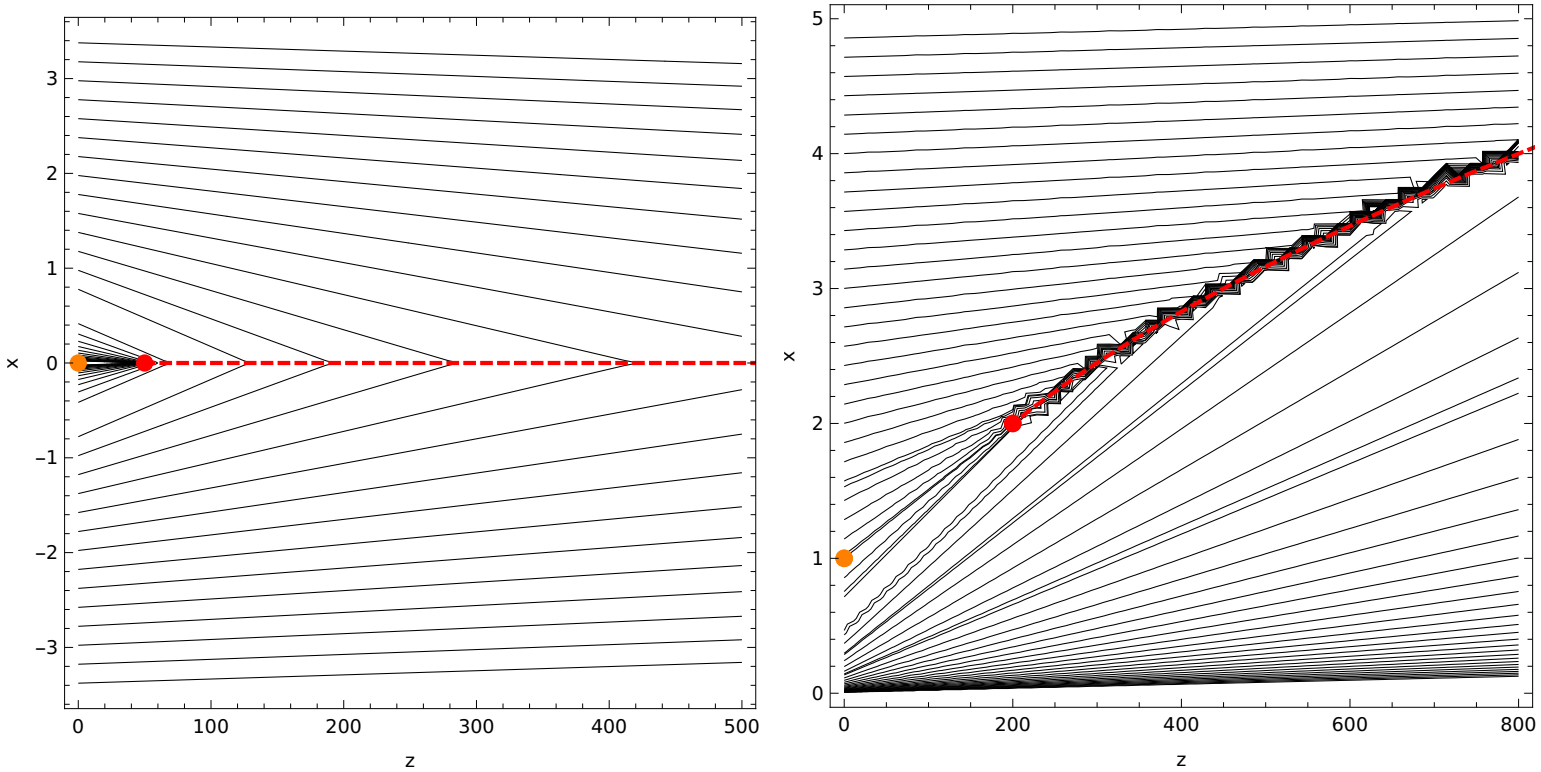
Wheeler de-Witt patch

In order to investigate the action proposal, it is necessary to compute the *Wheeler De-Witt* patch $\mathcal{W}$.



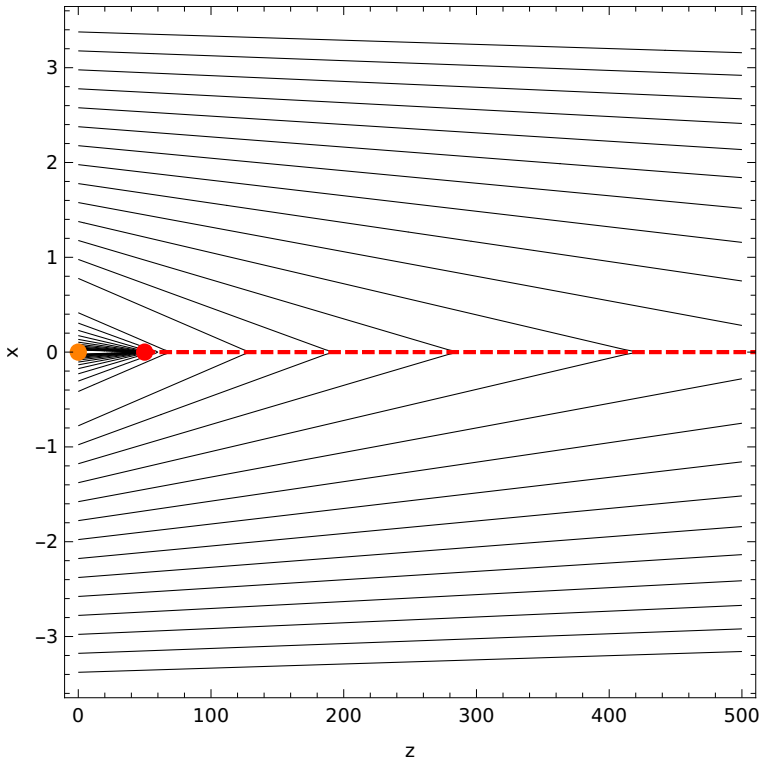
$\mathcal{W}$ is the set of bulk points which are not in causal contact with the chosen boundary slice $t = \mathtt{t}^{bdy}(x)$. I.e. the points in $\mathcal{W}$ are outside all lightcones starting on the boundary slice. $\mathcal{W}$ is bounded by *lightsheets*, generated by lightrays.

Caustics and Null-Null joints

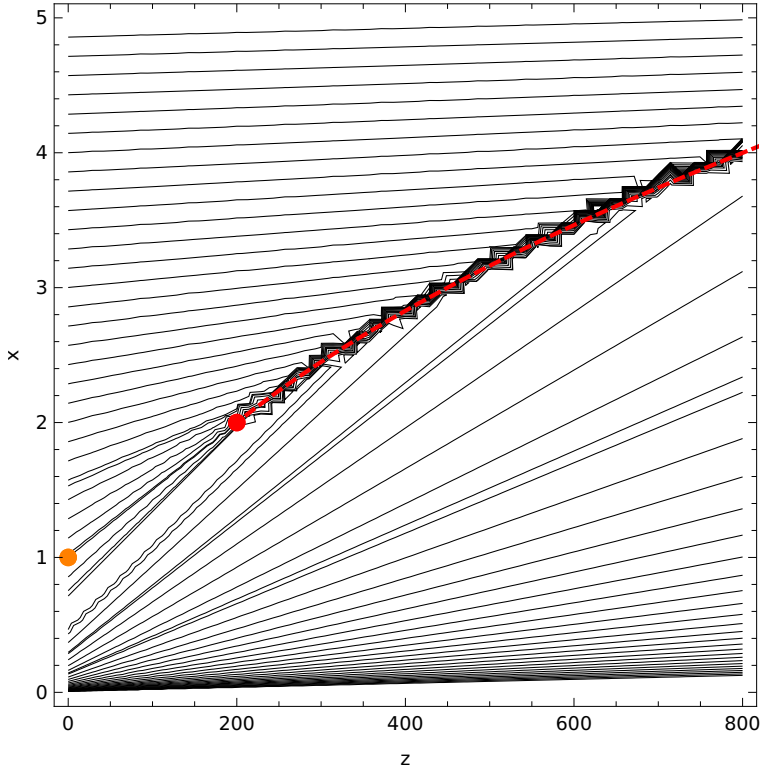


For general $\mathtt{t}^{bdy}(x)$, these lightsheets are only *piecewise smooth*: They have *caustics* and *creases* [Akers et al. 2018]. The *focussing theorem* implies that these features will occur at z-coordinates of order $\mathcal{O}(1/\sigma)$.

Caustics and Null-Null joints



$\mathtt{t}^{bdy}(x) = \frac{0.01}{1+x^2}$

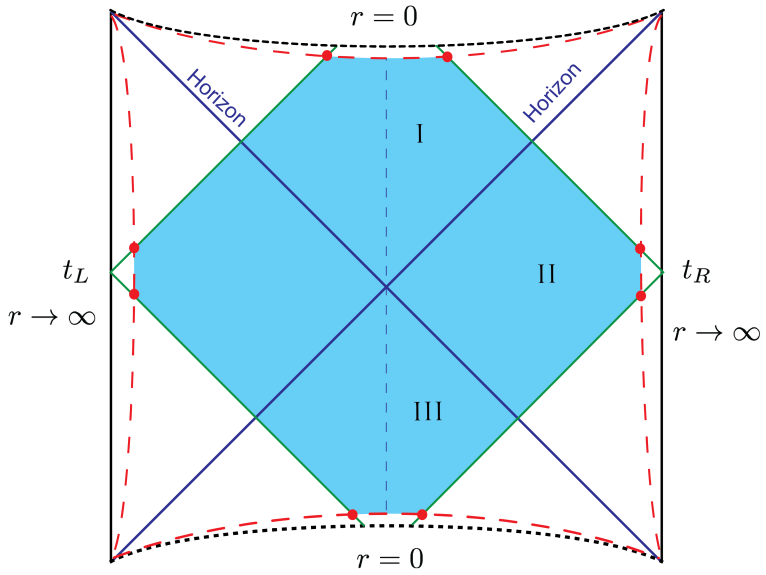


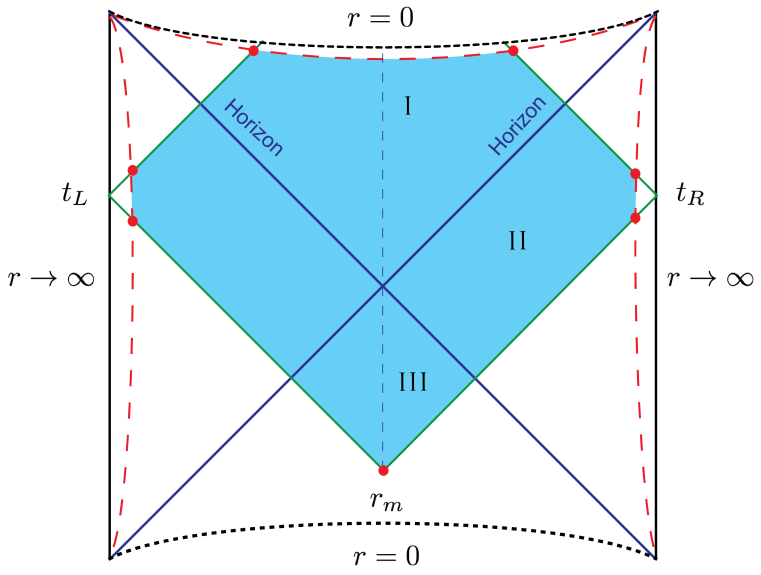
$\mathtt{t}^{bdy}(x) = \frac{-0.01}{1+x^2}$

• Caustic, - - crease/null-null-joint, • *"hyperbolic point"* at boundary, corresponding to caustic in bulk.

Similar situation in black holes

[Carmi et al. 2017] looked at time-evolution of (CA) complexity for black hole backgrounds. Two regimes separated by *critical time* t_c: $t < t_c$ (left) and $t_L + t_R > t_c$ (right) ($t = t_L + t_R$).





Figures from Carmi et al., *JHEP*, 11:188, 2017, CC-BY 4.0.

For $t < t_c$: $\mathcal{C}_A(t) = const.$

For $1 \gg t - t_c > 0$: $\dot{\mathcal{C}}_A \sim \log(t - t_c)$, $\mathcal{C}_A$ finite.

Similar situation in black holes

Results of [Carmi et al. 2017] seem to suggest that applying the operator $U(\delta t) = e^{-iH\delta t}$ for infinitesimal δt on the state $|TFD(t_c)\rangle$ leads to a complexity change of the state exceeding $\mathcal{C}(U(\delta t))$. Contradiction?

- See also [Couch et al. 2018] for a similar divergence in $\dot{\mathcal{C}}(t)$.
- This problem in [Carmi et al. 2017] (also) appears in higher dimensions.
- This problem appears *with and without* the counterterms $\sim \int \theta \log(|\theta \ell_c|) d\lambda \sqrt{\rho} dx$ [Carmi et al. 2017].
- [Carmi et al. 2017] proposed as solution that $\mathcal{C}(t)$ should be *averaged over timescales* $\sim 1/Temp \sim t_c$.

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