Holographic Entropy Cone from Marginal Independence

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[based on 1808.07871, 1812.08133 w/ M. Rangamani & M. Rota
+ 1912.01041 w/ S. Hernández-Cuenca, M. Rangamani & M. Rota
+ 3 w.i.p.'s w/ T. He, **S. Hernández-Cuenca**, F. Jia, & **M. Rota**]

Motivation

- Holography
  - Central question: how does gravity emerge from non-gravitational dual?
  - Hints that entanglement structure plays a crucial role…

$\rightsquigarrow$ characterize aspects of entanglement structure which correspond to *geometric* states

states in a holographic CFT which describe a classical bulk geometry

conveniently diagnosed by entanglement entropy

- Quantum Information
  - Understand information quantities derived from entanglement entropy

OUTLINE

- Motivation & Background
  - Entanglement entropy, holographic entropy cone & arrangement
- 2 important constructs
  - V-space ("proto-entropy subspace" / "min-cut subspace")
  - Pattern of marginal independence (PMI)
- 2 useful tools
  - Holographic graph models
  - Fine- / coarse- graining
- Main result
  - Thm & proof
  - Implications & open questions

Entanglement entropy

For CFT state $|\psi\rangle$ and bi-partition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$

$\rightsquigarrow$ reduced density matrix $\rho_A \equiv \mathrm{Tr}_{\bar{A}} |\psi\rangle\langle\psi|$

EE = $S(A) \equiv -\mathrm{Tr}\, \rho_A \, \log \rho_A$

- Decompose CFT into N elementary subsystems ("colors")

$$\mathcal{H} = \underbrace{\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \cdots}_{N} \otimes \mathcal{H}_{\overline{ABC\cdots}}$$

- $\rightsquigarrow$ entropy vector in D = 2^N - 1 dimensional entropy space

e.g. for N=3, $\vec{S} = \{S(A), S(B), S(C), S(AB), S(AC), S(BC), S(ABC)\}$

conceptually useful to consider large N…

Entropy relations

- Physically realizable entropy vectors are restricted

- Universal restrictions:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$

    ⇨ Mutual information positivity $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - Strong sub-additivity (SSA) $S(AB) + S(BC) \geq S(B) + S(ABC)$

    ⇨ Mutual information monotonicity $I(A:C|B) \equiv I(A:BC) - I(A:B) \geq 0$
  - … (expect more relations with increasing N)

- Further restrictions, depending on the system

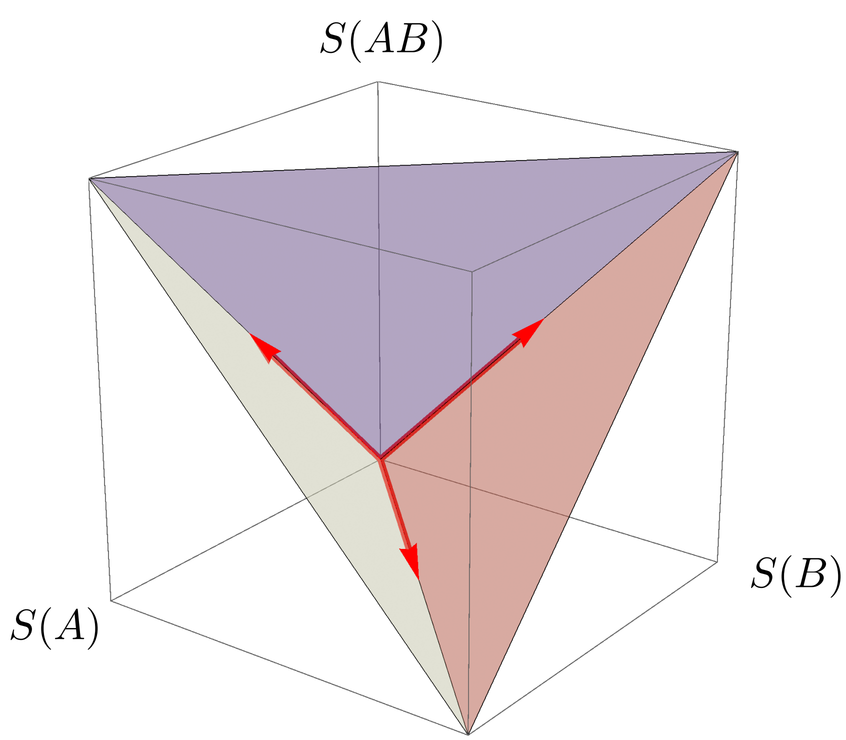
- Our task: understand the full set in holography

Entropy cone

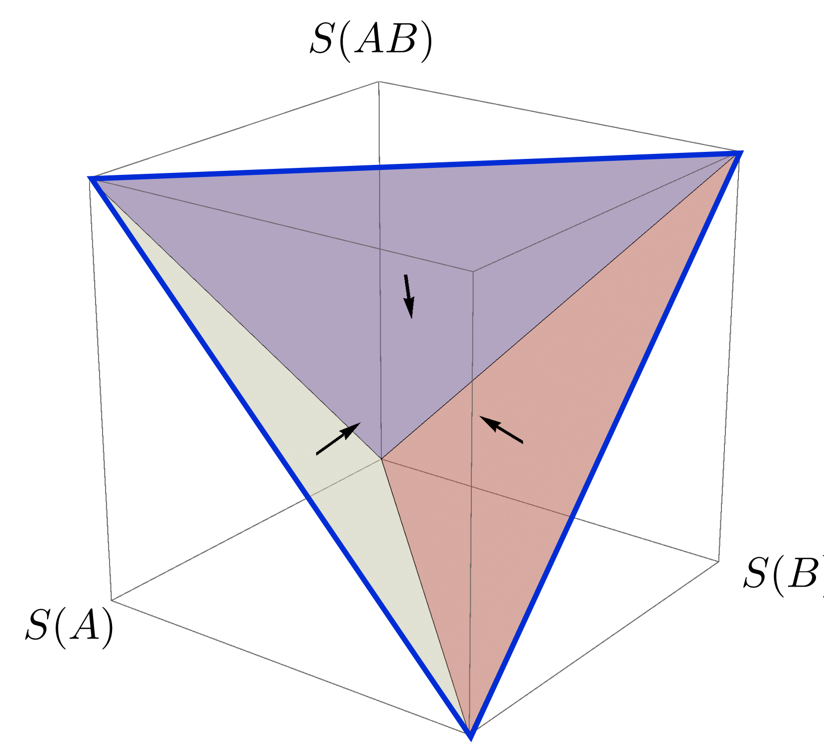
{All physically allowed entropy vectors} = convex cone in entropy space

2 useful characterizations:

convex hull of extreme rays



intersection of half-spaces (=polyhedron)
delineated by entropy inequalities

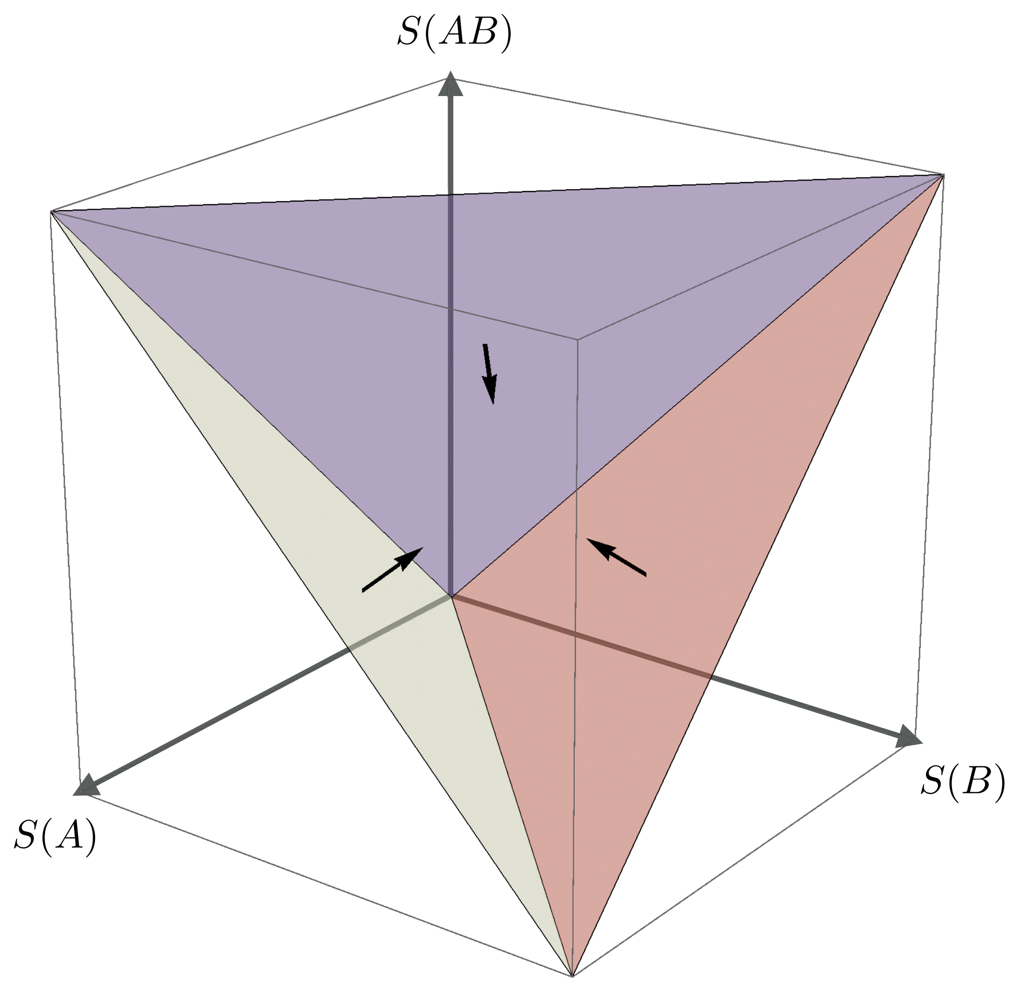


when restricted to geometric states in holography $\rightsquigarrow$ holographic entropy cone (HEC)

[Bao, Nezami, Ooguri, Stoica, Sully, Walter '15]

Entropy cone for N=2

- {geometric states in holographic CFT} ⊂ {all quantum states}
- but for N=2, quantum entropy cone = holographic entropy cone:



$$S(A) + S(B) \geq S(AB)$$

$$S(A) + S(AB) \geq S(B)$$

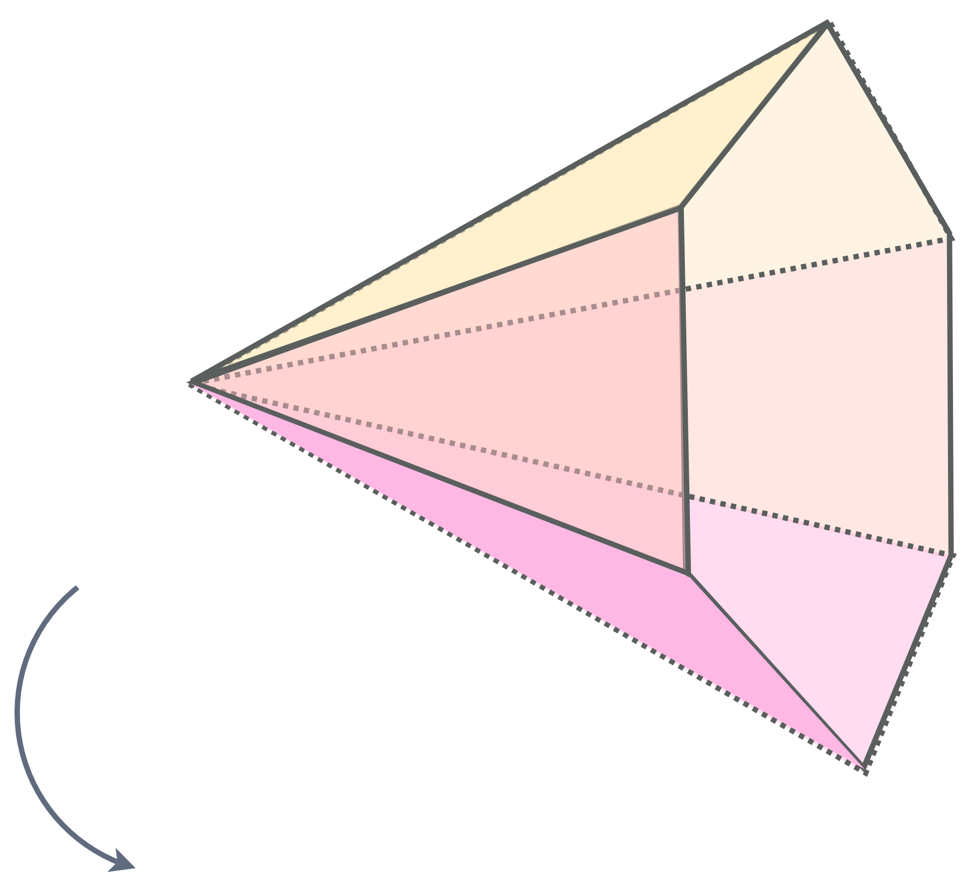
$$S(B) + S(AB) \geq S(A)$$

- For N>2, holographic entropy cone ⊂ quantum entropy cone,
  (⇒ specified by additional entropy inequalities)

Entropy cone for N=3

- Quantum entropy cone specified by {SA,

6 independent (non-redundant) inequalities {uplifts, permutations, purifications}

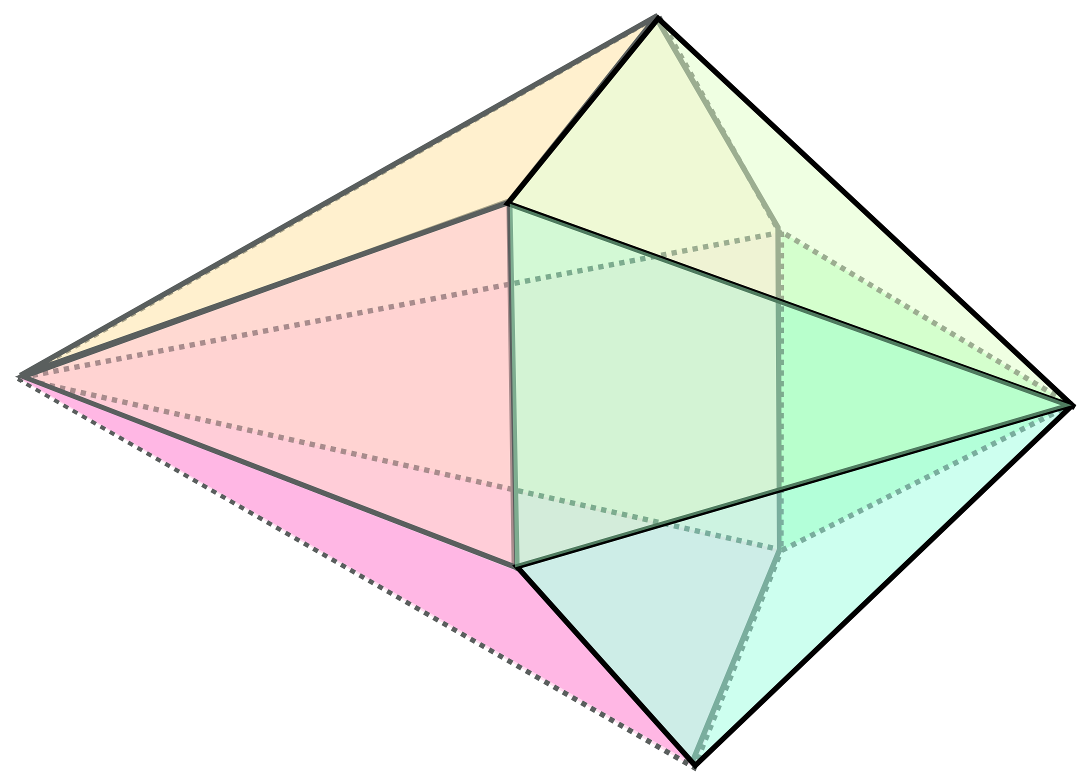


cartoon of 3-d cross-section of R^7 (not including the origin)

Entropy cone for N=3

- Quantum entropy cone specified by {SA, SSA}

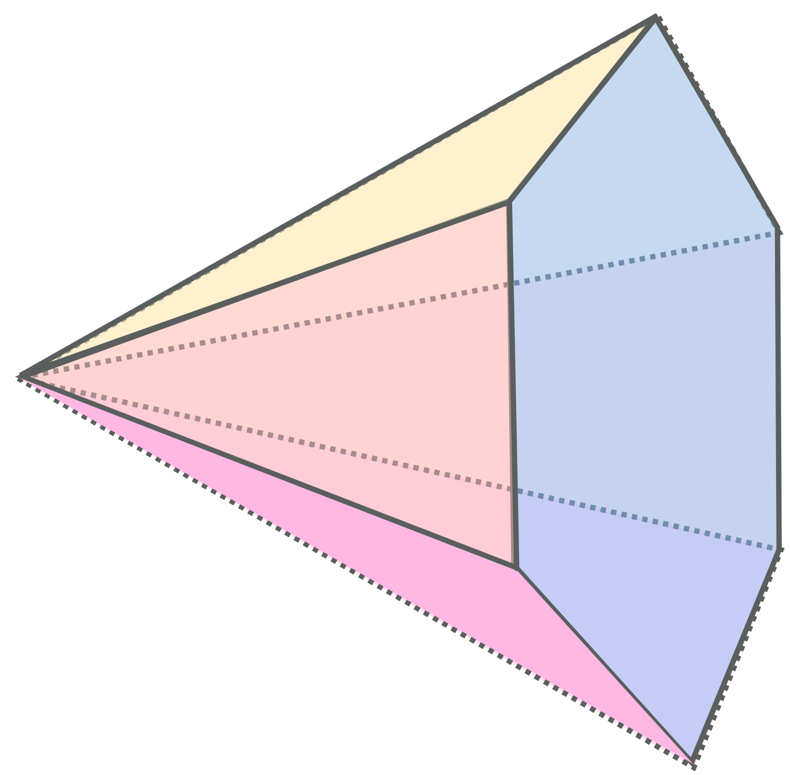
6 independent (non-redundant) inequalities {uplifts, permutations, purifications}



Entropy cone for N=3

- Quantum entropy cone specified by {SA, SSA}

6 independent (non-redundant) inequalities {uplifts, permutations, purifications}



- Holographic entropy cone specified by {SA, MMI}

single (permutation/purification symmetric) inequality

Entropy relations for N=3

Recall:

- Universal:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$

    ⇨ Mutual information positivity $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - Strong sub-additivity (SSA) $S(AB) + S(BC) \geq S(B) + S(ABC)$

    ⇨ Mutual information monotonicity $I(A:C|B) \equiv I(A:BC) - I(A:B) \geq 0$

- True in holography:
  - Monogamy of mutual information (MMI)

$$S(AB) + S(BC) + S(CA) \geq S(A) + S(B) + S(C) + S(ABC)$$

⇨ Tripartite information $I_3(A:B:C) \equiv I(A:B) + I(A:C) - I(A:BC) \leq 0$

NB. geometric proof similar to that of SSA [Hayden, Headrick, Maloney]

but distinguished via bit thread reformulation [Freedman, Headrick]

$\rightsquigarrow$ MMI more deeply rooted in bulk locality than SSA [VH]

Entropy cone for N=4,5,6

- Quantum entropy cone not known ∀ $N \geq 4$.

  ⇝ Important open problem in Quantum Information Science

- N=4 holographic entropy cone however still consists of only {SA, MMI} (now gives 20 independent inequalities)

- N=5 holographic entropy cone (HEC) has 5 further inequalities

  specified in [Bao, Nezami, Ooguri, Stoica, Sully, Walter, '15]

  & proved to be the complete set in [Hernández-Cuenca, '19]

  ⇝ N=5 HEC has 8 orbits of facets (total 372) and 19 orbits of extreme rays (total 2267)

- N=6 HEC (yet TBD) has many further inequalities!

  ⇝ N=6 HEC has ≥ 140 orbits of facets and ≥ 3910 orbits of extreme rays

  [Avis & Hernández-Cuenca '21], [w.i.p., Hernández-Cuenca]

?: How do we find HEC systematically & understand its meaning / implications?

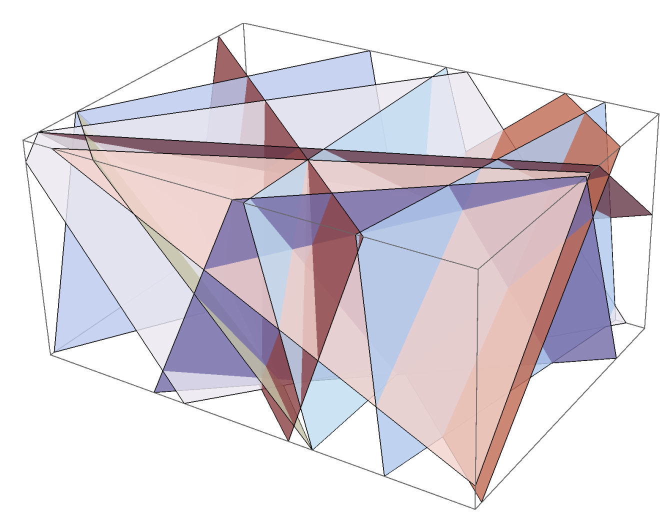
Original Strategy

- Enlarge the structure of interest:
  - Linear combinations of composite subsystem EEs such as $I(A:B)$ or $I_3(A:B:C)$ are called *information quantities*
  - General form, e.g. for N=3:

$$Q(\vec{S}) = q_A\, S(A) + q_B\, S(B) + q_C\, S(C) + q_{AB}\, S(AB) + q_{AC}\, S(AC) + q_{BC}\, S(BC) + q_{ABC}\, S(ABC)$$

  - $Q(\vec{S}) \geq 0$ gives entropy inequality, but it is useful to consider **all** interesting information quantities $Q(\vec{S})$, i.e. ones which:
    - can vanish for some configurations in geometric states
    - are independent of other such IQs (we'll call these *primitive*)
- Each $Q(\vec{S}) = 0$ specifies a hyperplane in entropy space

  $\rightsquigarrow$ full set of primitive IQs = *hyperplane arrangement*



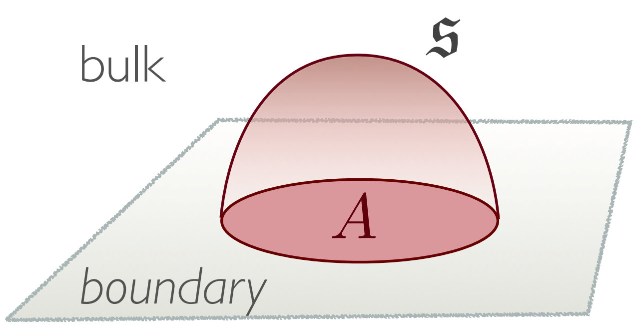
- Utilize holography…

Holographic Entanglement Entropy

Proposal [RT=Ryu & Takayanagi, '06] for static configurations, covariantized by [HRT=VH, Rangamani, Takayanagi, '07] for time-dependent situations:

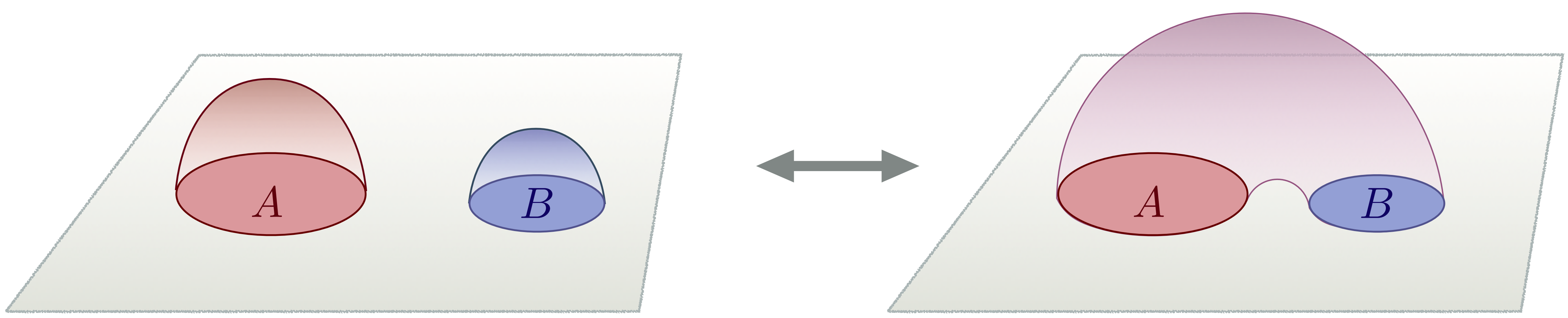
Entanglement entropy $S(A)$ for a boundary region A is captured by the area of a bulk extremal surface $\mathfrak{s}$ homologous to A;

for multiple candidates, choose least area one.



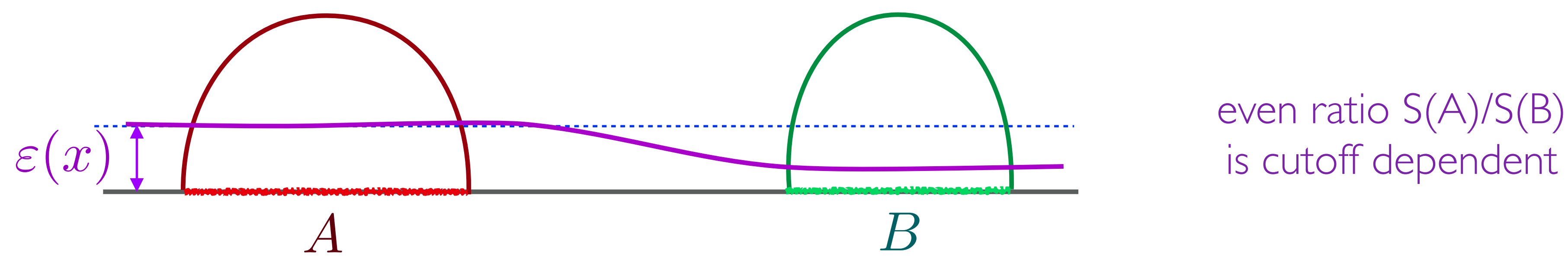
$$S(A) = \min_{\mathfrak{s} \sim A} \frac{\text{Area}(\mathfrak{s})}{4\, G_N}$$

Allows for phase transitions, e.g. jump in surface for S(AB):



Proto-entropy

- Entanglement entropy $S(A)$ is infinite whenever $\partial A \neq \emptyset$
  - $\Rightarrow$ can't localize in entropy space (unless we take a cutoff $\vec{S}_{\varepsilon(x)}$ — but depends on $\varepsilon(x)$)
- However, certain information quantities are UV-finite & ε-indep.
  - Ex.: saturation of SA: $I(A:B) = S(A) + S(B) - S(AB) = 0$



same parts of surfaces appear on both sides of the equality
$\Rightarrow$ cancel out independently of the cutoff

$\Rightarrow$ under varying cutoff, vectors $\vec{S}_{\varepsilon(x)}$ span lower-dimensional subspace of entropy space.

- Suggests hyperplanes are the natural / fundamental constructs
  - Think of QI relations as formal combinations of bulk extremal surfaces ($\rightsquigarrow$ ***proto-entropy***), rather than their areas…

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V-space

- **Def**: *V-space* = intersection of all {$Q=0$} hyperplanes
  - linear subspace of entropy space, of dimensionality $1 \leq d \leq D$

- Meaning & Utility
  - every configuration has a unique V-space associated to it
  - captures the essential features of multipartite entanglement structure implemented by the given configuration (insensitive to irrelevant details)
  - most refined (meaningfully localized in entropy space) such subspace
  - discrete structure

- Unifying construct:
  - extreme rays ⇒ 1-dim. V-space
  - facets ⇒ $(D-1)$-dim. V-space

PMI

- **Def**: *Pattern of Marginal Independence* (PMI) is a specification of full set of subsystems { X , Y } for which $I(X{:}Y) = 0$.
  - *marginals* = reduced density matrices
  - *independent* if factorized structure

- Meaning & Utility
  - every entropy vector $\vec{S}$ has a unique PMI
  - conceptually simpler (more primal) construct
  - likewise a linear subspace of entropy space, discrete structure
  - = intersection of all saturated SA or AL hyperplanes
  - $\Rightarrow$ contains the V-space

Key Q: what is the exact relation between V-space and PMI?

Marginal Independence Problem

- However, not all PMIs are possible, due to
  - mathematical inconsistency

    e.g. violates the identity $I_2(A:BC) + I_2(B:C) = I_2(B:AC) + I_2(A:C)$

  - physical inconsistency

    violates entropy inequality, e.g. SSA ⇒ $I_2(A:BC) = 0 \quad \Rightarrow \quad I_2(A:B) = 0$

- *Marginal Independence Problem* (MIP): what PMIs are realizable?
  - QMIP: what PMIs are realizable in QM?
    considered in [Hernández-Cuenca, VH, Rangamani, Rota]
  - HMIP: what PMIs are realizable by geometric states in holography?
    w.i.p. [He, Hernández-Cuenca, VH, Rota]
  - useful mathematical tools: hyperplane arrangement lattices & matroids

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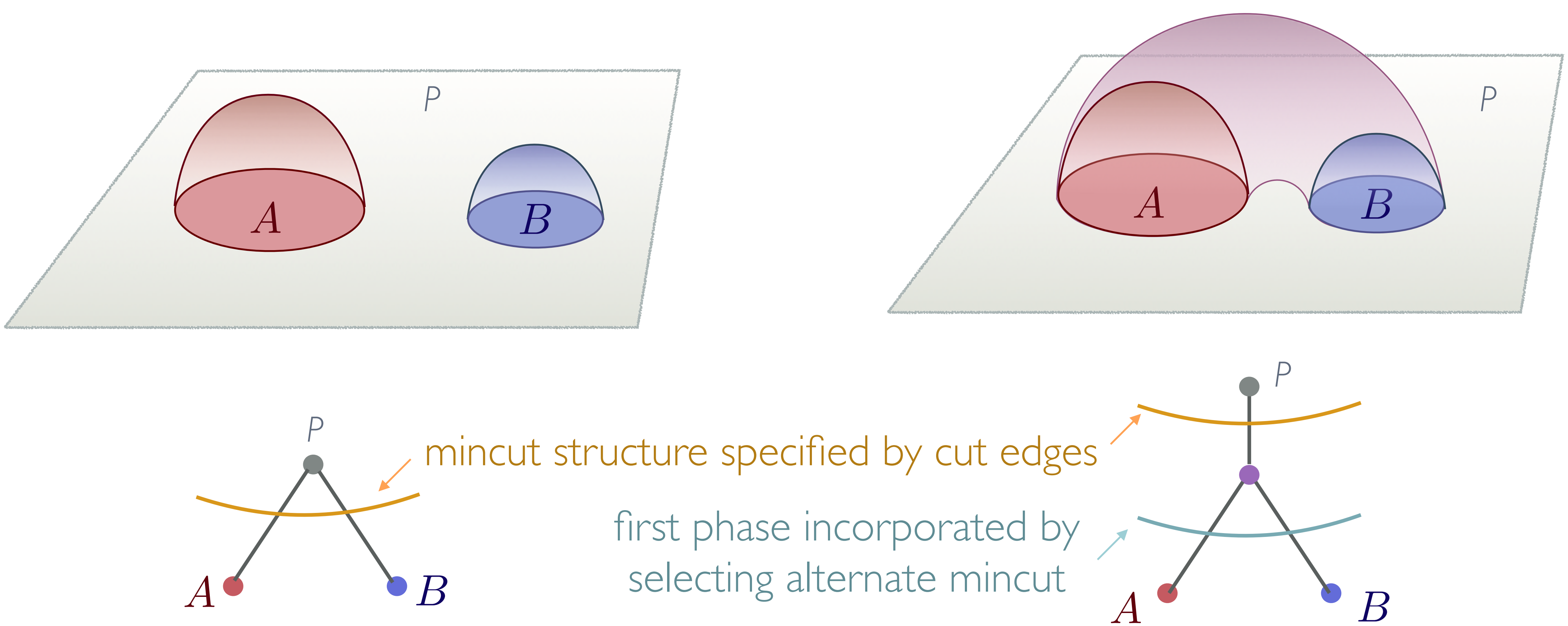
Holographic graph models

- Convenient representation of proto-entropies: abstracts away the discrete elements from a holographic configuration
  - vertices = cells in RT surface network
  - edges = pieces of RT surfaces separating neighboring regions, with weight = corresponding area

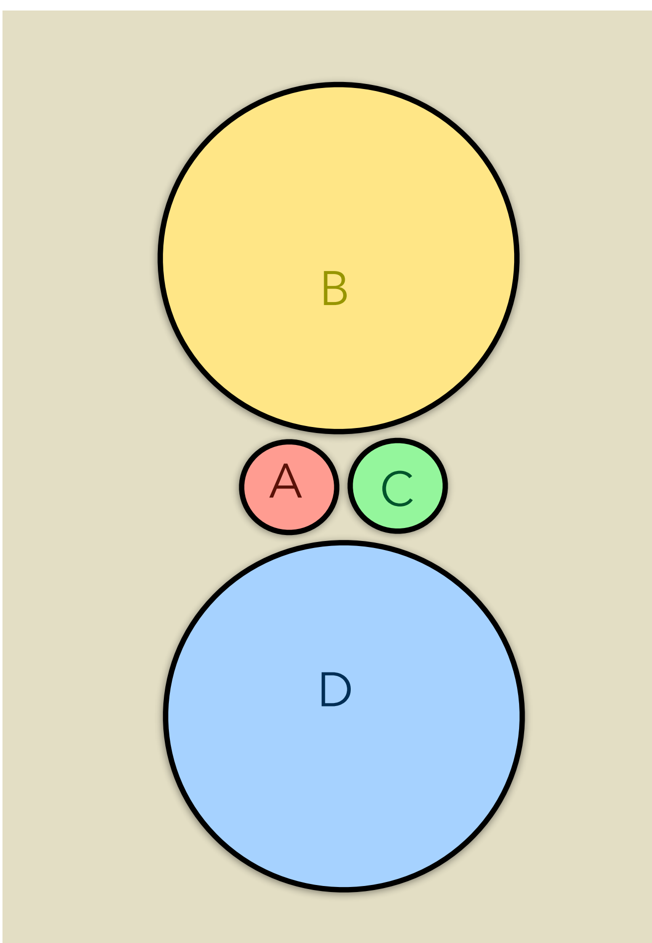
cf. [Bao, Nezami, Ooguri, Stoica, Sully, Walter]

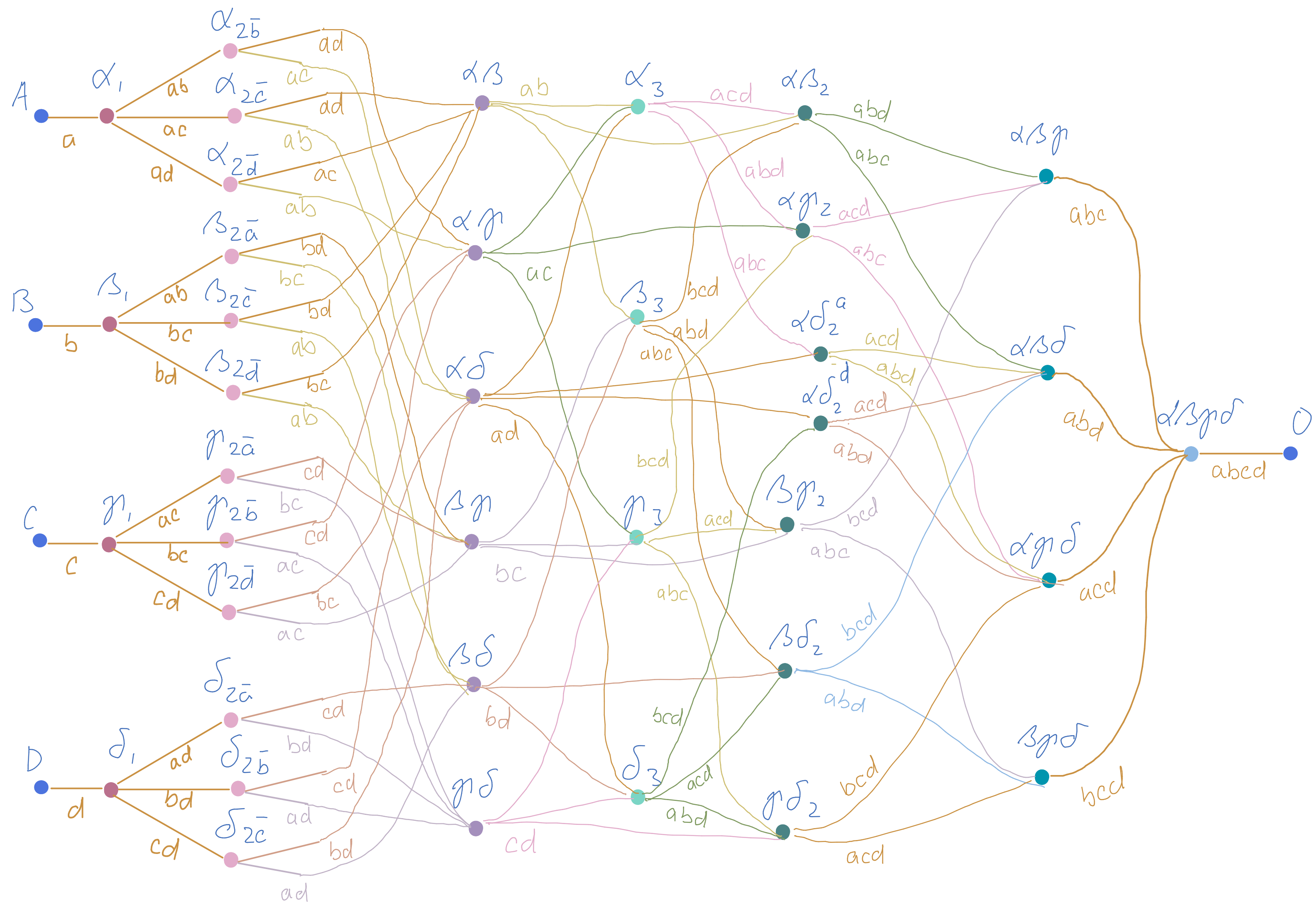


Holographic graph models

- Graphs can get complicated...

e.g. graph model for

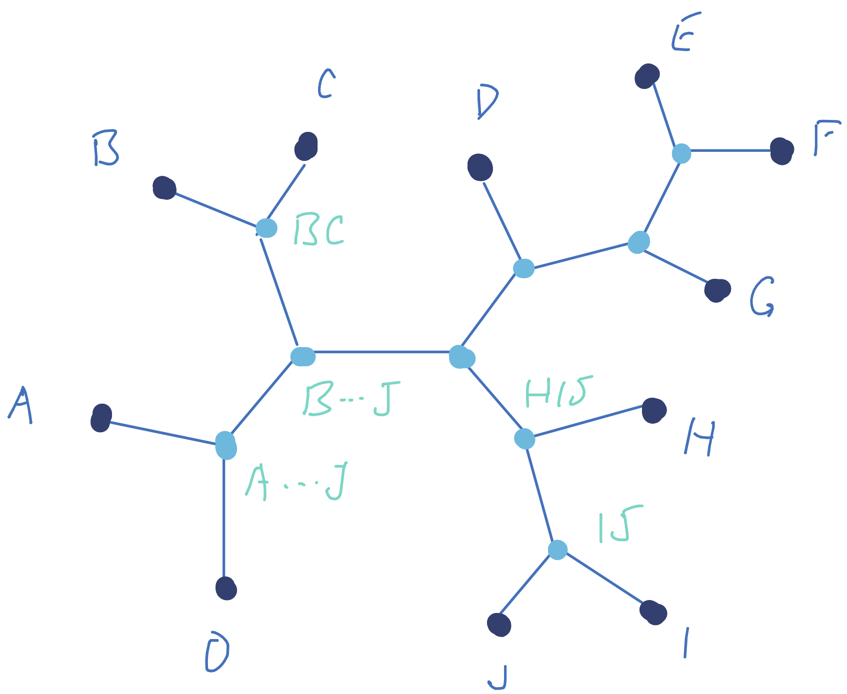




has 43 vertices & 88 edges

Simple tree graphs

- But tree graphs have much simpler structure



- each edge specified by a unique collection of boundary vertices
- PMI follows directly from specification of mincut structure

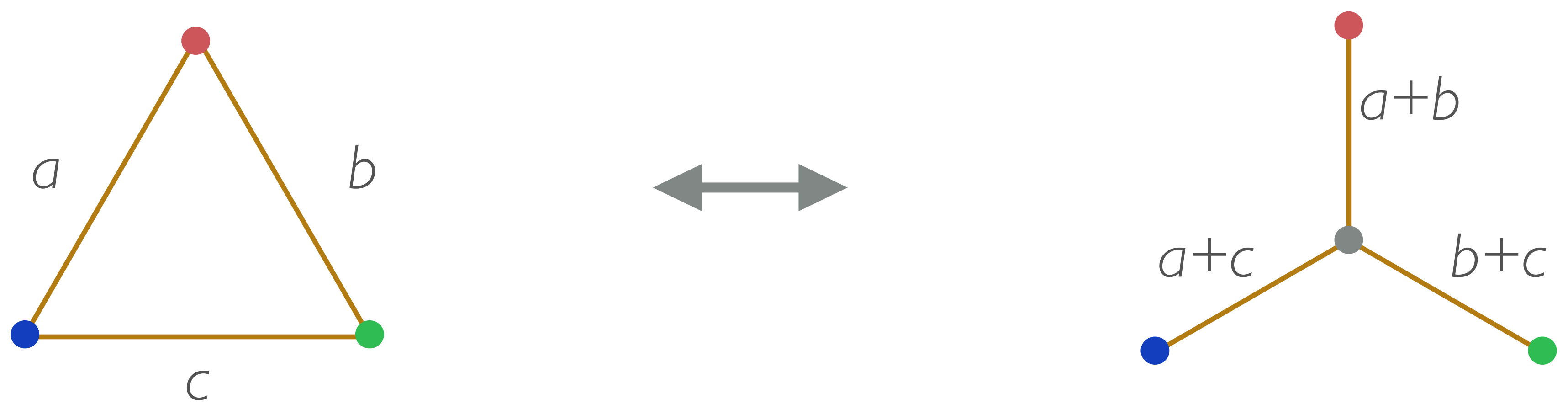
- **Def**: graph is *simple* if every edge defines some subsystem cut; or equivalently, if each boundary vertex has different color

- **Thm**: For *simple* tree graphs, V-space = PMI

Main Conjecture

- **Conjecture 1:** We can convert any holographic graph model into a tree while preserving the V-space.

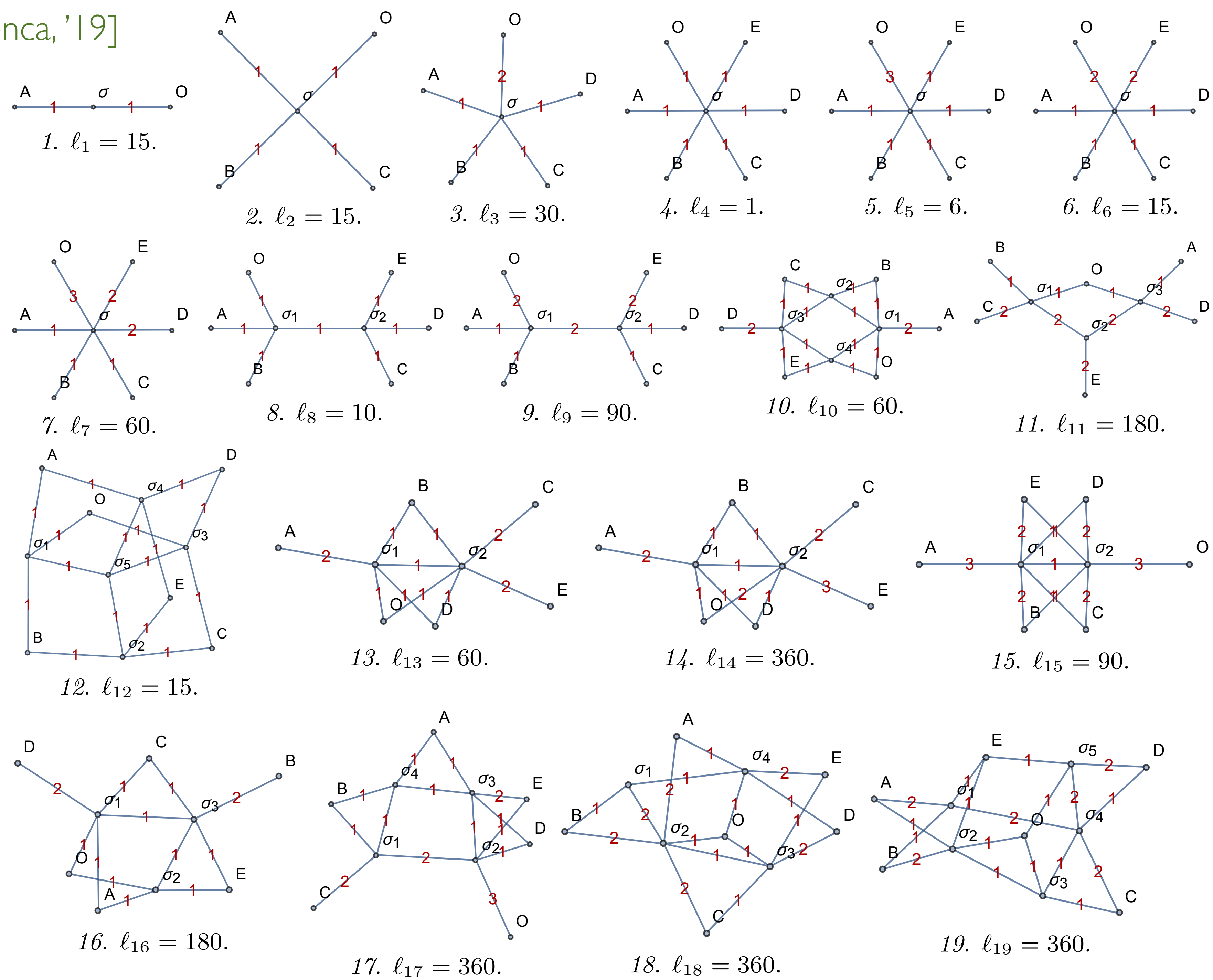
  - e.g.



  - similarly, can collapse any isolated k-cycle
  - Holds true for all ERs for N=5 HEC and all (hitherto-known) N=6 HEC ERs.
  - generically gives a non-simple tree, but can be trivially made simple by `fine-graining':

Graph representation of HEC_5 ERs

[Hernández-Cuenca, '19]



- All deformable to tree graphs by splitting boundary vertices

Coarse-graining & fine-graining

- Change N → N'
  - Changes dimensionality D → D' of entropy space
  - Aspects of entanglement structure preserved (inherited)

- Coarse-graining = declare multiple colors indistinguishable
  - projection of entropy vectors
  - corresponding projection of linear subspaces (V-space & PMI)

- Fine-graining = reverse of coarse-graining
  - Hence can obtain simple graph from non-simple one by fine-graining

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Main theorem

- Consider ERs of N-party holographic entropy cone (HEC_N)
  - The HEC_N is a convex hull of these; so ERs (in principle) determine all holographic entropy inequalities
- **Thm**: Assuming Conjecture 1, every ER of HEC_N is obtained as a projection of ER of a subadditivity cone $SAC_{N'}$ (for some $N' \geq N$)
- Idea of Pf:
  - Start w/ HEC_N ER & obtain graph representation G (has 1-d V-space)
  - Use Conjecture1& fine-grain to transform into a simple tree G'
  - Resulting V-space = PMI in D'-dimensional entropy space
  - Reduce to 1-d PMI (if uplifting increased V-space dimensionality)
  - = ER for $SAC_{N'}$

Implications

- HEC_N (delimiting geometric states in holography) is fully determined solely from $SAC_{N'}$ (which bounds all quantum states, regardless of holography)!
  - All faces and internal flats of HEC_N are projections of faces of $SAC_{N'}$
  - So the complicated multipartite entanglement structure is rooted in fine-grained PMI
  - Nonetheless, non-SA facets of HEC are superbalanced [He, VH, Rangamani], so can't be correlation measures [w.i.p. w/ Hernández-Cuenca & Jia]

- This in principle allows us to construct the full HEC_N for any N
  - In practice complicated: requires correct set of ERs of $SAC_{N'}$ for all N', projecting, taking convex hull, extracting ERs, and constructing facets...
  - But conceptually demystifies the HEC (and entanglement structure of holographic states)

Future directions

- But not all ERs of SAC are physical (cf. QMIP).
  - e.g. can violate SSA, so not even realizable in QM
  - **Q**: what characterizes the set of SAC ERs?

- **Conjecture 2**: All SAC ERs compatible w/ QM are holographic
  - hence realizable graphs

- **Speculation**: All SAC ERs consistent w/ SSA are quantum mechanical

- **Corrolary**: All SAC ERs consistent w/ SSA are holographic