New method for renormalon subtraction using Fourier transform

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Based on arXiv: 2012.15760 [hep-ph] YH, Yukinari Sumino, Hiromasa Takaura

INTRODUCTION

- **Perturbative expansion and divergence**

We consider $[X(Q)]_{\mathrm{PT}} = c_0\alpha_s + c_1\alpha_s^2 + \cdots$

$X(Q)$: dimensionless observable

Q : typical scale of X

PT expansion is good approximation where $Q \gg \Lambda_{\mathrm{QCD}}$.

But not invincible

All order PT series of observable $X(Q)$ is divergent

- **Renormalons** : main source of divergence of PT series

$$[X(Q)]_{\mathrm{PT}} = \sum_{n=0}^{\infty} c_n \alpha_s^{n+1} \xrightarrow[(n \gg 1)]{\text{Large order behavior}} \boxed{c_n \sim n!(b_0/u_*)^n}$$

$b_0 = (11 - 2n_f/3)/(4\pi)$ $\quad$ u_* : $\mathcal{O}(1)$ constant

Divergent behavior gives theoretical ambiguities δX to prediction of PT.

$$\boxed{\delta X \approx \left(\Lambda_{\mathrm{QCD}}/Q\right)^{2u_*}}$$

- **Remove ambiguities in OPE formulation**

$$\delta X \approx \left(\Lambda_{\mathrm{QCD}}/Q\right)^{2u_*}$$

Renormalon ambiguities cancel with the same size ambiguity of non-perturbative term in OPE.

1985 Mueller

・Operator product expansion (OPE)

$$[X(Q)]_{\mathrm{OPE}} = [X(Q)]_{\mathrm{PT}} + \frac{\langle \mathrm{NP} \rangle}{Q^d} + \cdots$$

absorb

After renormalon subtraction, we can obtain the convergent prediction of PT expansion.

MOTIVATION : convergent prediction of PT expansion

How to remove renormalons from finite PT series ?

→ We need non-trivial approximation. $\{c_0, c_1, \cdots, c_k\}$

Our new method

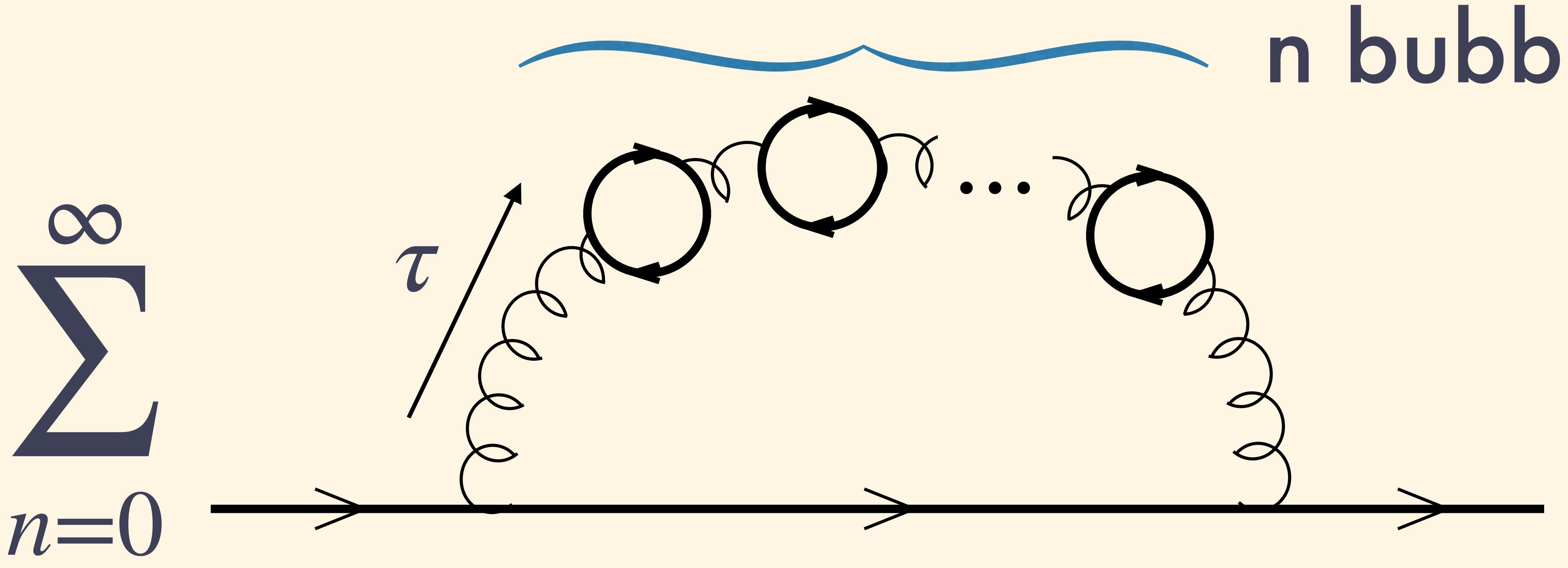
w/ reasonable approximation accuracy in a systematic & practical way

HOW TO ? : momentum integral rep. using Fourier transform

→ subtract renormalon by taking PV of integral

Concept of our method

Renormalon : stem from dynamics of low momentum gluon



IR : $\alpha_s \sum_{n=0}^{\infty} \int_{\tau^2 \lesssim \Lambda_{QCD}^2} d\tau^2 (\tau^2)^{u_*-1} [-b_0 \alpha_s \log(\tau^2)]^n \sim \sum_{n=0}^{\infty} n!(b_0/u_*)^n \alpha_s^{n+1}$

Resum of $[\log(\tau^2)]^n$ using RG equation $\rightarrow \int_{\tau^2 \lesssim \Lambda_{QCD}^2} d\tau^2 (\tau^2)^{u_*-1} \alpha_s(\tau) \approx \frac{1}{b_0} \int_{\tau^2 \lesssim \Lambda_{QCD}^2} d\tau^2 \frac{(\tau^2)^{u_*-1}}{\log(\tau^2/\Lambda_{QCD}^2)}$

: Divergence from renormalon = coming from singularity of $\alpha_s(\tau)$

→ *removed by taking PV of momentum integral*

Construct momentum integral : Fourier transform

Extension of method which works for QCD potential. 2005 Sumino

$$\tilde{X}(\tau) = \int d^3x\, e^{-i\vec{\tau}\cdot\vec{x}} r^{2au'} X(r^{-a}) = \frac{4\pi}{\tau^{3+2au'}} \sum_{n=0}^{\infty} \tilde{c}_n \alpha_s(\tau^a)^{n+1}$$

$(r = |\vec{x}| = Q^{-1/a} \ll \Lambda_{\overline{\rm MS}}^{1/a})$

a & u' : parameters of FT

: $\log(\tau)$ terms are resummed using RG equation

$$X(Q) \;\rightarrow\; \frac{r^{-2au'-1}}{2\pi^2} \int_{0,\mathrm{PV}}^{\infty} d\tau\, \tau \sin(\tau r) \tilde{X}(\tau) + \text{Ambiguities}$$

Finite

ATTENTION : $\tilde{X}(\tau)$ should not have renormalons.

Consistent to OPE !

Finite prediction is truely convergent ?

$$[X(Q)]_{\mathrm{Finite}} \equiv \frac{r^{-2au'-1}}{2\pi^2}\int_{0,\mathrm{PV}}^{\infty} d\tau\, \tau \sin(\tau r)\tilde{X}(\tau) \quad ; \tilde{X}(\tau) = \int d^3x e^{-i\vec{\tau}\cdot\vec{x}} r^{2au'} X(r^{-a})$$

Finite

$[X(Q)]_{\mathrm{Finite}}$ is convergent as long as renormalons in $\tilde{X}(\tau)$ are eliminated.

Non-trivial condition

$$\delta X(Q) \approx (\Lambda_{\overline{\mathrm{MS}}}/Q)^{2u^*} \xrightarrow{\mathrm{FT}} \delta\tilde{X}(\tau) = \int d^3x e^{-i\vec{\tau}\cdot\vec{x}} r^{2au'} \delta X(r^{-a})$$

from OPE

$$\propto \sin(\pi a(u_* + u'))$$

Proper choice of a & u' suppress the dominant renormalons in $\tilde{X}(\tau)$!

- **Practical way to calculate $[X(Q)]_{\rm Finite}$**

$\tilde{X}(\tau)$ from N^{k}LO PT expansion of $X(Q) = \sum_{n=0}^{k} c_n \alpha_s^{n+1}$

$$\tilde{X}(\tau) = \int d^3x\, e^{-i\vec{\tau}\cdot\vec{x}} r^{2au'} X(r^{-a}) \ \rightarrow \tilde{X}^{(k)}(\tau) = \frac{4\pi}{\tau^{3+2au'}} \sum_{n=0}^{k} \tilde{c}_n \alpha_s (\tau^a)^{n+1}$$

: Convergent series since renormalons are eliminated.

Precision will be improved by going to higher order.

Test analysis : application to $\Gamma(B \to X_u l\bar{\nu})$

remove $\mathcal{O}(\Lambda_{\rm QCD}^2)$ renormalon

OPE of semileptonic B decay width

$$\Gamma_{\rm OPE} = C m_b^5 \left[X_{\bar{b}b} - \frac{\mu_\pi^2}{2m_b^2} + \frac{\mu_G^2}{2m_b^2} + \mathcal{O}(\Lambda_{\rm QCD}^3/m_b^3) \right]$$

(neglecting $\mathcal{O}(\alpha_s)$ correction to NP terms)

$$m_b = m_b^{pole}\ , \quad C = \frac{G_F^2 |V_{ub}|^2}{192\pi^3}$$

- $X_{\bar{b}b}$ contains the renormalons of $\mathcal{O}(\Lambda_{\rm QCD}), \mathcal{O}(\Lambda_{\rm QCD}^2), \cdots$.

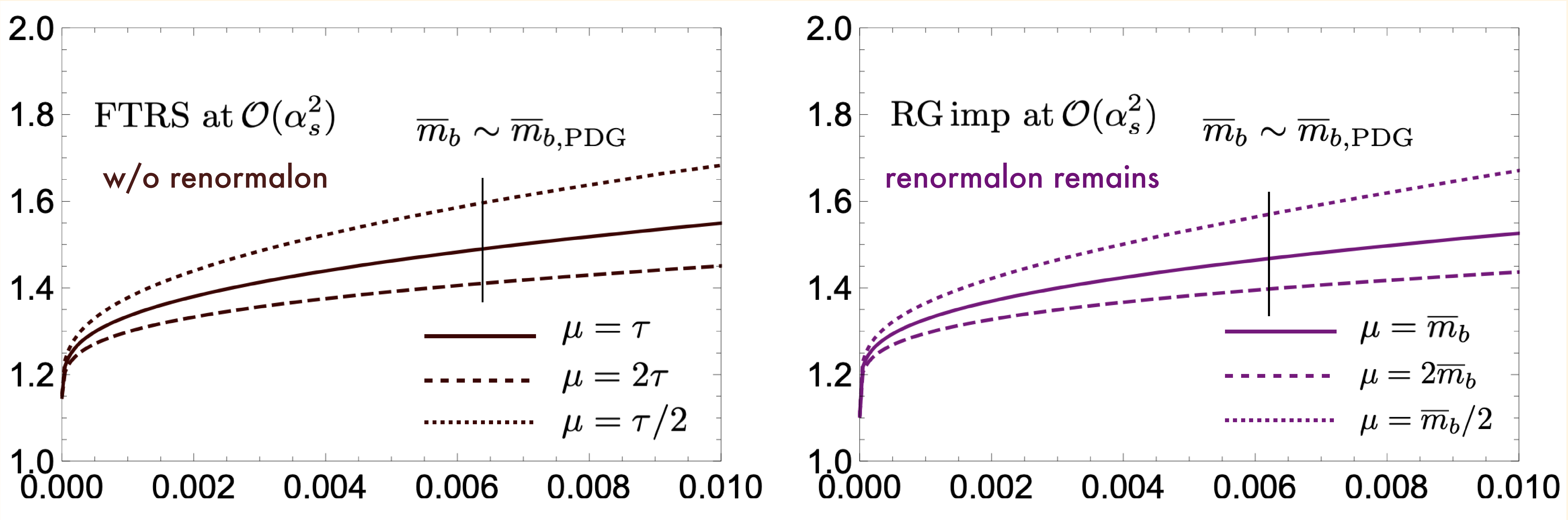
  (We know PT expansion up to 2 loops $\mathcal{O}(\alpha_s^2)$)

- $\mathcal{O}(\Lambda_{\rm QCD})$ renormalon : eliminated when expressing $m_b \to \bar{m}_b = m_b^{\overline{\rm MS}}(m_b^{\overline{\rm MS}})$

  1994 Bigi, Shifman, Uraltsev

Analysis : Study scale dep. of Γ by removing $\mathcal{O}(\Lambda_{\rm QCD}^2)$ renormalon

・Analysis at 2 loop level (at the current accuracy)



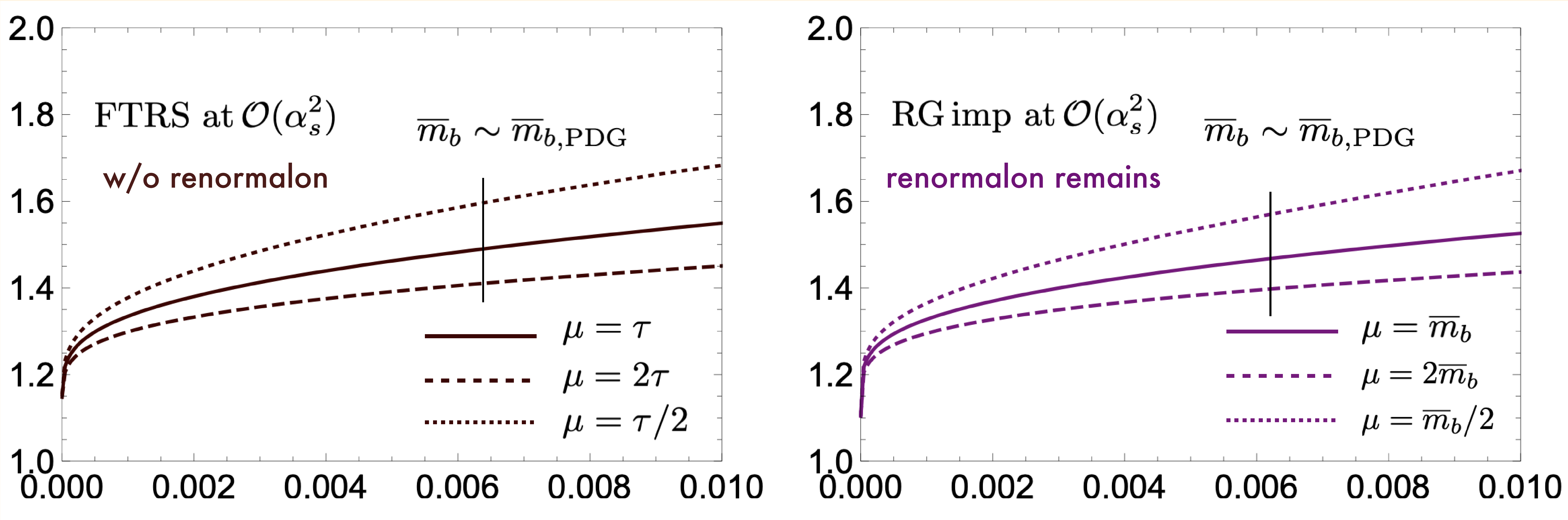
x axis : $\Lambda^2_{\overline{\mathrm{MS}}}/\bar{m}_b^2$

y axis : $\dfrac{\Gamma}{C\bar{m}_b^5}$

: Scarcely different

(Lack of known coeffs)

- Analysis at 2 loop level (at the current accuracy)



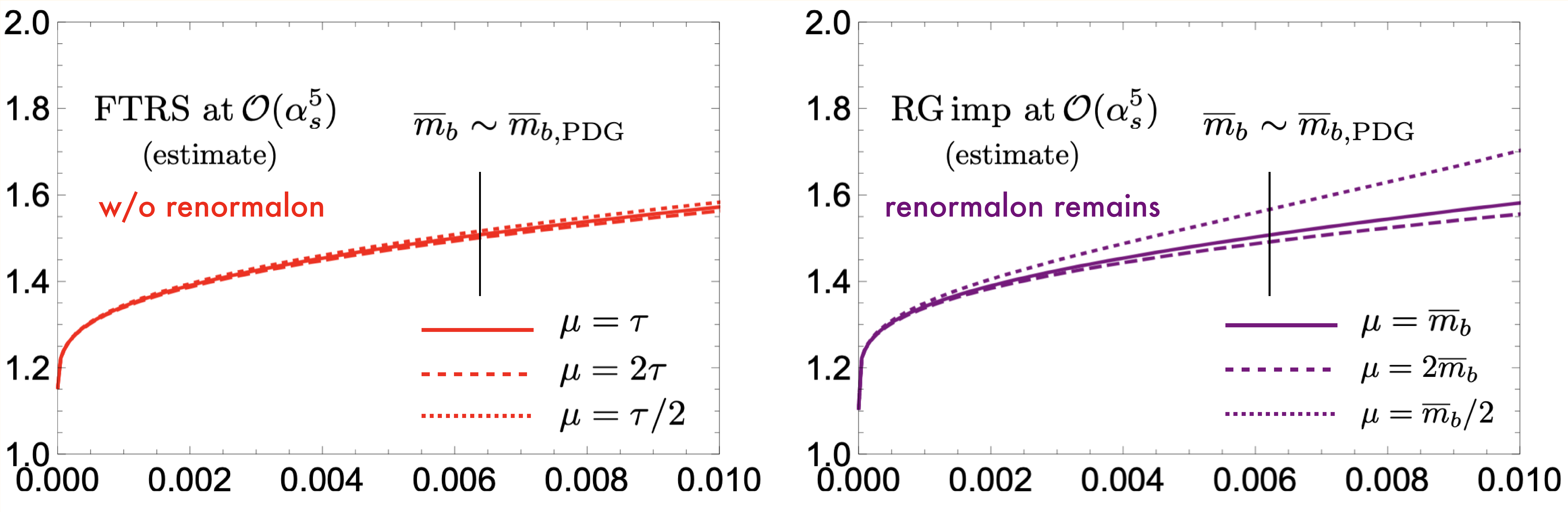
x axis : $\Lambda_{\overline{\mathrm{MS}}}^2/\bar{m}_b^2$

y axis : $\dfrac{\Gamma}{C\bar{m}_b^5}$

: Scarcely different

(Lack of known coeffs)

- 5 loop estimate using 5 loop QCD beta function



Left shows convergence

$\because$ Renormalons are suppressed in momentum space PT series.

Conclusions

- Renormalon subtraction is achievable in the framework of OPE.

  However we need non-trivial approximation to obtain the finite prediction of PT.

- Our new method : using Fourier transform

  → calculate finite prediction with a reasonable accuracy of approximation in a practical and systematic way.

  Specific feature : we can remove multiple renormalons simultaneously.

- Our test analyses show a good consistency.

  → more accurate prediction of different observables is obtained by going to higher order.

Next goal : application to bottomonium energy spectrum

Back up slides

- **Theoretical ambiguity from renormalon**

We need regularization of divergent behavior of PT expansion

e.g.) Borel transform (and resummation)

$$[X(Q)]_{\rm PT} = \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}$$

: divergent due to $c_n \sim n!(b_0/u_*)^n \quad (n \gg 1)$

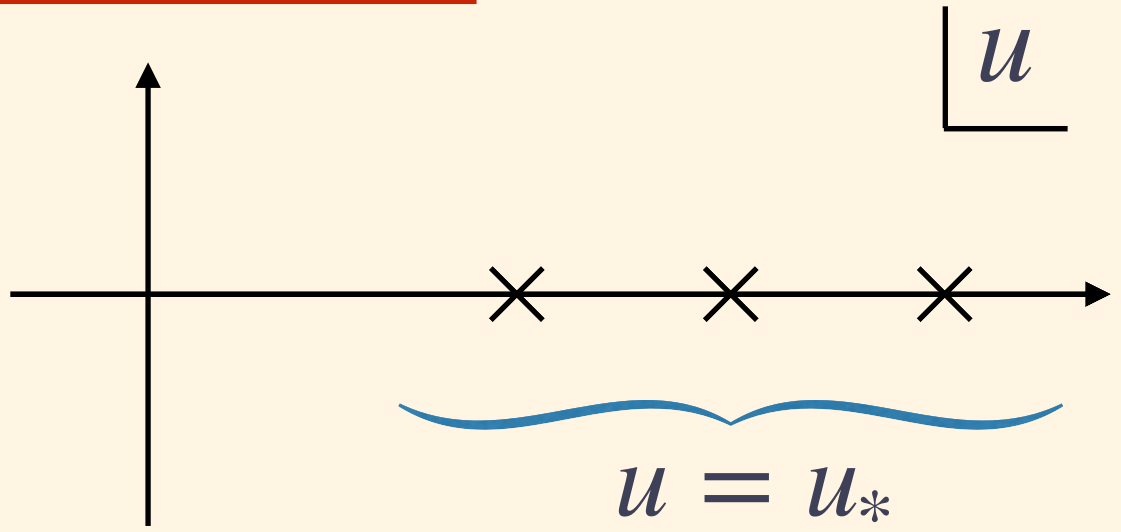
regularize ↓

$$[X(Q)]_{\rm Bl} = \boxed{\frac{1}{b_0}\int_{0,\rm PV}^{\infty} du\, e^{-\frac{u}{b_0 \alpha_s}} B_X(u)} + \boxed{\text{ambiguities}}$$

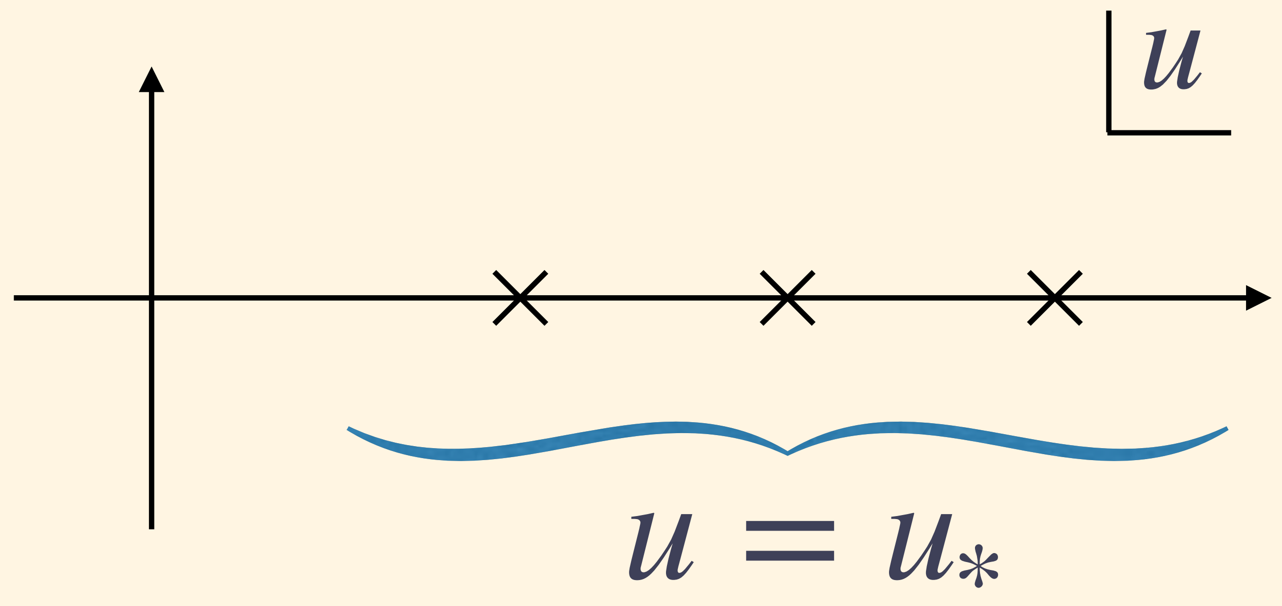
Finite

$$; B_X(u) = \sum_{n=0}^{\infty} \frac{c_n}{n!}\left(\frac{u}{b_0}\right)^n$$

: singular due to renormalon



- **Ambiguities induced by renormalons**



$$\text{ambiguities} \approx \pm i\left(\Lambda_{\overline{\rm MS}}/Q\right)^{2u_*}$$

u_* : Location of pole singularities of $B_X(u)$

$\Lambda_{\overline{\rm MS}}$: Dynamical scale in $\overline{\rm MS}$ scheme

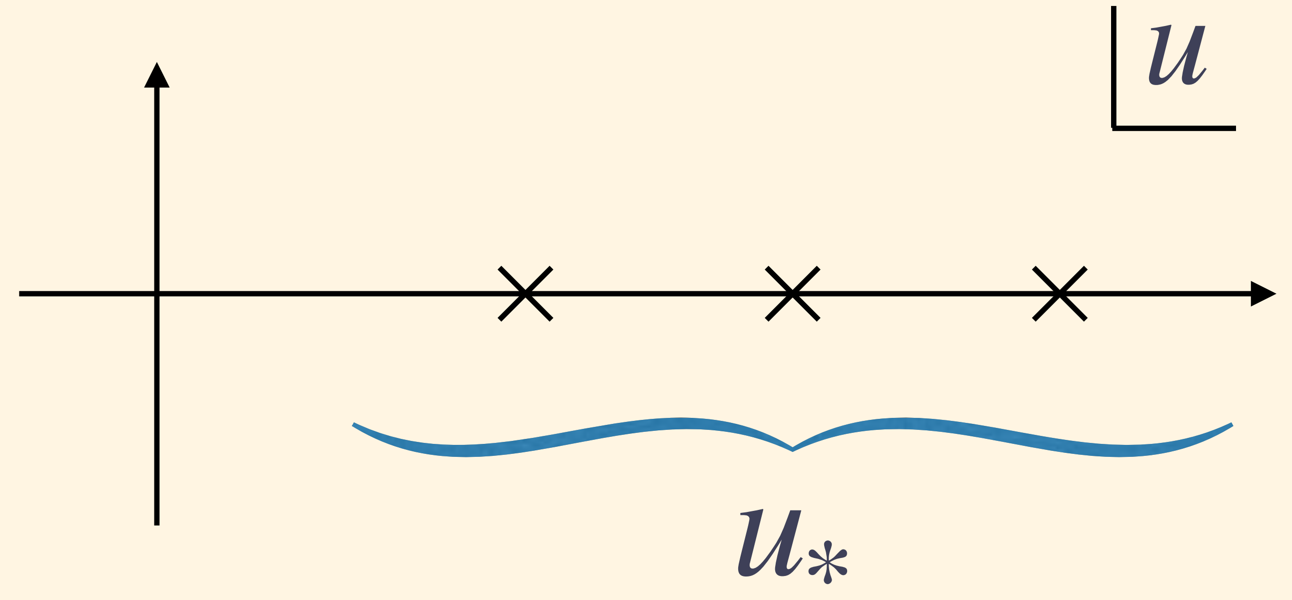
(Here we neglect $\mathcal{O}(\alpha_s)$ correction & anomalous dim.)

Operator product expansion (OPE) is expected to achieve renormalon subtraction.

1985 Mueller

- Renormalon ambiguities cancel with the same size ambiguity of non-perturbative term in OPE.

- **Remove ambiguities in OPE formulation**



Ansatz : Renormalon ambiguities are absorbed by NP corrections in OPE

1985 Mueller

1993 Beneke

- Renormalon of $X(Q)$ $\to \pm \sum_{u_*} N_{u_*} \left(\Lambda_{\overline{\mathrm{MS}}}/Q\right)^{2u_*} \alpha_s^{\gamma_0/b_0}(1+\mathcal{O}(\alpha_s))$

- Operator product expansion (OPE) ↕ Same Q-dep.

$$[X(Q)]_{\mathrm{OPE}} = [X(Q)]_{\mathrm{PT}} + \frac{\langle \mathrm{NP} \rangle}{Q^d} \alpha_s^{\gamma_0/b_0}(1+\mathcal{O}(\alpha_s)) + \cdots$$

$$\langle \mathrm{NP} \rangle = (const.)\Lambda_{\overline{\mathrm{MS}}}^d$$

$$\to [X(Q)]_{\mathrm{PV}} + \frac{\langle \mathrm{NP} \rangle_{\mathrm{RS}}}{Q^d}(1+\mathcal{O}(\alpha_s)) + \cdots$$

: renormalon-subtracted prediction

- **Numerical calculation of $[X(Q)]_{\rm Finite}$**

$$[X(Q)]_{\rm Finite} \equiv \frac{r^{-2au'-1}}{2\pi^2}\int_{0,\rm PV}^{\infty} d\tau\, \tau \sin(\tau r)\tilde{X}(\tau)$$

Decompose

$$= \quad X_0(Q) \quad + \quad X_{\rm pow}(Q)$$

$X_0(Q)$: exponentially damped part

$X_{\rm pow}(Q)$: can be expanded to the appropriated order of $1/Q^n$

(should be expanded up to order of the eliminated renormalon)

• Remove multiple renormalons

e.g.) subtracting renormalons at $u_* = 1,\ 2,\ \cdots$. $(\mathcal{O}(\Lambda_{\rm QCD}^2),\ \mathcal{O}(\Lambda_{\rm QCD}^4), \cdots)$

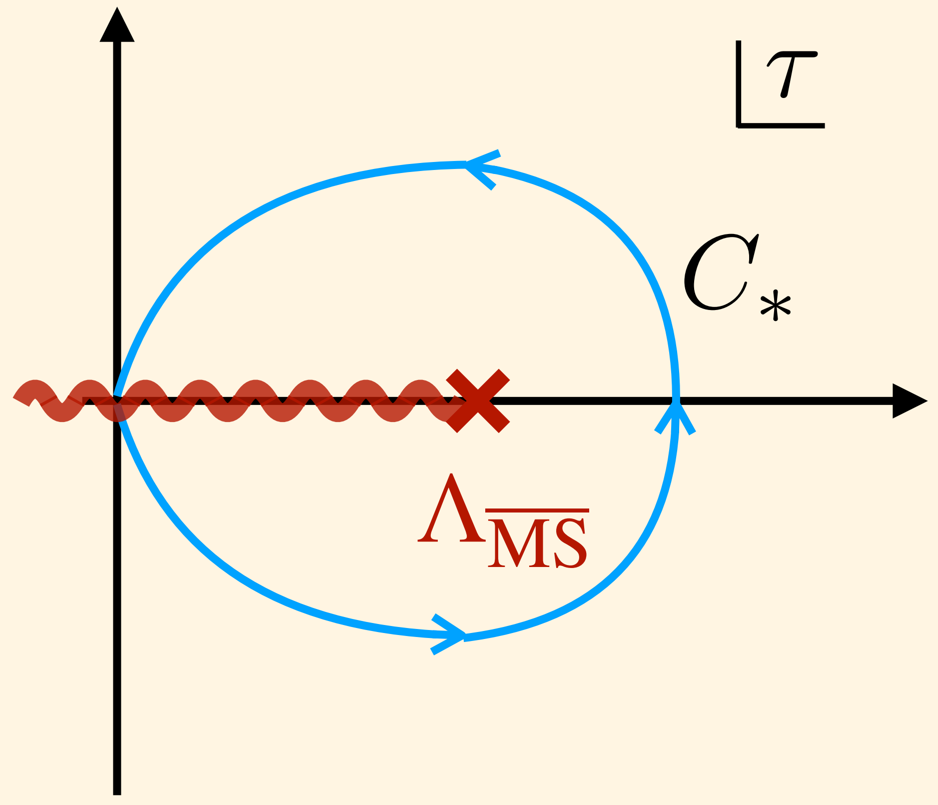
→ we adjust $a = 1,\ u' = -1$.

$$\delta\tilde{X}(\tau) \propto \sin(\pi a(u_* + u'))\Big|_{a=1,u'=-1} \propto \sin(\pi u_*)$$

: $\delta\tilde{X} = 0$ at $u_* = 1,\ 2,\ \cdots$.

Ambiguities induced by renormalons : desired 1/Q dependence.

$$= \frac{1}{2\pi^2 Q}\int_{C_*} d\tau\, \tau \sin(\tau/Q)\tilde{X}(\tau) = R_2\frac{\Lambda_{\overline{\rm MS}}^2}{Q^2} + R_4\frac{\Lambda_{\overline{\rm MS}}^4}{Q^4} + \cdots .$$



Test analysis : application to $\Gamma(B \to X_u l\bar{\nu})$

- For simplicity, Wilson coeff of μ_π^2 and μ_G^2 = 1. → $\mathcal{O}(\Lambda_{\rm QCD}^2)$ renormalon is suppressed.
- We set $a = 1,\ u' = -1$. ($\mathcal{O}(\Lambda_{\rm QCD}^2 \alpha_s)$ renormalon remains.)

$$\Gamma_{\rm FTRS} = C\bar{m}_b^5\left[1 + \int_{0,\rm PV}^{\infty} d\tau\, \tau \sin(\tau/\bar{m}_b)\tilde{X}_{\bar{b}b}(\tau)\right] \quad ; \tilde{X}_{\bar{b}b}(\tau) = \frac{4\pi}{\tau}\sum_{n=0}^{k} \tilde{s}_n \alpha_s(\tau)^{n+1}$$

- $\tilde{s}_n$ is combination of coeffs of (1) $X_{\bar{b}b}$, (2) pole-$\overline{\rm MS}$ relation, (3) QCD beta function

$k = 1$ $k = 3$ $k = 4$

Analyses at ❶ exact 2loop level $(k = 1)$

❷ estimated 5loop level using QCD beta function $(k = 4)$

Remove ambiguities with anomalous dim & α_s corrections

- Form of $\delta X(Q) = N_{u_*}\left(\Lambda_{\overline{\rm MS}}/Q\right)^{2u_*}\alpha_s^{\gamma_0/b_0}(1+\mathcal{O}(\alpha_s))$ 1993 Beneke

$$= N_{u_*}\left(\Lambda_{\overline{\rm MS}}/Q\right)^{2u_*}(1+q_1\log(1/Q^2)+q_2\log(1/Q^2)^2+\cdots)$$

We can define the Fourier transform so that $\delta\tilde{X}(\tau)$ doesn't have $\log(\tau)$ corrections.

(Detailed discussion is includes in full paper now we're preparing.)