Determination of the QCD coupling constant from the static energy and the free energy

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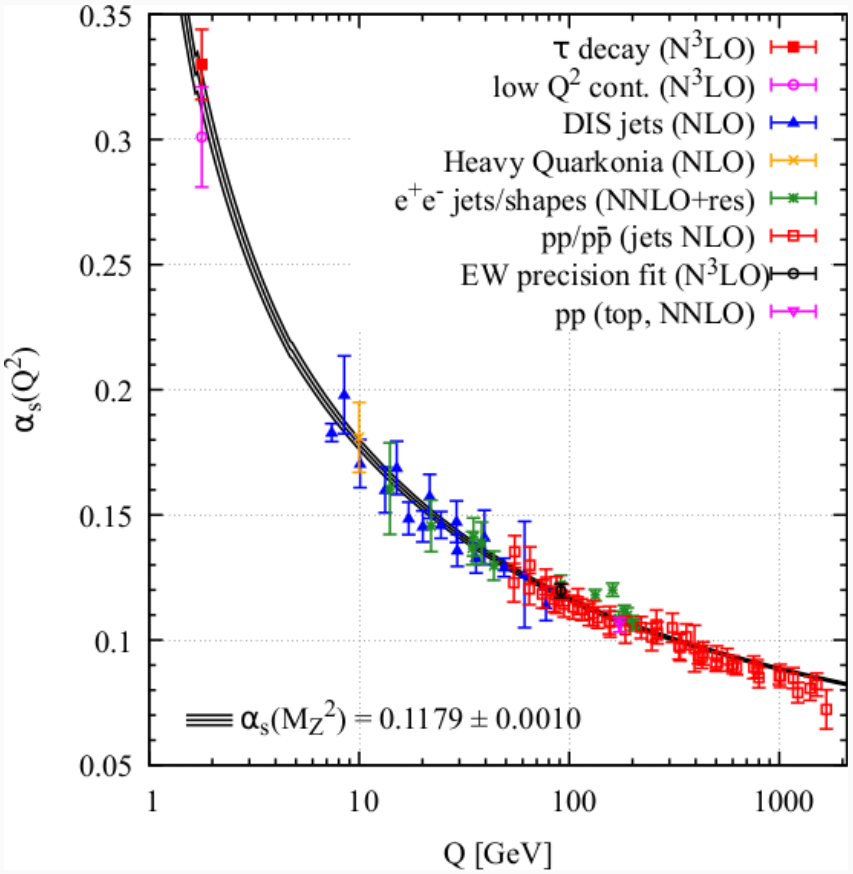
QWG 2021
03/18/2021

TUMQCD: PR D90 (2014) ⇒ *TUMQCD: PR D100 (2019) no.11*
J. Komijani, P. Petreczky, JHW: *PPNP 113 (2020)*

Outline

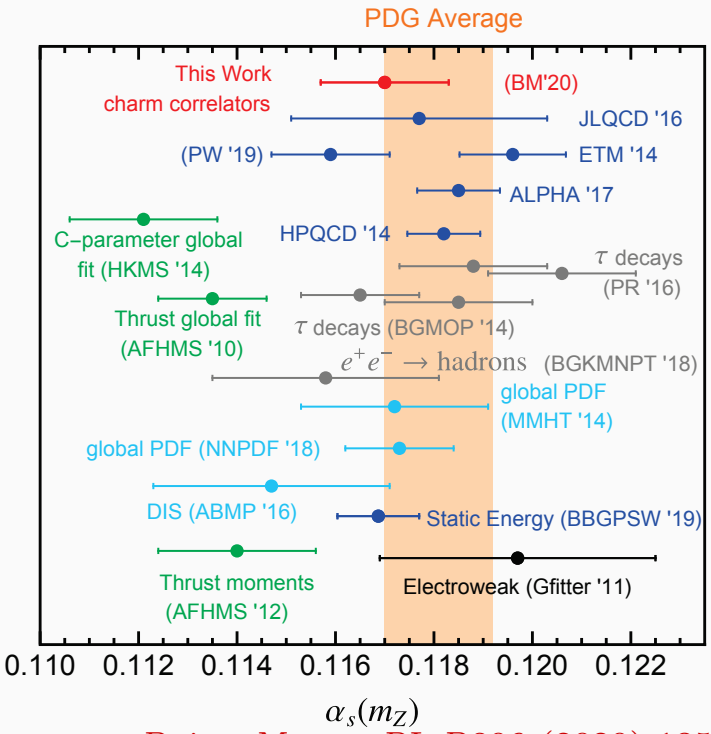
Overall picture



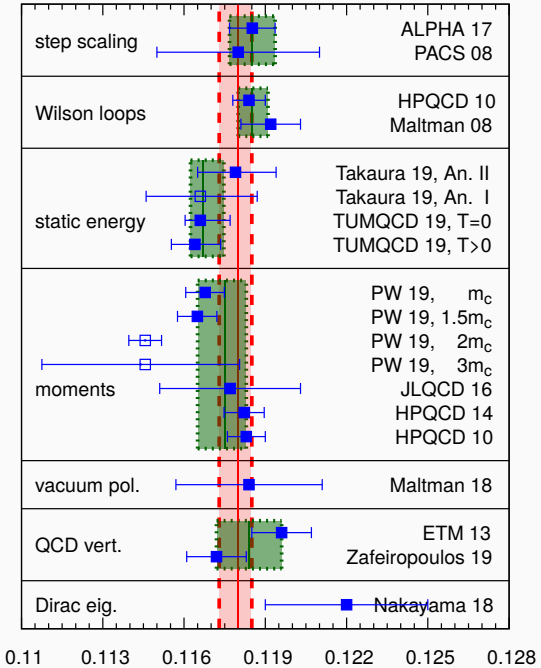
source: PDG 2019

- QCD has only one parameter: $\alpha_s(Q^2)$ (or Λ_{QCD}, respectively)
- $\alpha_s(Q^2)$ is determined experimentally via different processes
- The overall picture from high to low scales seems consistent

Lattice determinations of α_s in context



source: Boito, Mateu, PL B806 (2020) 135482



source: *Komijani, Petreczky, JHW, PPNP 113 (2020)*

- PDG has increased the global error of α_s since 2014
- Spread hints at **underestimated systematic uncertainties**
- Lattice QCD dominates the global average and error

$\Rightarrow$ urgent need for **reexamining errors of lattice** calculations

Conceptual idea of lattice determinations of α_s

- We sacrifice a few hadronic observables on the lattice to determine the quark masses and set the lattice scale (ultimately through f_π)
- We compute hadronic observables $O(\nu)$ on the lattice **at sufficiently high scales** ν for the weak-coupling approach to be applicable

  Window problem: $\Lambda_{\mathrm{QCD}} \ll \nu \ll 1/a$ hard to realize in practice

- We compare continuum extrapolated lattice results for $O(\nu)$ to perturbative continuum results in $\overline{\mathbf{MS}}$ scheme to determine parameters

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The QCD static energy of a (static) quark-antiquark pair (2010-2021)

- The scale is set by the (inverse) size of the system, $\nu = 1/r$
- Other scales are involved, i.e. the ultra-soft scale $\mu_{us} = \alpha_s/r$
- The singlet potential contributes
  - renormalon subtraction (DR)
  - diverges as $\sim 1/a$ towards the continuum limit in lattice reg.

Bibliography : α_s from the QCD static energy

- TUM group[1] using 3 sea quark flavors (2010-2019)
- TUM data re-analyzed[2] (2020)
- Frankfurt/Jena group[3] using 2 sea quark flavors (2012-2018)
- JLQCD-affiliated group[4] using 3 sea quark flavors (2018)

Extension to 4 sea quark flavors by TUMQCD collaboration ongoing

[1] Brambilla et al., Phys. Rev. Lett. 105 (2010) 212001
Bazavov et al., Phys. Rev. D86 (2012) 114031
Bazavov et al., Phys. Rev. D90 (2014) 7, 074038
Bazavov et al. [TUMQCD], Phys. Rev. D100 (2019) 11, 114511
[2] Ayala et al., JHEP 09 (2020) 016
[3] Jansen et al. [ETMC], JHEP 1201, 025 (2012)
Karbstein et al., JHEP 1409, 114 (2014)
Karbstein et al., Phys. Rev. D98 (2018) 11, 114506
[4] Takaura et al., JHEP 1904, 155 (2019)
Takaura et al., Phys. Lett. B789, 598-602 (2019)

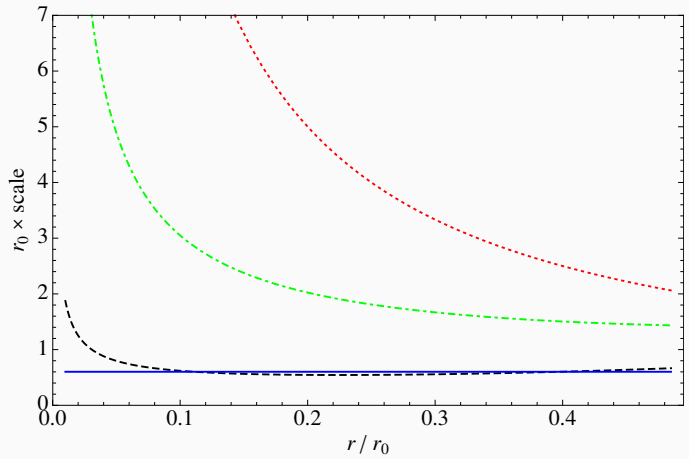
Static quark-antiquark energy

- Static energy determined from large-time behavior of Wilson loops

$$E(r) = \lim_{t\to\infty} \mathrm{i}\frac{d}{dt}\langle \ln W(t,\boldsymbol{r})\rangle\,, \qquad W(t,\boldsymbol{r}) = \exp\left\{ \mathrm{i}g \oint_{\boldsymbol{r},t} dz^\mu A_\mu \right\}$$

- Known in perturbation theory @ N^{3}LL (dimensional regularization)
- Nonperturbatively calculable in the lattice regularization
- For $r \ll 1/\Lambda_{\mathrm{QCD}}$ both schemes must agree, with hierarchy of scales

$$\frac{1}{r} \gg V_o - V_s \gg \Lambda_{\mathrm{QCD}}, \qquad \left\{ \begin{array}{c} V_s \\ V_o \end{array} \right\} \approx -\left\{ \begin{array}{c} -C_F \\ +\frac{1}{2N_c} \end{array} \right\} \frac{\alpha_s}{r}$$



Static quark-antiquark energy in perturbation theory

- Static energy determined from large-time behavior of Wilson loops

$$E(r) = \Lambda_s - \frac{C_F \alpha_s}{r}\left(1 + \#\alpha_s + \#\alpha_s^2 + \#\alpha_s^3 + \#\alpha_s^3 \ln \alpha_s + \#\alpha_s^4 \ln^2 \alpha_s + \#\alpha_s^4 \ln \alpha_s + \dots\right) \quad @\ \mathrm{N^3LL}$$

- US contributions to the static energy can be understood in pNRQCD

$$E(r) = \Lambda_s + \underbrace{V_s(r, \mu_{us})}_{\sim \ln^k(r\mu_{us}),\ k=1,2,\dots} - i\frac{g^2}{N_c} V_A^2 \int_0^\infty dt\, \underbrace{e^{-it(V_o - V_s)} \langle \mathrm{tr}\ \boldsymbol{r} \cdot \boldsymbol{E}(t) \boldsymbol{r} \cdot \boldsymbol{E}(0)\rangle (\mu_{us})}_{\sim \ln^k(r\mu_{us}),\ \ln^k\left(\frac{V_o - V_s}{\mu_{us}}\right),\ k=1,2,\dots} + \dots$$

include the singlet potential and the ultra-soft contribution

- Potential nonrelativistic QCD (pNRQCD) Lagrangian[5] at NLO

$$\mathcal{L}_{\mathrm{pNRQCD}} = \mathcal{L}_{\mathrm{light}} + \int d^3 r\ \mathrm{tr}\ \left\{ S^\dagger [i\partial_0 - V_s(r, \nu, \mu_{us})] S +)^\dagger [iD_0 - V_o(r, \nu, \mu_{us})] O \right\}$$

$$+ V_A(r, \nu, \mu_{us})\ \mathrm{tr}\ \left\{ O^\dagger \boldsymbol{r} \cdot g\boldsymbol{E} S + S^\dagger \boldsymbol{r} \cdot g\boldsymbol{E} O \right\}$$

$$+ V_B(r, \nu, \mu_{us})\ \mathrm{tr}\ \left\{ O^\dagger \boldsymbol{r} \cdot g\boldsymbol{E} O + O^\dagger O \boldsymbol{r} \cdot g\boldsymbol{E} \right\} + \dots$$

[5] Brambilla et al., Nucl. Phys. B566 (2000) 275

Dealing with the mass renormalon of the potential

- The singlet potential is affected by an r-independent renormalon

$$E(r) = \Lambda_s + V_s(r, \nu, \mu_{us}) + \delta_{us}(r, \nu, \mu_{us}),$$

- May absorb renormalon into residual mass:

$$RS(\nu) = \Lambda_s - \Lambda_s^{\rm rs}(\nu), \quad V_s^{\rm rs}(r, \nu, \mu_{us}) = V_s(r, \nu, \mu_{us}) + \Lambda_s^{\rm rs}(\nu),$$
$$E(r) = RS(\nu) + V_s^{rs}(r, \nu, \mu_{us}) + \delta_{us}(r, \nu, \mu_{us}),$$

- Or, to avoid both large logs $\ln(r\nu)$ or an r-dependent renormalon term

$\Rightarrow$ compute the force, resum the logarithm via $\nu = 1/r$, and integrate[6]

$$F\left(r, \frac{1}{r}\right) = \left.\frac{\partial E(r,\nu)}{\partial r}\right|_{\nu=\frac{1}{r}}$$

$$E(r) = \int_{\bar{r}}^{r} dr' F\left(r', \frac{1}{r'}\right) + \text{const}$$

[6] Garcia i Tormo, MPLA 28 133028

Three-loop force

The force at N^{3}LO with $\nu = 1/r$

$$
\begin{aligned}
F(r, 1/r) = \frac{C_F}{r^2}\alpha_s(1/r)\Bigg[& 1 \\
& + \frac{\alpha_s(1/r)}{4\pi}\left(\tilde{a}_1 - 2\beta_0\right) \\
& + \frac{\alpha_s^2(1/r)}{(4\pi)^2}\left(\tilde{a}_2 - 4\tilde{a}_1\beta_0 - 2\beta_1\right) \\
& + \frac{\alpha_s^3(1/r)}{(4\pi)^3}\Big(\tilde{a}_3 - 6\tilde{a}_2\beta_0 - 4\tilde{a}_1\beta_1 - 2\beta_2 \\
& + a_3^L \ln\frac{C_A\alpha_s(1/r)}{2}\Big) \\
& + \mathcal{O}(\alpha_s^4, \alpha_s^4 \ln^2\alpha_s)\Bigg]
\end{aligned}
$$

... with leading ultra-soft resummation

$$
\begin{aligned}
F(r, 1/r) = \frac{C_F}{r^2}\alpha_s(1/r)\Bigg[& 1 \\
& + \frac{\alpha_s(1/r)}{4\pi}\left(\tilde{a}_1 - 2\beta_0\right) \\
& + \frac{\alpha_s^2(1/r)}{(4\pi)^2}\left(\tilde{a}_2 - 4\tilde{a}_1\beta_0 - 2\beta_1\right) \\
& - \frac{\alpha_s^2(1/r)}{(4\pi)^2}\frac{a_3^L}{2\beta_0}\ln\frac{\alpha_s(\mu_{us})}{\alpha_s(1/r)} \\
& + \frac{\alpha_s^2(1/r)\alpha_s(\mu_{us})}{(4\pi)^3}a_3^L\ln\frac{C_A\alpha_s(1/r)}{2r\mu_{us}} \\
& + \frac{\alpha_s^3(1/r)}{(4\pi)^3}\left(\tilde{a}_3 - 6\tilde{a}_2\beta_0 - 4\tilde{a}_1\beta_1 - 2\beta_2\right) + \mathcal{O}(\alpha_s^4)\Bigg]
\end{aligned}
$$

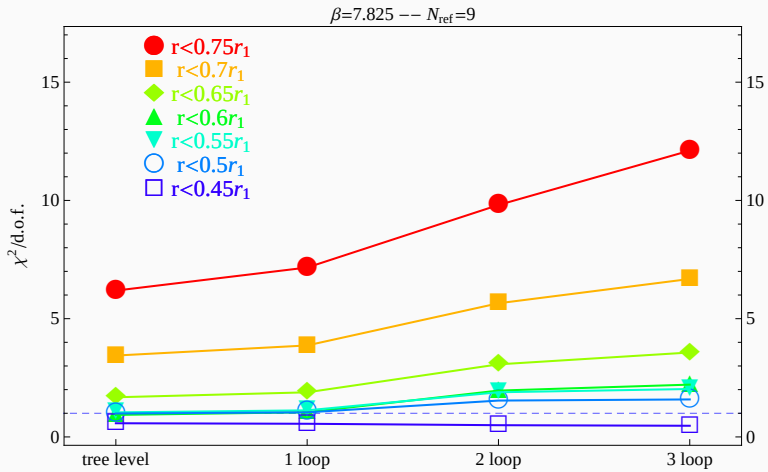
The ultra-soft scale $\mu_{us} = \frac{C_A}{2}\frac{\alpha_s(1/r)}{r}$ gives rise to $\ln\alpha_s(1/r)$ term[a]

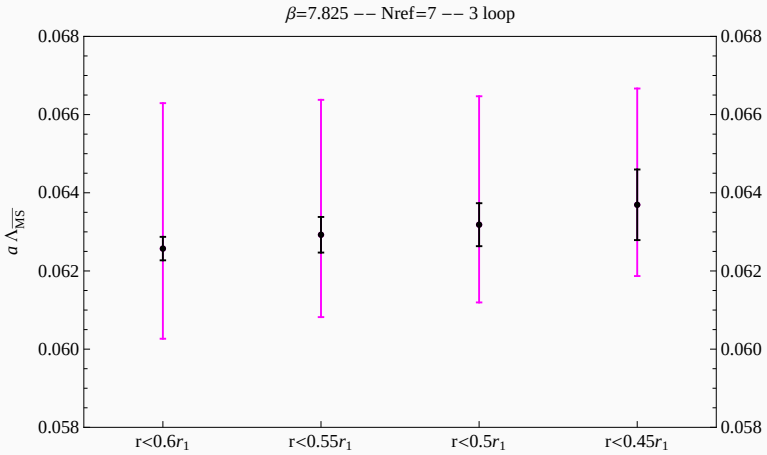
$$\ln\left[\frac{V_o - V_s}{\nu}\right] - \ln\left[r\nu\right] = \ln\left[\frac{C_A}{2}\alpha_s(1/r)\right] = \ln\left[r\mu_{us}\right]$$

Leading ultra-soft resummation recovers F^{N^3LO} in limit $\mu_{us} \to 1/r$

[a] Brambilla et al., Phys. Rev. D60 (1999) 091502

Fitting lattice results of the static energy (2014)





Different perturbative orders

- χ^2/dof reduces for higher orders at shorter distances

$\Rightarrow$ Weak-coupling suitable for static energy for $r \lesssim 0.15\,\mathrm{fm}$

- At shortest distances little sensitivity to perturbative order

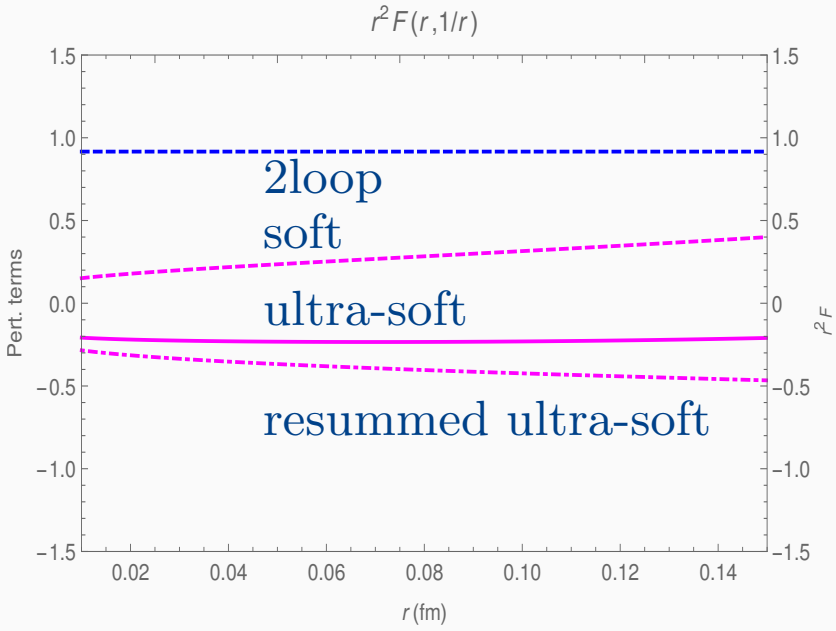
When going to shorter distances

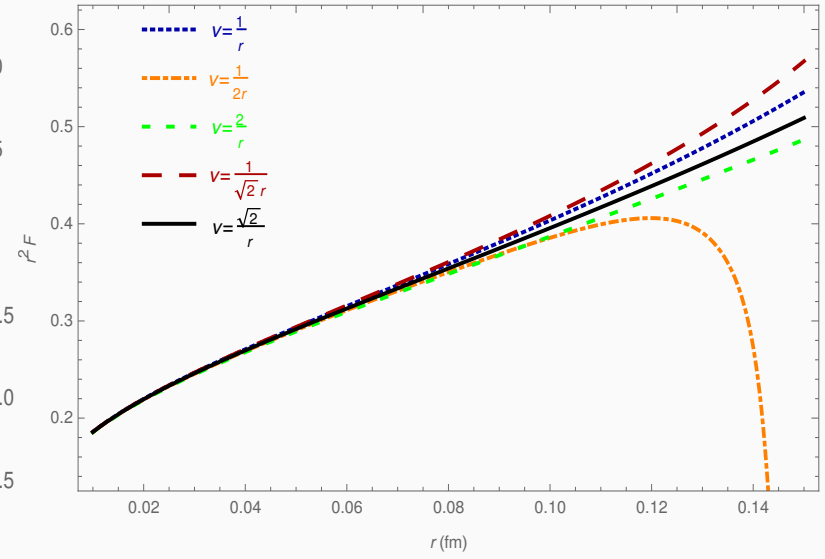
- Statistical errors increase
- Perturbative errors decrease

Perturbative errors estimated from

- scale variation $\nu = 1/\sqrt{2}r$ to $\sqrt{2}/r$
- soft higher order term $\pm\#\alpha_s^4/r$

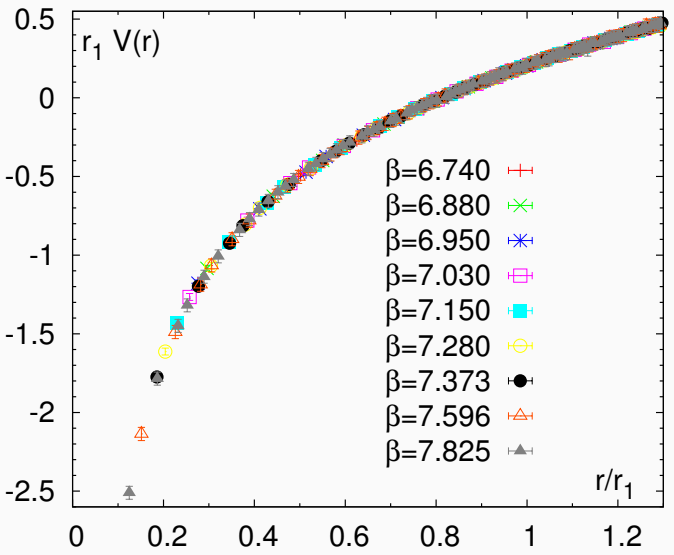
Perturbative uncertainty in the 2019 edition



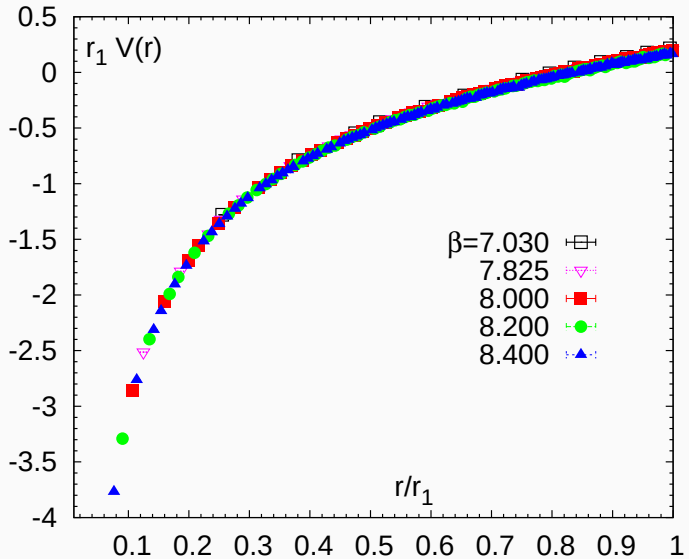


- Ultra-soft logs are small – use *three-loop with leading US resummation*
- Soft scale variation generates the dominant uncertainty at three loops
- More conservative soft scale variation in 2019 edition: $\nu = 1/2r$ to $2/r$
- Nonmonotonic scale dependence is a common effect in EFTs, whenever the error is estimated from scale variation, is minimal for $\nu \approx 1/\sqrt{2}r$
- Variation of ultra-soft resummation included in the error budget

Static energy on the lattice: 2014 vs 2019



2014 edition[7], $a^{-1} \leq 4.9\,\text{Gev}$



2019 edition[8], $a^{-1} \leq 7.9\,\text{Gev}$

- Smallest distance $r = 0.04\,\text{fm}$
- Perturbative errors dominant
- Very light pion $m_\pi = 160\,\text{MeV}$
- Consistent with 2012 edition[9]

- Three extra fine lattice spacings at $T = 0$
- Include for shortest distances singlet free energies at $T > 0$ $\Rightarrow a^{-1} \lesssim 22\,\text{GeV}$

[7] Bazavov et al., Phys. Rev. D90 (2014) 7, 074038
[8] Bazavov et al. [TUMQCD], Phys. Rev. D100 (2019) 11, 114511
[9] Bazavov et al., Phys. Rev. D86 (2012) 114031

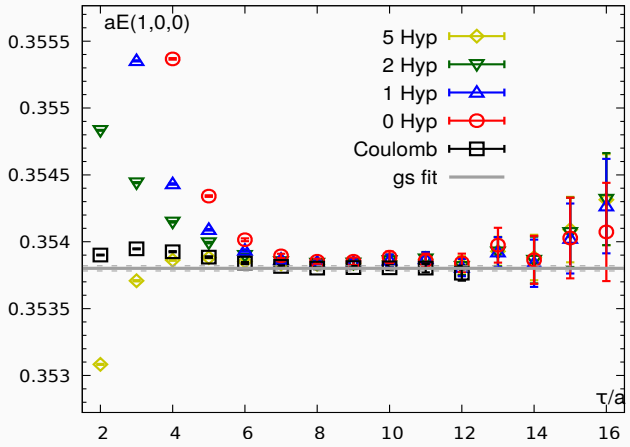
Wilson loops vs Wilson line correlators in Coulomb gauge

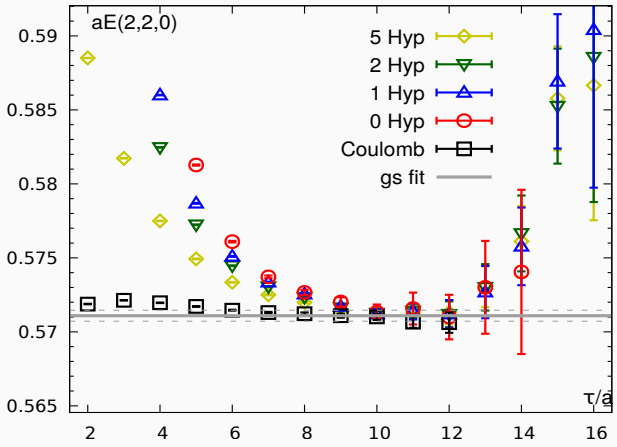
Wilson loops on the lattice

- \+ Explicit gauge invariance
- – Cusp divergences due to corners
- – Extra cusp divergences for off-axis separation
- – Self-energy divergences due to spatial Wilson lines

Wilson line correlator on the lattice

- – Must fix some gauge, i.e. Coulomb gauge
- \+ No corners, no cusps
- \+ On- and off-axis separation have same divergence
- \+ No spatial Wilson lines

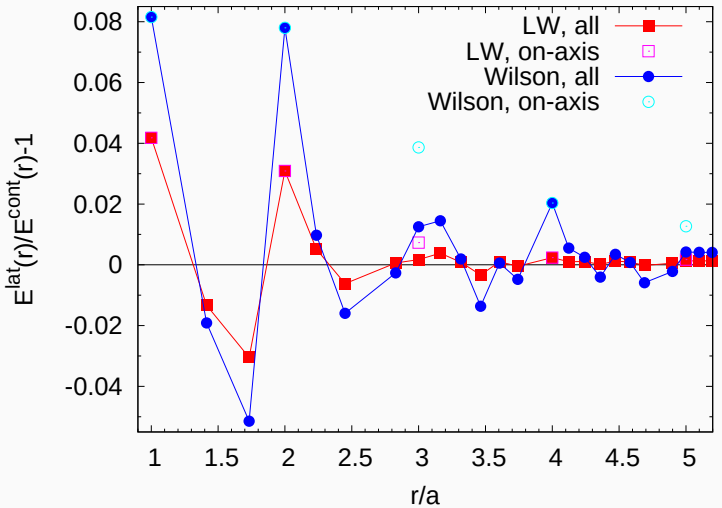


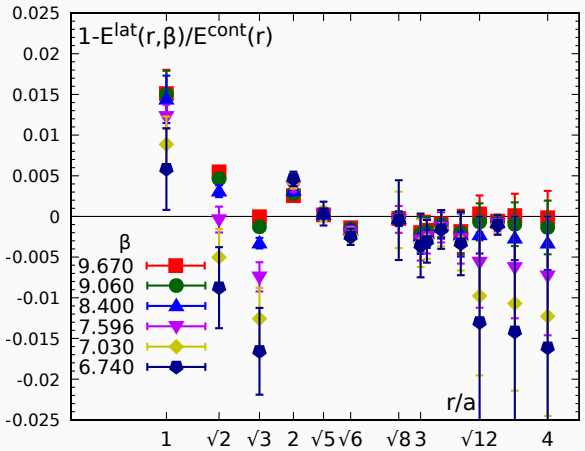


- Same ground state for both, but Wilson lines technically favorable
- Distortions at small distance and time for both operators

Lattice artifacts in the static quark-antiquark energy

- The static energy at short distances has percent-level lattice artifacts

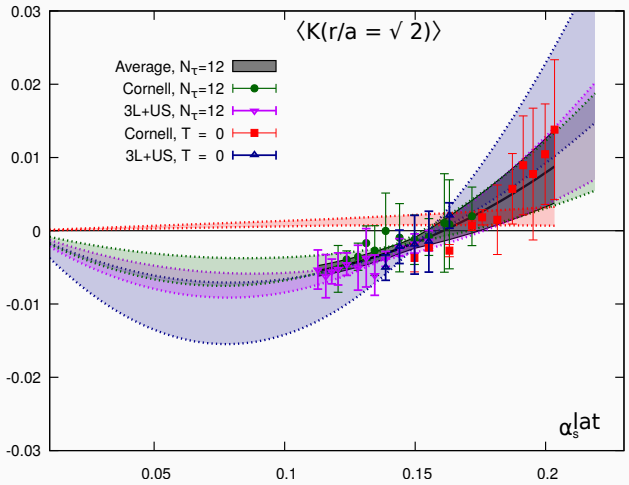


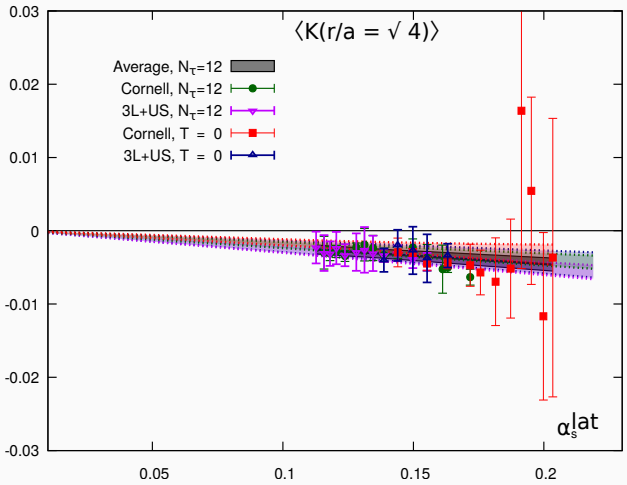


- Improved gauge action (Lüscher–Weisz) – reduced symmetry breaking
- Tree-level correction (TLC): $\frac{E^{\rm lat}(r)}{E^{\rm cont}(r)}$ for OGE without running coupling
- After tree-level correction – smaller, similar pattern of lattice artifacts[10]
- Impasse: only data with $r/a \geq \sqrt{8}$ omitting $r/a = \sqrt{12}$ are safe
- Way out: estimate the artifacts ⇒ nonperturbative correction (NPC)

[10] Bazavov et al, Phys. Rev. D98 (2018) no.5, 054511

Nonperturbative correction of the static energy

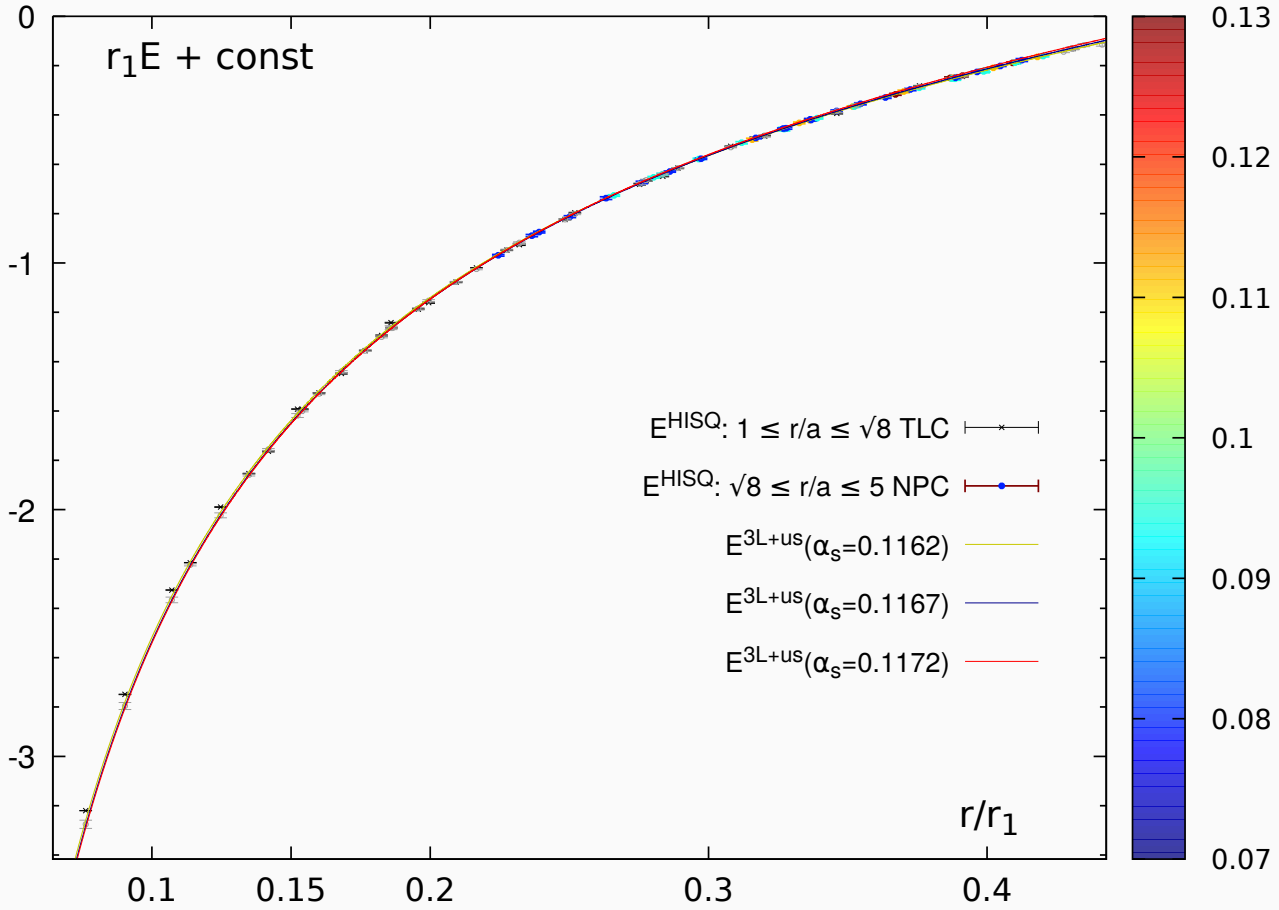




- Estimate continuum static energy using fine lattice using $r/a \geq \sqrt{5}$, interpolate (Cornell ansatz), determine corrections for coarser lattices
- Estimate continuum static energy for $r/a \geq 1$ at N^3LO *with leading ultra-soft resummation*, marginalizing over Λ_{QCD} in some window

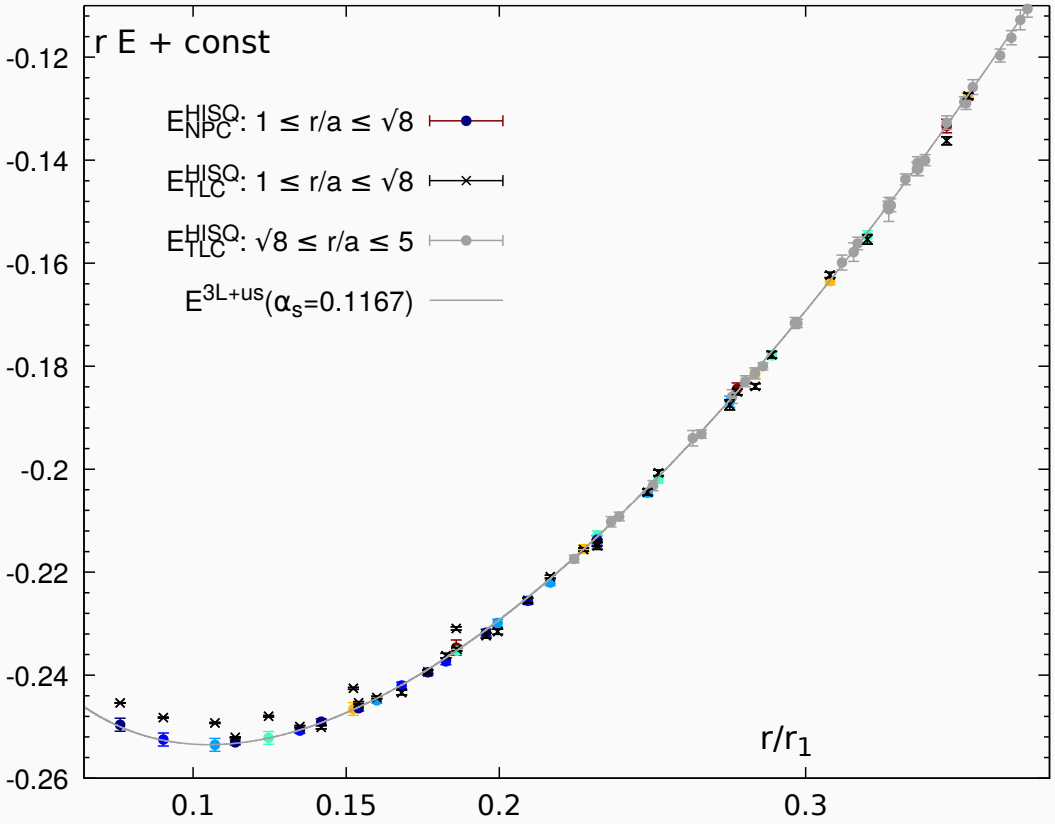
$\Rightarrow$ Extrapolate the corrections in boosted coupling α_s^{lat} to finer lattices

Impact of the lattice artifacts



- Restrict lattice data to $r < 0.14\,\mathrm{fm} \approx 0.45 r_1$ (perturbative regime)
- Analyze TLC data at $r/a \geq \sqrt{8} \Rightarrow$ artifacts are statistically irrelevant

Impact of the lattice artifacts



- TLC data at $r/a < \sqrt{8}$ yield α_s smaller by up to 2σ at bad χ^2/dof
- NPC data at $r/a < \sqrt{8}$ are well-described by fit for $r/a \geq \sqrt{8}$
- Combined analysis of lattice data with $a \leq 0.06\,\mathrm{fm}$, i.e., $a/r_1 \leq 0.2$
- Statistical and systematic errors reduced, while χ^2/dof is unchanged

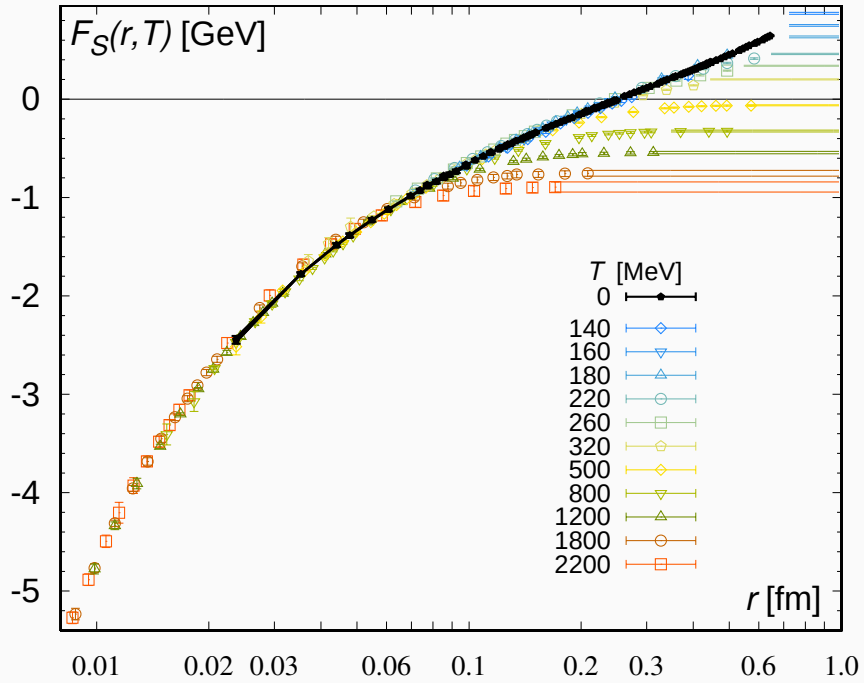
Systematic errors in the 2019 edition

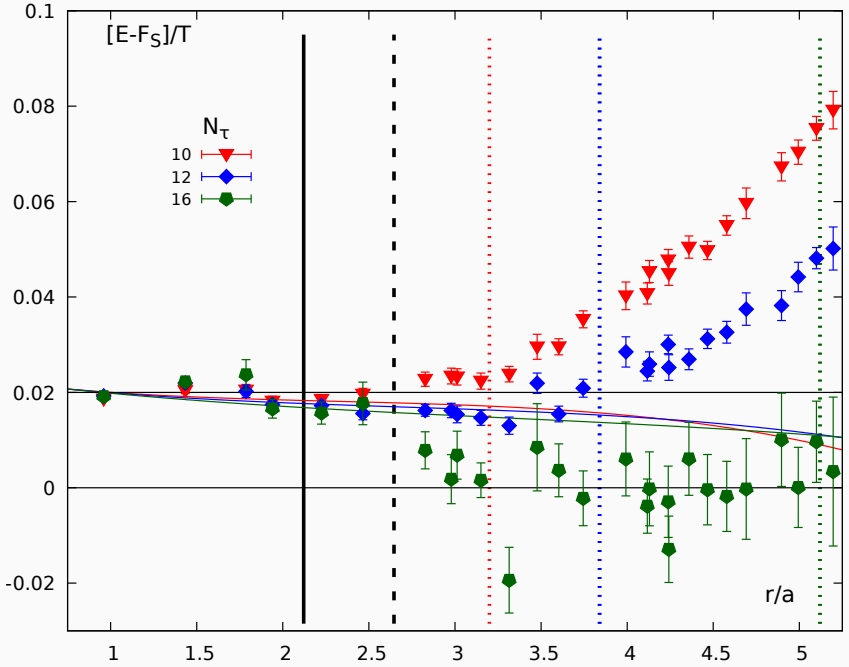
$\min(r/a)$	$\max(r)$ fm	α_s^{3L}	$\delta^{\rm stat}$	$\delta^{\rm pert}_{2014}$	$\delta^{\rm pert}_{2019}$	α_s^{2L}
$\sqrt{8}$	0.097	0.1166	0.0007	$^{+0.0007}_{-0.0003}$	$^{+0.0016}_{-0.0005}$	0.1167
$\sqrt{8}$	0.131	0.1167	0.0005	$^{+0.0008}_{-0.0003}$	$^{+0.0019}_{-0.0006}$	0.1168
1	0.055	0.1164	0.0005	$^{+0.0003}_{-0.0001}$	$^{+0.0008}_{-0.0003}$	0.1164
1	0.073	0.1166	0.0004	$^{+0.0004}_{-0.0001}$	$^{+0.0010}_{-0.0003}$	0.1166
1	0.098	0.1167	0.0003	$^{+0.0005}_{-0.0002}$	$^{+0.0012}_{-0.0004}$	0.1167
1	0.131	0.1167	0.0003	$^{+0.0007}_{-0.0003}$	$^{+0.0015}_{-0.0005}$	0.1168

- Must keep $r \lesssim 0.1$ fm to enable the full (factor 2) soft scale variation
- For soft scale $1/\sqrt{2}r \leq \nu \leq \sqrt{2}/r$ stable against variation of $\max(r)$
- No leading ultra-soft resummation: α_s^{3L} lower by $\sim 70\%$ of lower $\delta^{\rm pert}$
- Include $r/a < \sqrt{8}$ to reduce all perturbative errors
- r_1 lattice scale error: ± 1.7 MeV for $\Lambda_{\rm QCD}$, ± 0.0001 for $\alpha_s(m_z, N_f = 5)$

$$\Lambda_{\rm QCD}^{N_f=3} = 314^{+16}_{-8}\,\text{MeV}, \qquad \alpha_s(m_z, N_f = 5) = 0.11660^{+0.00110}_{-0.00056}$$

$T > 0$ data in the 2019 edition

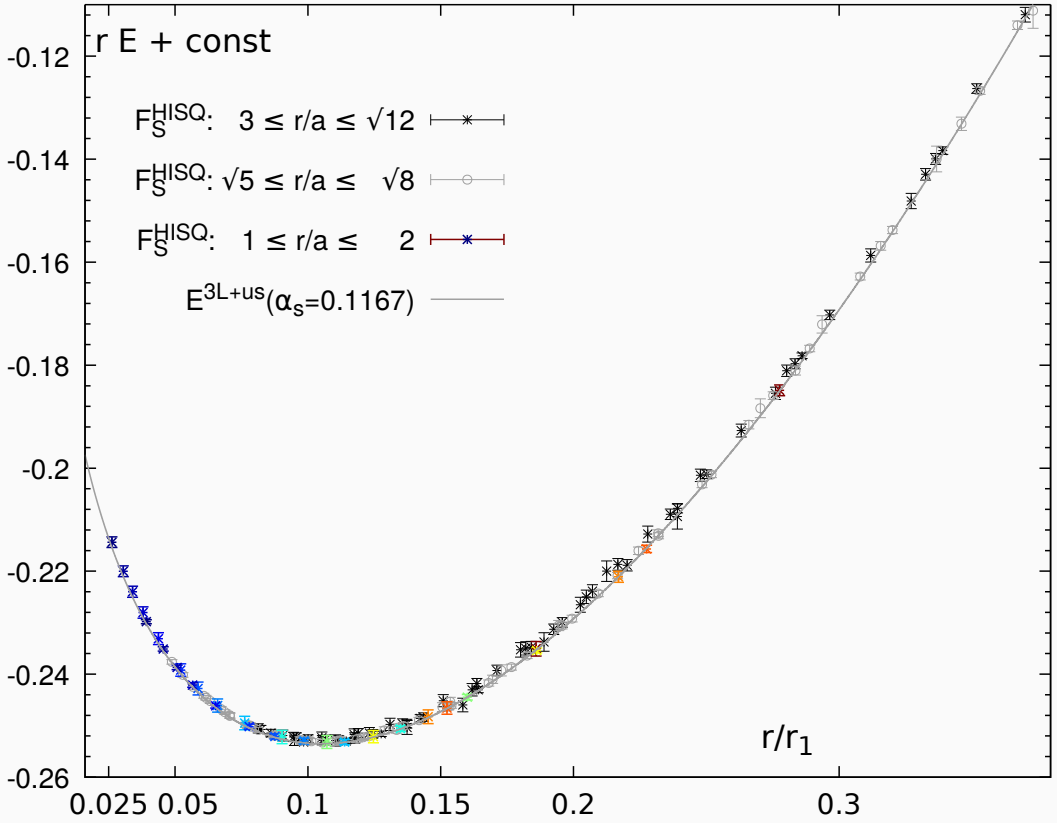




- Singlet free energy for $T > 0$ with much finer lattice spacing[11] (no pion)
- F_S defined via Coulomb gauge thermal Wilson line correlator: $\tau = 1/T$
- $T > 0$ effects exponentially suppressed for $\alpha_s/r \gg T$, i.e., $r/a \ll \alpha_s N_\tau$
- Nonconstant $T > 0$ effects are numerically small for $r/a \lesssim 0.3 N_\tau$ due to compensation between static gluons vs nonstatic gluons and quarks

[11] Bazavov et al, Phys. Rev. D98 (2018) no.5, 054511

α_s from $T > 0$



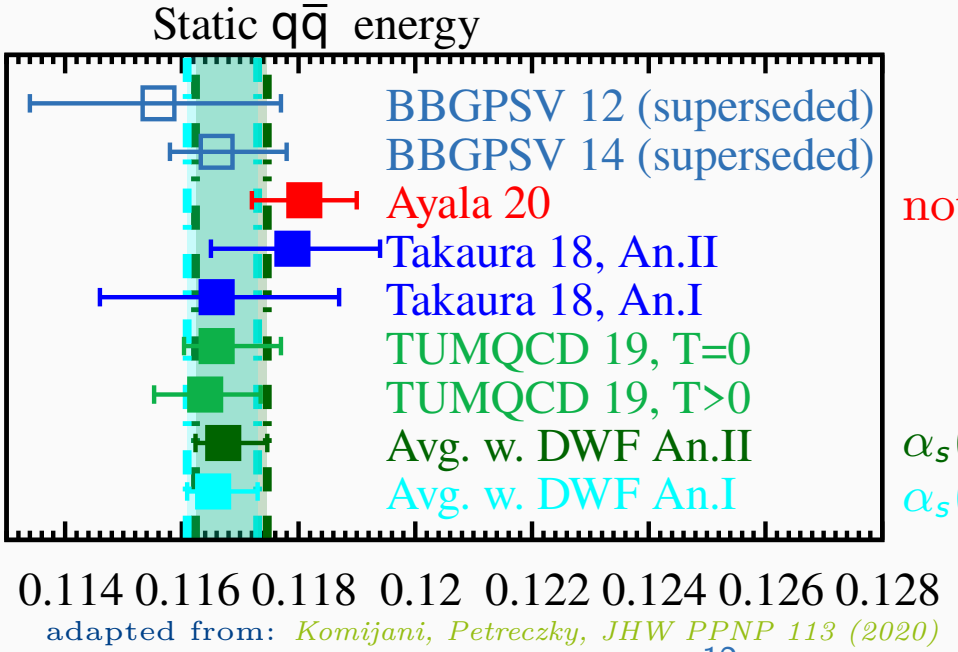
- Restrict $N_\tau = 12$ data to $r/a \le 2$ or 3, i.e., $r \le 0.17/T$ or $0.25/T$
- Cannot avoid the nonperturbative correction for the lattice artifacts
- Restrict to tiny distances $r \le 0.03$ fm to reduce the perturbative error

$T = 0$ vs $T > 0$

N_τ	$\max(r/a)$	$\max(r)$ fm	α_s^{3L}	$\delta^{\rm stat}$	$\delta^{\rm pert}_{2014}$	$\delta^{\rm pert}_{2019}$	α_s^{2L}
64	2	0.057	0.1165	0.0006	$^{+0.0003}_{-0.0001}$	$^{+0.0008}_{-0.0003}$	0.1164
64	2	0.096	0.1166	0.0005	$^{+0.0004}_{-0.0002}$	$^{+0.0011}_{-0.0003}$	0.1166
12	2	0.030	0.1164	0.0008	$^{+0.0002}_{-0.0001}$	$^{+0.0004}_{-0.0002}$	0.1163
12	2	0.057	0.1165	0.0007	$^{+0.0002}_{-0.0001}$	$^{+0.0006}_{-0.0002}$	0.1164
12	2	0.091	0.1167	0.0006	$^{+0.0003}_{-0.0001}$	$^{+0.0009}_{-0.0003}$	0.1167
64	3	0.055	0.1164	0.0005	$^{+0.0003}_{-0.0001}$	$^{+0.0008}_{-0.0003}$	0.1164
64	3	0.134	0.1167	0.0003	$^{+0.0006}_{-0.0003}$	$^{+0.0014}_{-0.0005}$	0.1168
12	3	0.030	0.1166	0.0006	$^{+0.0002}_{-0.0001}$	$^{+0.0005}_{-0.0002}$	0.1165
12	3	0.055	0.1167	0.0005	$^{+0.0002}_{-0.0001}$	$^{+0.0006}_{-0.0002}$	0.1167
12	3	0.133	0.1168	0.0004	$^{+0.0005}_{-0.0002}$	$^{+0.0012}_{-0.0004}$	0.1169

Complete agreement between α_s from $T = 0$ or $T > 0$

α_s from the QCD static energy on the lattice



not included in averages

$\alpha_s(m_Z) = 0.11671^{+76}_{-47}$

$\alpha_s(m_Z) = 0.11654^{+76}_{-45}$

adapted from: *Komijani, Petreczky, JHW PPNP 113 (2020)*

- Perfect agreement between superseded TUM[12] and TUMQCD results[13]
- Agreement with results for Wilson loops on JLQCD ensembles[14]

⇒ but potential issues with continuum limit and quark mass effects[15]

- Higher α_s in re-analysis[16] of TUMQCD data (hyperasymptotic exp.)

[12] Bazavov et al., Phys. Rev. D86 (2012) 114031
Bazavov et al., Phys. Rev. D90 (2014) 7, 074038
[13] Bazavov et al. [TUMQCD], Phys. Rev. D100 (2019) 11, 114511
[14] Takaura et al., JHEP 1904, 155 (2019)
Takaura et al., Phys. Lett. B789, 598-602 (2019)
[15] Komijani, Petreczky, JHW PPNP 113 (2020)
[16] Ayala et al., JHEP 09 (2020) 016

Coefficients of the force – color factors and beta function

- Color factors: $C_F = \frac{N_c^2-1}{2N_c}$, $C_A = N_c$, $T_F = \frac{1}{2}$
- Beta function:

$$\frac{d\,\alpha_s(\nu)}{d\ln\nu} = \alpha_s\beta(\alpha_s) = -\frac{\alpha_s^2}{2\pi}\sum_{n=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n\beta_n = -2\alpha_s\left[\beta_0\frac{\alpha_s}{4\pi} + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots\right]$$

- Relevant coefficients explicitly contributing to the force:

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F N_f,$$

$$\beta_1 = \frac{34}{3}C_A^2 - \frac{20}{3}C_A N_f T_F - 4C_F N_f T_F,$$

$$\beta_2 = \frac{2857}{54}C_A^3 - \left(\frac{1415}{27}C_A^2 + \frac{205}{9}C_A C_F - 2C_F^2\right)N_f T_F + \left(\frac{158}{27}C_A + \frac{44}{9}C_F\right)N_f^2 T_F^2$$

Coefficients of the force – other coefficients (I)

- Coefficients $\tilde{a}_i$:

$$\tilde{a}_1 = a_1 + 2\gamma_E \beta_0,$$

$$\tilde{a}_2 = a_2 + \left(\frac{\pi^2}{3} + 4\gamma_E^2\right)\beta_0^2 + \gamma_E\left(4a_1\beta_0 + 2\beta_1\right),$$

$$\tilde{a}_3 = a_3 + \left(8\gamma_E^3 + 2\gamma_E\pi^2 + 16\zeta(3)\right)\beta_0^3 + 2\gamma_E\beta_2$$

$$+ \left[\left(12\gamma_E^2 + \pi^2\right)\beta_0^2 + 4\gamma_E\beta_1\right]a_1 + \left[6a_2\gamma_E + \frac{5}{2}\left(4\gamma_E^2 + \frac{\pi^2}{3}\right)\beta_1\right]\beta_0$$

- Coefficients a_i:

$$a_1 = \frac{31}{9}C_A - \frac{20}{9}T_F N_f,$$

$$a_2 = \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right)C_A^2 - \left(\frac{1798}{81} + \frac{56}{3}\zeta(3)\right)C_A T_F N_f$$

$$- \left(\frac{55}{3} - 16\zeta(3)\right)C_F T_F N_f + \left(\frac{20}{9}T_F N_f\right)^2$$

Coefficients of the force – other coefficients (II)

- Coefficient a_3:

$$a_3 = a_3^{(3)} N_f^3 + a_3^{(2)} N_f^2 + a_3^{(1)} N_f + a_3^{(0)},$$

$$a_3^{(3)} = -\left(\frac{20}{9}\right)^3 T_F^3,$$

$$a_3^{(2)} = \left(\frac{12541}{243} + \frac{368}{3}\zeta(3) + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} - \frac{416}{3}\zeta(3)\right) C_F T_F^2,$$

$$a_3^{(1)} = (-709.717)\, C_A^2 T_F + \left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5)\right) C_A C_F T_F$$
$$+ \left(\frac{286}{9} + \frac{296}{3}\zeta(3) - 160\zeta(5)\right) C_F^2 T_F + (-56.83(1))\, \frac{18 - 6N_c^2 + N_c^4}{96 N_c^2},$$

$$a_3^{(0)} = 502.24(1) C_A^3 - 136.39(12) \frac{N_c^3 + 6N_c}{48} + \frac{8}{3}\pi^2 C_A^3 \left(-\frac{5}{3} + 2\gamma_E + 2\log 2\right)$$

- Coefficient a_3^L: $a_3^L = \frac{16\pi^2}{3} C_A^3$

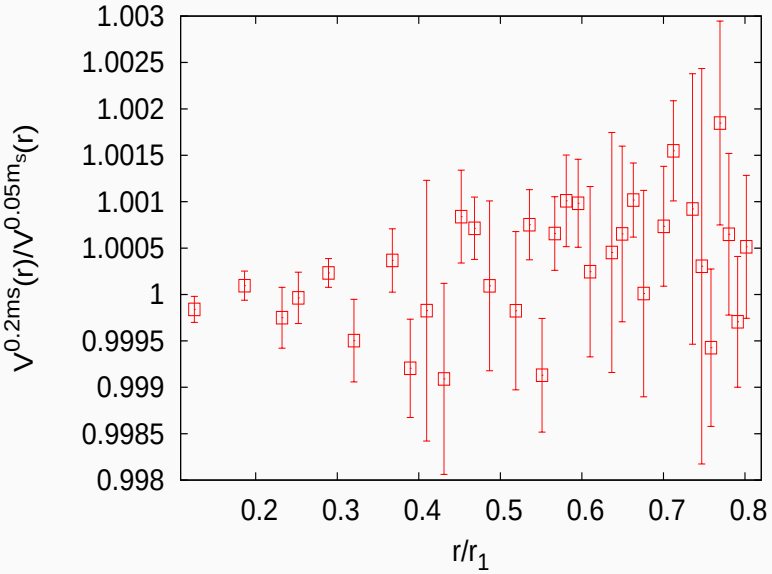
Quark mass dependence and topology

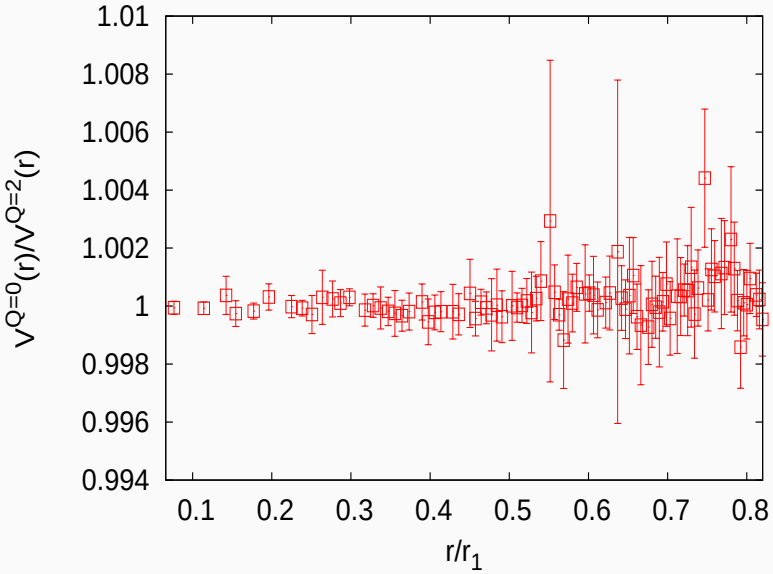
- Combine gauge ensembles with different light sea quark mass

⇒ No statistically significant quark mass effects up to $r \approx 0.5 r_1$

- Fine gauge ensembles with fully suppressed topological tunneling

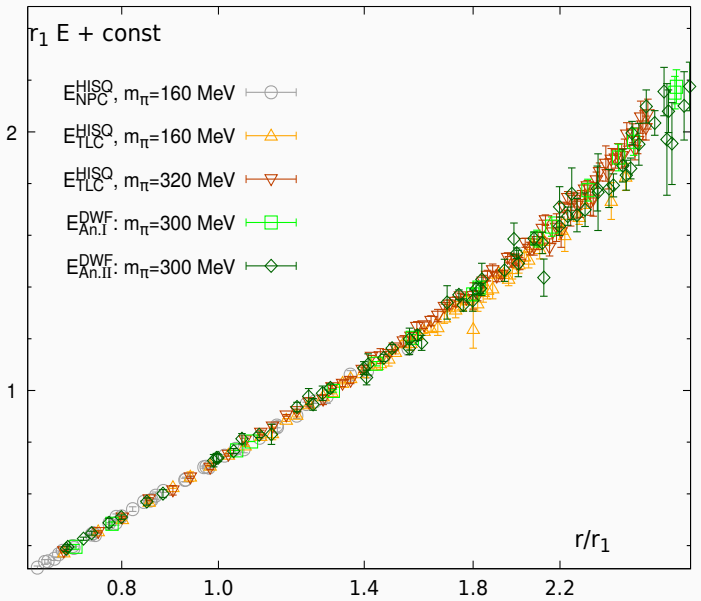
⇒ No statistically significant difference between static energy in different topological sectors up to $r \approx 0.5 r_1$ observed[17]

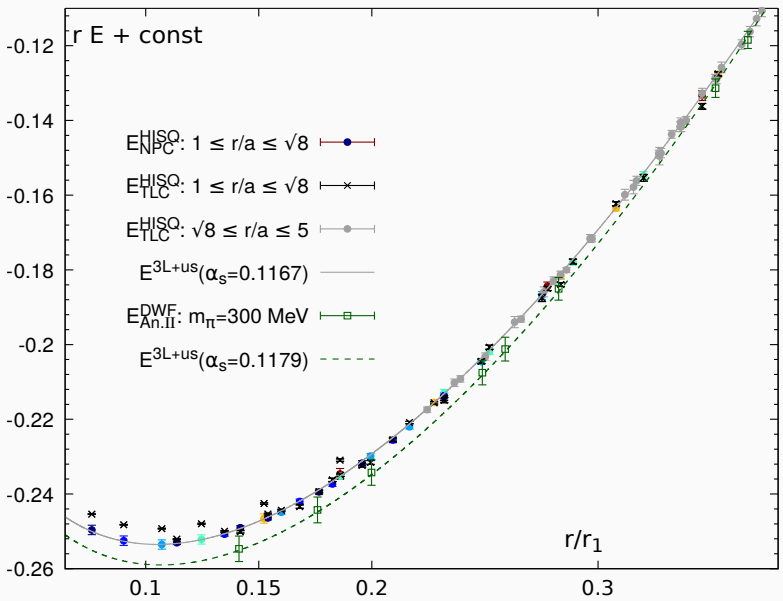




[17] Bazavov et al., PoS Confinement2018 (2019) 166

Quark mass dependence and continuum limit on JLQCD ensembles





- $E(r)$ distinguishes $m_\pi = 160\,\mathrm{MeV}$ or $m_\pi = 320\,\mathrm{MeV}$ for $r \gg 0.5 r_1$
- $E(r)$ for DWF at $m_\pi = 300\,\mathrm{MeV}$ and for HISQ at $m_\pi = 320\,\mathrm{MeV}$ agree
- For $r < 0.8 r_1$ continuum extrapolation for DWF via rescaled TLC combined with a smooth global fit using power series in a^2