

Determination of $\alpha(M_z)$ from a hyperasymptotic approximation to the energy of a static quark-antiquark pair

Xabier Lobregat

IFAE-BIST

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Work in collaboration with César Ayala and Antonio Pineda
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Outline of the talk

- Introduction
 - Static energy of a quark antiquark pair
 - Terminants
- The fits
- Conclusions

The static energy of a quark-antiquark pair

- $E^{\text{latt}}(r) - E^{\text{latt}}(r_{\text{ref}}) = E^{\text{th}}(r) - E^{\text{th}}(r_{\text{ref}})$
- In pNRQCD the singlet static energy admits the expression
$$E^{\text{th}}(r) = V(r, \nu_{\text{us}} = \nu_s) + \delta V_{\text{RG}}(r, \nu_s, \nu_{\text{us}}) + \delta E_{\text{us}}(r, \nu_{\text{us}})$$
- V has a $u = 1/2$ renormalon
- This renormalon is r independent
- We use the derivative of the static energy ($\frac{d}{dr} V \equiv F$)

$$\begin{aligned}\mathcal{F} &\equiv \frac{d}{dr} E^{\text{th}}(r) \\ &= F(r, \nu_{\text{us}} = \nu_s) + \frac{d}{dr} \delta V_{\text{RG}}(r, \nu_s, \nu_{\text{us}}) + \frac{d}{dr} \delta E_{\text{us}}(r, \nu_{\text{us}})\end{aligned}$$

- We fit

$$E^{\text{latt}}(r) - E^{\text{latt}}(r_{\text{ref}}) = \int_{r_{\text{ref}}}^r dr' \mathcal{F}(r')$$

The principal value Borel sum

- Not a new idea
- Based on Phys.Rev.D99,074019 we use the principal value Borel sums

$$\mathcal{F} = F(r, \nu_{\text{us}} = \nu_{\text{s}}) \Big|_{\text{PV}} + \frac{d}{dr} \delta V_{\text{RG}}(r, \nu_{\text{s}}, \nu_{\text{us}}) \Big|_{\text{N}^3\text{LL}} + \frac{d}{dr} E_{\text{us}}(r, \nu_{\text{us}}) \Big|_{\text{PV}}$$

- We define the principal value Borel sum

$$R = \sum_{n=0}^{\infty} r_n \alpha^{n+1}(\mu), \quad R_{\text{PV}} = \text{PV} \int_0^{\infty} dt e^{-t/\alpha(\mu)} \sum_{n=0}^{\infty} \frac{r_n}{n!} t^n$$

Terminants

- Leading singularity in $t = \frac{2\pi d}{\beta_0}$; $N_P(d) = \frac{2\pi|d|}{\beta_0\alpha(\mu)} \{1 - c\alpha(\mu)\}$

$$R_{PV} \approx \sum_{n=0}^{N_P(d)} r_n \alpha^{n+1}(\mu) + T(d, N_P, \mu) + \mathcal{O}\left(e^{-\frac{2\pi|d|}{\beta_0\alpha(\mu)}(1 + \log \frac{|d_{\text{next}}|}{|d|})}\right)$$

- In our case $d = 3$ and the terminant takes the form

$$T = \sqrt{\alpha(\mu)} K^{(P)} r \mu^3 e^{-\frac{6\pi}{\beta_0\alpha(\mu)}} \left(\frac{\beta_0\alpha(\mu)}{4\pi}\right)^{-3b} \left(1 + \bar{K}_1^{(P)}\alpha(\mu) + \mathcal{O}(\alpha^2(\mu))\right)$$

where $\eta_c = -3b + \frac{6\pi c}{\beta_0} - 1$; $b = \frac{\beta_1}{2\beta_0^2}$ and

$$K^{(P)} = -\frac{Z_3^F 2^{1-3b} \pi 3^{3b+1/2}}{\Gamma(1+3b)} \beta_0^{-1/2} \left[-\eta_c + \frac{1}{3}\right]$$

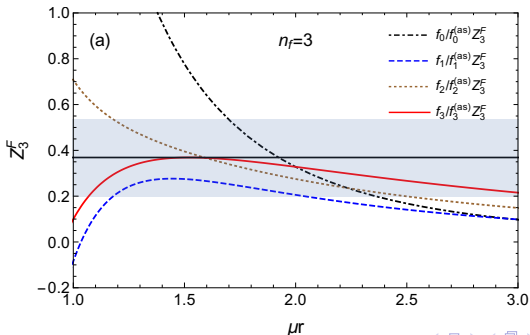
Normalization of the $u = 3/2$ renormalon

- The PV Borel sum of the force requires knowledge of Z_3^F (recall $Z_3^F = 2Z_3^V$)

$$r^2 f_n^{\text{as}} = Z_3^F (r\mu)^3 \left(\frac{\beta_0}{6\pi}\right)^n \frac{\Gamma(n+1+3b)}{\Gamma(1+3b)} \left\{ 1 + \frac{3b}{n+3b} b_1 + \mathcal{O}\left(\frac{1}{n^2}\right) \right\}$$

- We estimate it dividing by the exact value ($x = 1.52$)

$$Z_3^F|_{n_f=3} = 0.37_{-0.16}^{-0.06}(\Delta x) + 0.02(N^2\text{LO}) - 0.05(\mathcal{O}(1/n)) + 0.005(\text{us}) = 0.37(17)$$



$$F = \frac{d}{dr} V$$

$$V = \sum_{n=0}^{\infty} V_n(r, \nu_s, \nu_{us}) \alpha^{n+1}(\nu_s)$$

$$V_n(r, \nu_s, \nu_{us}) = -\frac{C_F}{r} \frac{1}{(4\pi)^n} a_n(r, \nu_s, \nu_{us})$$

$$a_0 = 1$$

$$a_1(r, \nu_s) = a_1 + 2\beta_0 \log(\nu_s e^{\gamma_E} r)$$

$$a_2(r, \nu_s) = a_2 + \frac{\pi^2}{3} \beta_0^2 + (4a_1\beta_0 + 2\beta_1) \log(\nu_s e^{\gamma_E} r) + 4\beta_0^2 \log^2(\nu_s e^{\gamma_E} r)$$

$$\begin{aligned} a_3(r, \nu_s, \nu_{us}) = & a_3 + a_1\beta_0^2\pi^2 + \frac{5\pi^2}{6}\beta_0\beta_1 + 16\zeta_3\beta_0^3 \\ & + \left(2\pi^2\beta_0^3 + 6a_2\beta_0 + 4a_1\beta_1 + 2\beta_2 \right) \log(\nu_s e^{\gamma_E} r) + \frac{16}{3}C_A^3\pi^2 \log(\nu_{us} e^{\gamma_E} r) \\ & + \left(12a_1\beta_0^2 + 10\beta_0\beta_1 \right) \log^2(\nu_s e^{\gamma_E} r) + 8\beta_0^3 \log^3(\nu_s e^{\gamma_E} r) \end{aligned}$$

Details on the fits I

- Data from Phys.Rev.D100,114511
- $\beta = 8.4$ lattice spacing $a = 0.025 \text{ fm} = 0.125 \text{ GeV}^{-1}$
- We consider the ranges
 - Set I: $0.353 \text{ GeV}^{-1} \leq r \leq 0.499 \text{ GeV}^{-1}$; 8 points
 - Set II: $0.353 \text{ GeV}^{-1} \leq r \leq 0.612 \text{ GeV}^{-1}$; 17 points
 - Set III: $0.353 \text{ GeV}^{-1} \leq r \leq 0.8002 \text{ GeV}^{-1}$; 31 points
 - Set IV: $0.353 \text{ GeV}^{-1} \leq r \leq 1 \text{ GeV}^{-1}$; 50 points
- $r_{\text{ref}} = 0.353 \text{ GeV}^{-1}$
- The central values for the soft and ultrasoft scales are
 - $\nu_s = 1/r$
 - $\nu_{\text{us}} = \frac{C_A \alpha(\nu_s)}{2r}$

Details on the fits II

We will explore the following orders

- LL/LO

$$\mathcal{F}_{\text{LO}}(r) = F(r, \nu_{\text{us}} = \nu_s) \Big|_{\text{LO in } \alpha(\nu_s)}$$

- NLL/NLO

$$\mathcal{F}_{\text{NLO}}(r) = F(r, \nu_{\text{us}} = \nu_s) \Big|_{\text{NLO in } \alpha(\nu_s)}$$

- N²LL

$$\mathcal{F}_{\text{N}^2\text{LL}}(r) = F(r, \nu_{\text{us}} = \nu_s) \Big|_{\text{N}^2\text{LO in } \alpha(\nu_s)} + \frac{d}{dr} \delta V_{\text{RG}}(r, \nu_s, \nu_{\text{us}}) \Big|_{\text{N}^2\text{LL}}$$

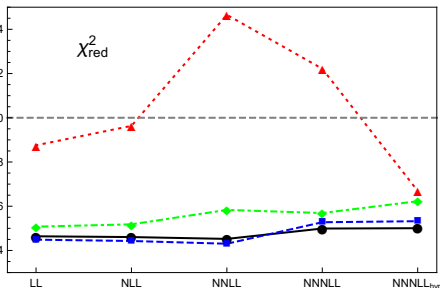
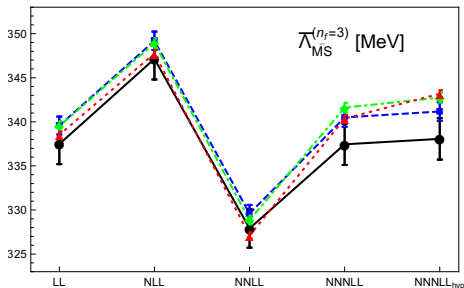
- N³LL

$$\mathcal{F}_{\text{N}^3\text{LL}}(r) = F(r, \nu_{\text{us}} = \nu_s) \Big|_{\text{N}^3\text{LO in } \alpha(\nu_s)} + \frac{d}{dr} \delta V_{\text{RG}}(r, \nu_s, \nu_{\text{us}}) \Big|_{\text{N}^3\text{LL}} + \frac{d}{dr} \delta E_{\text{us}}(r, \nu_{\text{us}}) \Big|_{\text{LO in } \alpha(\nu_{\text{us}})}$$

- N³LL_{hyp}

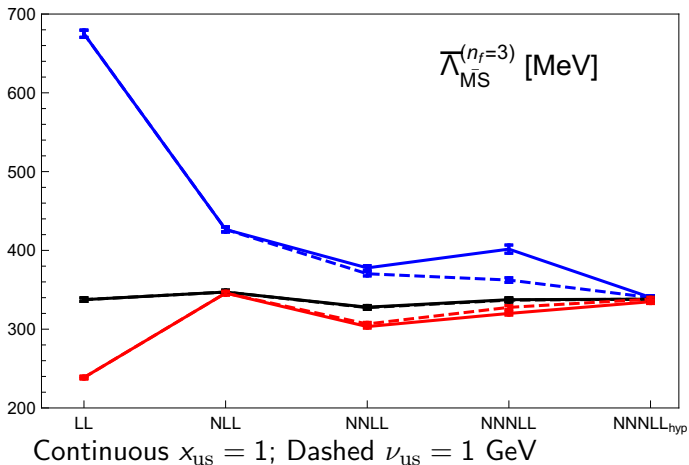
$$\begin{aligned} \mathcal{F}_{\text{N}^3\text{LL}}(r) = & F(r, \nu_{\text{us}} = \nu_s) \Big|_{\text{N}^3\text{LO in } \alpha(\nu_s)} + T(d=3, N_{\text{P}}=3, \nu_s) + \frac{d}{dr} \delta V_{\text{RG}}(r, \nu_s, \nu_{\text{us}}) \Big|_{\text{N}^3\text{LL}} \\ & + \frac{d}{dr} \delta E_{\text{us}}(r, \nu_{\text{us}}) \Big|_{\text{LO in } \alpha(\nu_{\text{us}})} - T(d=3, N_{\text{P}}=0, \nu_{\text{us}}) \end{aligned}$$

Results



- $r \in [0.353, 0.499] \times \text{GeV}^{-1}$
- $r \in [0.353, 0.612] \times \text{GeV}^{-1}$
- $r \in [0.353, 0.8002] \times \text{GeV}^{-1}$
- $r \in [0.353, 1] \times \text{GeV}^{-1}$

Variations on $\nu_S \equiv x_S/r$ and $\nu_{US} \equiv x_{US} \frac{C_A \alpha(\nu_S)}{2r}$



Final numbers

$$\text{Set I} \quad \Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 338(2)_{\text{stat}}(10)_{\text{h.o.}}(8)_{\Gamma_{\text{ref}}} \text{ MeV} = 338(12) \text{ MeV}$$

$$\text{Set II} \quad \Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 341(1)_{\text{stat}}(11)_{\text{h.o.}}(6)_{\Gamma_{\text{ref}}} \text{ MeV} = 341(14) \text{ MeV}$$

$$\text{Set III} \quad \Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 343(1)_{\text{stat}}(13)_{\text{h.o.}}(7)_{\Gamma_{\text{ref}}} \text{ MeV} = 343(14) \text{ MeV}$$

$$\text{Set IV} \quad \Lambda_{\overline{\text{MS}}}^{(n_f=3)} = 343(0)_{\text{stat}}(13)_{\text{h.o.}}(9)_{\Gamma_{\text{ref}}} \text{ MeV} = 343(16) \text{ MeV}$$

Therefore our central value result for the strong coupling

- $\alpha^{(n_f=3)}(M_\tau) = 0.3151(65)$
- $\alpha^{(n_f=5)}(M_Z) = 0.1181(8) \Lambda_{\overline{\text{MS}}}^{(4)}(M_\tau \rightarrow M_Z) = 0.1181(9)$

Conclusions

- We have obtained the normalization of the $d = 3$ renormalon of V

$$Z_3^F|_{n_f=3} = 0.37(17)$$

- Making use of the hyperasymptotic expansion of principal value Borel sums and N^3LL resummation we have obtained an estimate of the QCD strong coupling

$$\alpha^{(n_f=5)}(M_z) = 0.1181(9)$$

Formulas I

$$F \equiv \frac{d}{dr} V, \quad F = \sum_{n=0}^{\infty} f_n(r, \nu_s, \nu_{us}) \alpha^{n+1}(\nu_s)$$

$$f_0 = \frac{C_F}{r^2}, \quad f_1 = \frac{C_F}{4\pi r^2} \{a_1(r, \nu_s) - 2\beta_0\}$$

$$f_2 = \frac{C_F}{(4\pi)^2 r^2} \{a_2(r, \nu_s) - 4a_1(r, \nu_s)\beta_0 - 2\beta_1\}$$

$$f_3 = \frac{C_F}{(4\pi)^3 r^2} \left\{ a_3(r, \nu_s, \nu_{us}) - 6a_2(r, \nu_s)\beta_0 - 4a_1(r, \nu_s)\beta_1 - 2\beta_2 - \frac{16}{3}C_A^3\pi^2 \right\}$$

$$a_1(r, \nu_s) = a_1 + 2\beta_0 \log(\nu_s e^{\gamma_E} r)$$

$$a_2(r, \nu_s) = a_2 + \frac{\pi^2}{3}\beta_0^2 + (4a_1\beta_0 + 2\beta_1) \log(\nu_s e^{\gamma_E} r) + 4\beta_0^2 \log^2(\nu_s e^{\gamma_E} r)$$

$$\begin{aligned} a_3(r, \nu_s, \nu_{us}) = & a_3 + a_1\beta_0^2\pi^2 + \frac{5\pi^2}{6}\beta_0\beta_1 + 16\zeta_3\beta_0^3 \\ & + \left(2\pi^2\beta_0^3 + 6a_2\beta_0 + 4a_1\beta_1 + 2\beta_2 \right) \log(\nu_s e^{\gamma_E} r) + \frac{16}{3}C_A^3\pi^2 \log(\nu_{us} e^{\gamma_E} r) \\ & + \left(12a_1\beta_0^2 + 10\beta_0\beta_1 \right) \log^2(\nu_s e^{\gamma_E} r) + 8\beta_0^3 \log^3(\nu_s e^{\gamma_E} r) \end{aligned}$$

Formulas II

$$a_1 = \frac{31C_A - 20T_F n_f}{9}$$

$$a_2 = \frac{400n_f^2 T_F^2}{81} - C_F n_f T_F \left(\frac{55}{3} - 16\zeta(3) \right) \\ + C_A^2 \left(\frac{4343}{162} + \frac{16\pi^2 - \pi^4}{4} + \frac{22\zeta(3)}{3} \right) - C_A n_f T_F \left(\frac{1798}{81} + \frac{56\zeta(3)}{3} \right)$$

$$a_3 = a_3^{(3)} n_f^3 + a_3^{(2)} n_f^2 + a_3^{(1)} n_f + a_3^{(0)}$$

Formulas III

$$\begin{aligned}
 a_3^{(3)} &= -\left(\frac{20}{9}\right)^3 T_F^3 \\
 a_3^{(2)} &= \left(\frac{12541}{243} + \frac{368\zeta(3)}{3} + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} - \frac{416\zeta(3)}{3}\right) C_F T_F^2 \\
 a_3^{(1)} &= \frac{d_F^{abcd} d_A^{abcd}}{N_A} \left\{ \pi^2 \left(\frac{1264}{9} - \frac{976}{3} \zeta(3) + \log(2)[64 + 672\zeta(3)] \right) + \pi^4 \left(-\frac{184}{3} + \frac{32}{3} \log(2) - 32 \log^2(2) \right) + \frac{10\pi^6}{3} \right\} \\
 &\quad + T_F \left\{ C_F^2 \left(\frac{286}{9} + \frac{296}{3} \zeta(3) - 160\zeta(5) \right) + C_A C_F \left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5) \right) \right. \\
 &\quad + C_A^2 \left(-\frac{58747}{486} + \pi^2 \left[\frac{17}{27} - 32\theta_4 + \log(2) \left\{ -\frac{4}{3} - 14\zeta(3) \right\} - \frac{19}{3} \zeta(3) \right] - 356\zeta(3) \right) \\
 &\quad \left. + \pi^4 \left[-\frac{157}{54} - \frac{5}{9} \log(2) + \log^2(2) \right] + \frac{1091}{6} \zeta(5) + \frac{57}{2} \zeta^2(3) + \frac{761}{2520} \pi^6 - 48y_6 \right\} \\
 a_3^{(0)} &= \frac{d_F^{abcd} d_A^{abcd}}{N_A} \left\{ \pi^2 \left(\frac{7432}{9} - 4736\theta_4 + \log(2) \left[\frac{14752}{3} - 3472\zeta(3) \right] - \frac{6616}{3} \zeta(3) \right) \right. \\
 &\quad + \pi^4 \left(-156 + \frac{560}{3} \log(2) + \frac{496}{3} \log^2(2) \right) + \frac{1511\pi^6}{45} \left. \right\} + C_A^3 \left\{ \frac{385645}{2916} + \pi^2 \left(-\frac{953}{54} + \frac{584}{3} \theta_4 + \frac{175}{2} \zeta(3) \right) \right. \\
 &\quad + \log(2) \left[-\frac{922}{9} + \frac{217}{3} \zeta(3) \right] \left. \right\} + \frac{584}{3} \zeta(3) + \pi^4 \left(\frac{1349}{270} - \frac{20}{9} \log(2) - \frac{40}{9} \log^2(2) \right) - \frac{1927}{6} \zeta(5) - \frac{143}{2} \zeta^2(3) \\
 &\quad - \frac{4621\pi^6}{3024} + 144y_6 \left. \right\}
 \end{aligned}$$

Formulas IV

$$\frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{18 - 6N_c^2 + N_c^4}{96N_c^2}$$

$$\frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N_c(N_c^2 + 6)}{48}$$

$$\theta_n = \text{Li}_n(1/2) + \frac{(-\log(2))^n}{n!}$$

$$y_6 = \zeta(-5, -1) + \zeta(6)$$

Where $\zeta(z)$ is the Riemann zeta function, $\zeta(z_1, z_2)$ is the multiple zeta function, and $\text{Li}_n(z)$ is the polylogarithm.

Formulas V

$$\left. \frac{d}{dr} \delta V_{\text{RG}}(r, \nu_s, \nu_{us}) \right|_{\text{N}^2\text{LL}} = -\frac{1}{6\beta_0} C_F C_A^3 \frac{1}{r^2} \alpha^3(\nu_s) \log\left(\frac{\alpha(\nu_{us})}{\alpha(\nu_s)}\right)$$

$$\left. \frac{d}{dr} \delta V_{\text{RG}}(r, \nu_s, \nu_{us}) \right|_{\text{N}^3\text{LL}} = C_F C_A^3 \frac{1}{r^2} \alpha^3(\nu_s) \left\{ -\frac{1}{6\beta_0} \log\left(\frac{\alpha(\nu_{us})}{\alpha(\nu_s)}\right) \right.$$

$$\left. + \frac{\pi}{4\beta_0} K \{ \alpha(\nu_{us}) - \alpha(\nu_s) \} + \frac{1}{8\pi} \left(2 - \frac{1}{\beta_0} [a_1 + 2\beta_0 \log(r\nu_s e^{\gamma_E})] \right) \alpha(\nu_s) \log\left(\frac{\alpha(\nu_{us})}{\alpha(\nu_s)}\right) \right\}$$

$$K \equiv \frac{8\beta_1}{3\beta_0} \frac{1}{(4\pi)^2} - \frac{1}{27\pi^2} (C_A (47 + 6\pi^2) - 10T_F n_f)$$

$$L \equiv \log \frac{\nu_{us}}{\nu_s}$$

$$\text{N}^2\text{LL} : \alpha^{3+n}(\nu_s) L^n \sim \alpha^{3+n}(\nu_s) \log^n(\alpha(\nu_s)) \sim \alpha^3(\nu_s) \text{ if } \alpha(\nu_s) L \sim 1$$

$$\text{N}^3\text{LL} : \alpha^{4+n}(\nu_s) L^n \sim \alpha^{4+n}(\nu_s) \log^n(\alpha(\nu_s)) \sim \alpha^4(\nu_s) \text{ if } \alpha(\nu_s) L \sim 1$$

$$\left. \frac{d}{dr} \delta E_{us} \right|_{\text{LO in } \alpha(\nu_{us})} = C_F C_A^3 \frac{1}{12\pi r^2} \alpha^3(\nu_s) \alpha(\nu_{us}) \left\{ \log\left(\frac{C_A \alpha(\nu_s)}{2r\nu_{us}}\right) + \log(2) + 1/6 \right\}$$

Formulas VI

$$T = \sqrt{\alpha(\mu)} K^{(P)} r \mu^3 e^{-\frac{6\pi}{\beta_0 \alpha(\mu)}} \left(\frac{\beta_0 \alpha(\mu)}{4\pi} \right)^{-3b} \left(1 + \bar{K}_1^{(P)} \alpha(\mu) + \mathcal{O}(\alpha^2(\mu)) \right)$$

where $\eta_c = -3b + \frac{6\pi c}{\beta_0} - 1$; $b = \frac{\beta_1}{2\beta_0^2}$ and

$$K^{(P)} = -\frac{Z_3^F 2^{1-3b} \pi 3^{3b+1/2}}{\Gamma(1+3b)} \beta_0^{-1/2} \left[-\eta_c + \frac{1}{3} \right]$$

$$\bar{K}_1^{(P)} = \frac{\beta_0/(3\pi)}{-\eta_c + \frac{1}{3}} \left[-3b_1 b \left(\frac{1}{2} \eta_c + \frac{1}{3} \right) - \frac{1}{12} \eta_c^3 + \frac{1}{24} \eta_c - \frac{1}{1080} \right]$$

The t plane and the u plane

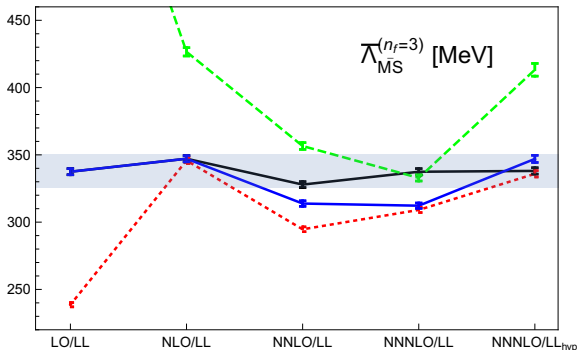
In the u plane

$$u = \frac{\beta_0}{4\pi} t$$

$$R_{\text{PV}} = \text{PV} \int_0^\infty du e^{-\frac{4\pi u}{\beta_0 \alpha}} B(u) = \text{PV} \int_0^\infty dt e^{-\frac{t}{\alpha}} \sum_{n=0}^{\infty} \frac{r_n}{n!} t^n$$

And the renormalons are located at $d = \pm 1, \pm 2, \pm 3, \pm 4, \dots$

$$t = \frac{2\pi d}{\beta_0} \quad u = d/2$$

Fixed order fits $\nu_{\text{us}} = \nu_{\text{s}}$ 

- Central value fits with log resummation
- $\nu_{\text{s}} = \nu_{\text{us}} = 1/r$
- $\nu_{\text{s}} = \nu_{\text{us}} = 2/r$
- $\nu_{\text{s}} = \nu_{\text{us}} = 1/(\sqrt{2}r)$