$\overline{\mathrm{MS}}$ Renormalization of the S-wave Quarkonium Wavefunctions at the Origin

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March 18, 2021

Based on **JHEP 12 (2020) 065** [arXiv:2007.01737]

QWG 2021 - The 14th International Workshop on Heavy Quarkonium

Bad convergence in NRQCD short-distance coefficients

- Decay constants and decay rates been computed to two loops:

$$f_{J/\psi} = \sqrt{2/m_{J/\psi}} c_v \langle 0|\chi^\dagger \boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}\psi|J/\psi\rangle,\ \Gamma(J/\psi \to \ell^+\ell^-) = \frac{4\pi}{3m_{J/\psi}}\alpha^2 e_c^2 f_{J/\psi}^2,$$
$$f_{\eta_c} = \sqrt{2/m_{\eta_c}} c_p \langle 0|\chi^\dagger\psi|\eta_c\rangle,\ \Gamma(\eta_c \to \gamma\gamma) = \frac{8\pi}{m_{\eta_c}^2}\alpha^2 e_c^4 |c_{\gamma\gamma}\langle 0|\chi^\dagger\psi|\eta_c\rangle|^2.$$

- Short-distance coefficients c depend on the scheme in which the NRQCD matrix elements are renormalized. In the $\overline{\rm MS}$ scheme:

$$c_v = 1 - \frac{8\alpha_s}{3\pi} + \left[-44.6 + 25.6\log\left(\frac{m_c}{\Lambda}\right) + 0.4n_f\right]\left(\frac{\alpha_s}{\pi}\right)^2 + O(\alpha_s^3),$$

Barbieri, Gatto, Kogerler, Kunszt, PLB57 (1975) 455
Celmaster, PRD19 (1979) 1517

Czarnecki and Melnikov, PRL80 (1998) 2531
Beneke, Signer, Smirnov, PRL80 (1998) 2535

$$c_p = 1 - \frac{2\alpha_s}{\pi} + \left[-50.6 + 37.3\log\left(\frac{m_c}{\Lambda}\right) + 0.1n_f\right]\left(\frac{\alpha_s}{\pi}\right)^2 + O(\alpha_s^3),$$

Braaten and Fleming, PRD 52 (1995) 181

Kniehl, Onishchenko, Piclum, Steinhauser, PLB638 (2006) 209

$$c_{\gamma\gamma} = 1 - \frac{(20-\pi^2)\alpha_s}{6\pi} + \left[-55.2 + 37.3\log\left(\frac{m_c}{\Lambda}\right) - 0.4n_f\right]\left(\frac{\alpha_s}{\pi}\right)^2 + O(\alpha_s^3).$$

Harris and Brown, PR105 (1957) 1656
Barbieri, d'Emilio, Curci, Remiddi, NPB 154 (1979) 535
Hagiwara, Kim, Yoshino, NPB 177 (1981) 461

Czarnecki and Melnikov, PLB519 (2001) 212
Feng, Jia, Sang, PRL115 (2015) 222001

- **Strong dependences on the NRQCD factorization scale Λ**
- **Large negative finite pieces at $\Lambda = m_c$ from loop corrections**

Scale dependence in NRQCD matrix elements

- To make accurate predictions at two-loop level, the *NRQCD matrix elements must also be computed to sufficient accuracy*, so that they have the **correct factorization scale dependences** which cancel with the Λ dependences in short-distance coefficients.
- The Λ dependences come from the **UV divergences** of the NRQCD matrix elements, which must be **renormalized in the same scheme** as the short-distance coefficients.
- Two-loop calculations of wavefunctions at the origin have been done within perturbative QCD, but they neglect nonperturbative long-distance effects, which are important for many quarkonium states.
- Potential models can include nonperturbative effects, but do not correctly reproduce UV divergences. In lattice QCD, conversion from lattice regularization to $\overline{\mathrm{MS}}$ is not available at two-loop level.
- Aim of this work is to compute S-wave quarkonium matrix elements in terms of wavefunctions at the origin that have the **correct scale dependences** and **include nonperturbative long-distance effects.**

NRQCD matrix elements in potential NRQCD

- We work in the **strongly coupled potential NRQCD** formalism. NRQCD matrix elements are given by wavefunctions at the origin as $|\langle 0|\chi^\dagger \boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}\psi|J/\psi\rangle|^2 = 2N_c|\psi_{J/\psi}(0)|^2$, $|\langle 0|\chi^\dagger\psi|\eta_c\rangle|^2 = 2N_c|\psi_{\eta_c}(0)|^2$ at leading order in $v^2, \Lambda_{\rm QCD}/m$.
  Pineda and Soto, NPB Proc. Suppl. 64 (1998) 428
  Brambilla, Pineda, Soto, Vairo, NPB566 (2000) 275, PRD 63 (2001) 014023, Rev. Mod. Phys. 77 (2005) 1423

  Later in the numerical results we include nonperturbative corrections of relative order v^2 and $\Lambda_{\rm QCD}/m$.
- $\psi(\mathbf{r})$ is a normalized bound-state solution of the Schrödinger equation

$$\Big[-\nabla^2/m + V(\mathbf{r},\nabla)\Big]\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$

  m : heavy quark pole mass
- $V(\mathbf{r},\nabla)$ is obtained by matching nonperturbatively to NRQCD order by order in the $1/m$ expansion $V = V^{(0)} + V^{(1)}/m + V^{(2)}/m^2 + \cdots$. $V^{(0)}$ is the static potential. $V^{(0)}$, $V^{(1)}$, $V^{(2)}$, ... have nonperturbative definitions that allow lattice QCD determinations as functions of r.
- **UV divergence of $\psi(0)$** can be determined from the *short-distance behavior of $\psi(r)$ at small r*.

UV divergences in the wavefunctions at the origin

- Small r behavior of $\psi(r)$ can be determined from the small r behavior of the potential, using methods from undergraduate level quantum mechanics (e.g. series solutions).
- Leading power in $1/m$: $V^{(0)} = -\alpha_s C_F/r + O(\alpha_s^2)$ at short distances. If the potential diverges like $1/r$ at small r, $\psi(r)$ is finite at $r = 0$: $\psi^{(0)}(r) = \psi^{(0)}(0) \times [1 + O(r)]$.
- Higher order potentials can diverge faster than $1/r$, which produces divergences in $\psi(r)$ at $r = 0$.
  - Order $1/m$: $V^{(1)} = -\alpha_s^2 C_F C_A/(2\boldsymbol{r^2}) + O(\alpha_s^3)$ at small r. Correction from $1/r^2$ potential is *logarithmically divergent:* $\delta\psi(r) = \psi^{(0)}(0) \times \left[-\frac{\alpha_s^2 C_F C_A}{2} \boldsymbol{\log r} + \text{finite}\right]$.
  - Order $1/m^2$: delta function potential $\delta^{(3)}(\mathbf{r})$ appears in $V^{(2)}$. Correction from $\delta^{(3)}(\mathbf{r})$ is power & logarithmically divergent: $\delta\psi(r) = \psi^{(0)}(0) \times [-\frac{1}{4\pi m}\frac{1}{\boldsymbol{r}} + \frac{\alpha_s C_F}{4\pi} \boldsymbol{\log r} + \text{finite}]$.

These effects are usually not included in potential models.

Strategy to compute $\psi(0)$ in $\overline{\mathrm{MS}}$

- Long-distance behaviors of potentials are given as functions of r, so $\psi(r)$ is most conveniently computed in *position space.*
- To renormalize $\psi(0)$ in the $\overline{\mathrm{MS}}$ scheme, corrections must be computed in dimensional regularization (DR) in $d = 4 - 2\epsilon$ spacetime dimensions, which is done in *momentum space.*
- **Solution :**
  1. Compute $\psi^{(0)}(r)$ using the static potential.
  2. Compute corrections to wavefunction $\delta\psi(r)$ in position space, regulate UV divergence using a position-space cutoff (finite-r regularization).
  3. **Compute scheme conversion from finite-r regularization to DR.**
  4. Renormalize in $\overline{\mathrm{MS}}$ scheme by subtracting $1/\epsilon$ poles.
- For **scheme conversion**, **position-space divergences** in $\psi(r)$ must be identified with the **DR counterpart** in **momentum space**.
- Similar method has been used in computing $\gamma^* \to Q\bar{Q}$ near threshold in perturbative QCD at two loops. Hoang and Teubner, PRD58 (1998) 114023; Hoang et al, EPJ direct 2 (2000) 3
- Spin-dependent part of the scheme conversion is known, which is reproduced in this work. Kiyo, Pineda, Signer, NPB 841 (2010) 231

Position-space calculation

- First compute leading order wavefunctions with the static potential: $\big[-\nabla^2/m + V^{(0)}(r)\big]\psi_n^{(0)}(r) = E_n^{(0)}\psi_n^{(0)}(r)$. $\psi_n^{(0)}(0)$ is UV finite.
- Calculate corrections from $\delta V = V^{(1)}/m + V^{(2)}/m^2 + \cdots$ using Rayleigh-Schrödinger perturbation theory. To first order:

$$\delta\psi_n(r) = -\int d^3r' \hat{G}_n(r,r')\delta V(r')\psi_n^{(0)}(r'),$$

$$\hat{G}_n(r,r') = \lim_{E\to E_n^{(0)}}\left[G(r,r';E) - \frac{\psi_n^{(0)}(r)\psi_n^{*(0)}(r')}{E_n^{(0)}-E}\right]$$

$$= \lim_{\eta\to 0}\frac{1}{2}\left[G(r,r';E_n^{(0)}+\eta) + G(r,r';E_n^{(0)}-\eta)\right]$$: reduced Green's function.

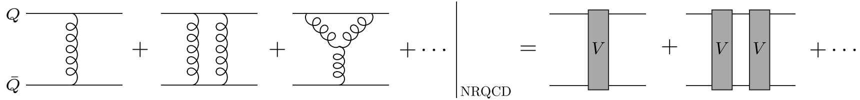
- UV divergence in $\delta\psi_n(0)$ can be regulated in position space by keeping r small but nonzero (finite-r regularization).
- Short-distance divergences occur from behavior of the integrand near $r'=0$. In the S-wave case, it is sufficient to replace $\psi_n^{(0)}(r') \to \psi_n^{(0)}(0) + O(r')$ in the integrand to isolate the UV divergence: $\delta\psi_n(r)\big|_{\rm UV} = -\psi_n^{(0)}(0) \times \int d^3r' \hat{G}_n(r,r')\delta V(r')$.

Position space to momentum space

- We want to identify UV divergences in $\int d^3r' \hat{G}_n(r,r')\delta V(r')$ in momentum space. For this purpose it suffices to replace $\hat{G}_n(r,r')$ with the full Green's function $G(r,r';E)$.
- To go to momentum space, we take the Fourier transform :
$$\int d^3r' G(r,r';E)\delta V(r') = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} e^{i\mathbf{p}'\cdot\mathbf{r}} \tilde{G}(\mathbf{p}',\mathbf{p})\delta\tilde{V}(\mathbf{p}).$$
- To isolate UV divergence, examine large $\mathbf{p}$ and $\mathbf{p}'$ behavior of integrand. This can be done from the iterative solution
$$\tilde{G}(\mathbf{p}',\mathbf{p}) = -\frac{(2\pi)^{d-1}\delta^{(d-1)}(\mathbf{p}'-\mathbf{p})}{E-\mathbf{p}^2/m} - \frac{\tilde{V}^{(0)}(\mathbf{p}'-\mathbf{p})}{(E-\mathbf{p}'^2/m)(E-\mathbf{p}^2/m)} + \cdots .$$
Higher order terms are increasingly less divergent. At two-loop level, UV divergence is contained in
$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}} \frac{\delta\tilde{V}(\mathbf{p})}{E-\mathbf{p}^2/m} + \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} e^{i\mathbf{p}'\cdot\mathbf{r}} \frac{\tilde{V}^{(0)}(\mathbf{p}'-\mathbf{p})\delta\tilde{V}(\mathbf{p})}{(E-\mathbf{p}'^2/m)(E-\mathbf{p}^2/m)}.$$
- We need to obtain the equivalent expression in $d-1$ spatial dimensions to set $\mathbf{r}=\mathbf{0}$ and regulate in DR.

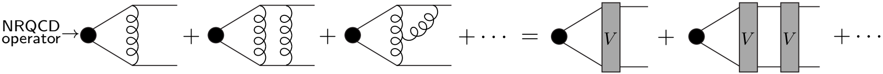
Momentum-space calculation in DR

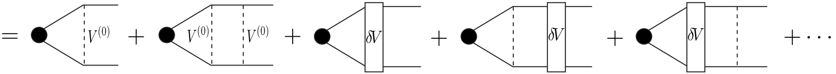
- Computation of momentum-space potentials in DR:



This provides $\tilde{V} = \tilde{V}^{(0)} + \delta\tilde{V}$ in $d = 4 - 2\epsilon$ spacetime dimensions.

- NRQCD matrix elements to two-loop accuracy in DR:





$$= \text{finite} + \int \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \frac{\delta\tilde{V}(\mathbf{p})}{E - \mathbf{p}^2/m} + \int \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \int \frac{d^{d-1}\mathbf{p}'}{(2\pi)^{d-1}} \frac{\tilde{V}^{(0)}(\mathbf{p}'-\mathbf{p})\delta\tilde{V}(\mathbf{p})}{(E - \mathbf{p}'^2/m)(E - \mathbf{p}^2/m)}$$

The UV divergent integrals are the DR counterpart of

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}} \frac{\delta\tilde{V}(\mathbf{p})}{E - \mathbf{p}^2/m} + \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} e^{i\mathbf{p}'\cdot\mathbf{r}} \frac{\tilde{V}^{(0)}(\mathbf{p}'-\mathbf{p})\delta\tilde{V}(\mathbf{p})}{(E - \mathbf{p}'^2/m)(E - \mathbf{p}^2/m)} \text{ at } \mathbf{r} = \mathbf{0}.$$

Numerical computation of $\psi(0)$

1. Compute finite-r regularized $\psi(0)$ numerically in position space:
   $\psi_n(0)\big|_{\text{finite-}r} = \psi_n^{(0)}(0) - \int d^3r' \hat{G}_n(r,r')\delta V(r')\psi_n^{(0)}(r')$.
2. Obtain $\boldsymbol{\psi_n(0)}\big|_{\overline{\mathbf{MS}}} = \psi_n(0)\big|_{\text{finite-}r} - \boldsymbol{\delta Z}\psi_n^{(0)}(0)$ by scheme conversion.
   $\boldsymbol{\delta Z}$ is the scheme conversion coefficient:

$$\boldsymbol{\delta Z} = \left[\int \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \frac{\delta\tilde{V}(\mathbf{p})}{E-\mathbf{p}^2/m} + \int \frac{d^{d-1}\mathbf{p}}{(2\pi)^{d-1}} \int \frac{d^{d-1}\mathbf{p}'}{(2\pi)^{d-1}} \frac{\tilde{V}^{(0)}(\mathbf{p}'-\mathbf{p})\delta\tilde{V}(\mathbf{p})}{(E-\mathbf{p}'^2/m)(E-\mathbf{p}^2/m)}\right]_{\overline{\text{MS}}}$$
$$-\left[\int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}} \frac{\delta\tilde{V}(\mathbf{p})}{E-\mathbf{p}^2/m} + \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} e^{i\mathbf{p}'\cdot\mathbf{r}} \frac{\tilde{V}^{(0)}(\mathbf{p}'-\mathbf{p})\delta\tilde{V}(\mathbf{p})}{(E-\mathbf{p}'^2/m)(E-\mathbf{p}^2/m)}\right]$$
$$= -\frac{\alpha_s C_F}{3m}\frac{\mathbf{1}}{\boldsymbol{r}}\mathbf{S}^2 + \alpha_s^2\left\{C_F^2\left[-\boldsymbol{L_\Lambda} + \frac{\mathbf{S^2}}{3}\left(\boldsymbol{L_\Lambda} + \frac{1}{6}\right)\right] - \frac{C_F C_A}{2}\left(\boldsymbol{L_\Lambda} - \frac{3}{4}\right)\right\},$$

$\boldsymbol{L_\Lambda} = \log(\boldsymbol{\Lambda r}) + \gamma_{\text{E}}$, $\mathbf{S}^2 = 2$ for spin triplet, $\mathbf{S}^2 = 0$ for spin singlet.

- $\boldsymbol{\delta Z}$ is **universal for all S-wave states** and is *unaffected by nonperturbative long-distance effects at two-loop level.*
- We obtain the **correct two-loop factorization scale dependence:**
  $\frac{d\log}{d\log\Lambda}\psi(0)\big|_{\overline{\text{MS}},\,\mathbf{S}^2=2} = \frac{d\log}{d\log\Lambda}\langle 0|\chi^\dagger\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}\psi|J/\psi\rangle = \alpha_s^2 C_F\left(\frac{C_F}{3}+\frac{C_A}{2}\right)$,
  $\frac{d\log}{d\log\Lambda}\psi(0)\big|_{\overline{\text{MS}},\,\mathbf{S}^2=0} = \frac{d\log}{d\log\Lambda}\langle 0|\chi^\dagger\psi|\eta_c\rangle = \alpha_s^2 C_F\left(C_F+\frac{C_A}{2}\right)$.

Numerical results

$\psi(0)|_{\overline{\rm MS}}$ for $1S$ and $2S$ charmonia, and $1S$, $2S$, and $3S$ bottomonia

- Included long-distance static and $1/m$ potentials from lattice QCD.
- Used modified renormalon subtracted (RS') charm and bottom masses.

Pineda, JHEP 06 (2001) 022
Peset, Pineda, Segovia, JHEP 09 (2018) 167

LO wavefunction $\psi^{(0)}(0)$ and relative correction $\delta\psi/\psi^{(0)}$ at $\Lambda = m$

State	$\psi^{(0)}(0)$ (GeV$^{3/2}$)	$\delta\psi(\mathbf{S}^2=2)/\psi^{(0)}$	$\delta\psi(\mathbf{S}^2=0)/\psi^{(0)}$
$1S$ charmonium	0.183	0.738	0.807
$2S$ charmonium	0.177	0.641	0.664
$1S$ bottomonium	0.496	0.554	0.644
$2S$ bottomonium	0.423	0.423	0.488
$3S$ bottomonium	0.400	0.399	0.461

Corrections to $\psi(0)$ are **large and positive**, while loop corrections to short-distance coefficients are **large and negative**

$\longrightarrow$ *cancellations in decay rates/constants!*

J/ψ and Υ decay constants

Vector decay constant $f_V = \frac{1}{m_V}\langle 0|\bar{Q}\boldsymbol{\epsilon}_V \cdot \boldsymbol{\gamma} Q|V\rangle$ of vector quarkonium V.

State	This work (GeV)	Lattice QCD (GeV)
J/ψ	$0.363^{+0.089}_{-0.088}$	0.4104 ± 0.0017 HPQCD, PRD102 (2020) 054511
$\psi(2S)$	$0.309^{+0.076}_{-0.076}$	
$\Upsilon(1S)$	$0.621^{+0.084}_{-0.070}$	0.6772 ± 0.0097 HPQCD, arXiv:2101.08103
$\Upsilon(2S)$	$0.447^{+0.028}_{-0.027}$	0.481 ± 0.039 HPQCD, PRD91 (2015) 074514
$\Upsilon(3S)$	$0.395^{+0.025}_{-0.024}$	

- Included order v^2 and $\alpha_s v^2$ corrections.
- Uncertainties from QCD scale, higher order nonperturbative corrections, numerical uncertainties.

- *Complete cancellation of NRQCD factorization scale dependence* to two-loop accuracy, *no dependence on Λ in f_V*. Results agree with lattice QCD calculations.
- Large cancellation between corrections to wavefunctions and two-loop corrections to short-distance coefficients, convergence is significantly improved: $f_{J/\psi} = \sqrt{\frac{4N_c}{m_{J/\psi}}}|\psi^{(0)}(0)|\big(\underset{\text{LO}}{1} - \underset{O(\alpha_s)}{0.06} - \underset{O(v^2)}{0.07} + \underset{O(\alpha_s^2,\,\delta\psi/\psi^{(0)})}{0.14}\big)$

J/ψ and Υ leptonic decay

Leptonic decay rate $\Gamma(V \to \ell^+\ell^-)$ of vector quarkonium V.

State	This work (keV)	PDG (keV)
J/ψ	$4.5^{+2.5}_{-1.9}$	5.53 ± 0.10
$\psi(2S)$	$2.7^{+1.5}_{-1.2}$	2.33 ± 0.04
$\Upsilon(1S)$	$1.11^{+0.32}_{-0.24}$	1.340 ± 0.018
$\Upsilon(2S)$	$0.54^{+0.07}_{-0.06}$	0.612 ± 0.011
$\Upsilon(3S)$	$0.41^{+0.05}_{-0.05}$	0.443 ± 0.008

- Included order v^2 and $\alpha_s v^2$ corrections.
- Uncertainties from QCD scale, higher order nonperturbative corrections, numerical uncertainties.

- *Complete cancellation of NRQCD factorization scale dependence* to two-loop accuracy, *no dependence on Λ in decay rates*. Results agree with measurements.
- Large cancellation between corrections to wavefunctions and two-loop corrections to short-distance coefficients, convergence is significantly improved: $\Gamma_{\Upsilon(1S)\to\ell^+\ell^-} = \frac{16N_c\pi\alpha^2 e_b^2}{3m_\Upsilon^2}|\psi^{(0)}(0)|^2\big(\underbrace{1}_{\text{LO}} + \underbrace{0.06}_{O(\alpha_s)} + \underbrace{0.06}_{O(\alpha_s^2,\,\delta\psi/\psi^{(0)})}\big)^2$

η_c and η_b decay constants

Decay constant $f_P = \frac{1}{m_P}\langle 0|\bar{Q}\gamma_0\gamma_5 Q|P\rangle$ of pseudoscalar quarkonium P.

State P	This work (GeV)	Lattice QCD (GeV)
η_c	$0.385^{+0.094}_{-0.093}$	0.3981 ± 0.0010 HPQCD, PRD102 (2020) 054511
$\eta_c(2S)$	$0.271^{+0.068}_{-0.069}$	
$\eta_b(1S)$	$0.691^{+0.141}_{-0.080}$	0.724 ± 0.012 HPQCD, arXiv:2101.08103
$\eta_b(2S)$	$0.471^{+0.030}_{-0.029}$	
$\eta_b(3S)$	$0.403^{+0.026}_{-0.025}$	

- Included order v^2 corrections.
- Uncertainties from QCD scale, higher order nonperturbative corrections, numerical uncertainties.

- *Complete cancellation of NRQCD factorization scale dependence* to two-loop accuracy, *no dependence on Λ in f_P*. η_c and η_b results agree well with lattice QCD calculations.
- Large cancellation between corrections to wavefunctions and two-loop corrections to short-distance coefficients, convergence is significantly improved: $f_{\eta_c} = \sqrt{\frac{4N_c}{m_{\eta_c}}}|\psi^{(0)}(0)|\big(\underset{\text{LO}}{1} - \underset{O(\alpha_s)}{0.01} - \underset{O(v^2)}{0.13} + \underset{O(\alpha_s^2,\,\delta\psi/\psi^{(0)})}{0.17}\big)$

η_c and η_b two-photon decay

Two-photon decay rate $\Gamma(P \to \gamma\gamma)$ of pseudoscalar quarkonium P.

State P	This work (keV)	PDG (keV)
η_c	$6.8^{+3.0}_{-2.5}$	5.06 ± 0.34
$\eta_c(2S)$	$3.0^{+1.4}_{-1.3}$	
$\eta_b(1S)$	$0.433^{+0.165}_{-0.065}$	
$\eta_b(2S)$	$0.194^{+0.022}_{-0.021}$	
$\eta_b(3S)$	$0.141^{+0.016}_{-0.015}$	

- Included order v^2 and $\alpha_s v^2$ corrections.
- Uncertainties from QCD scale, higher order nonperturbative corrections, numerical uncertainties.

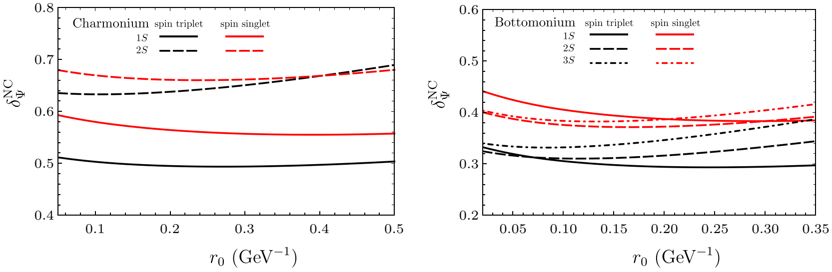
- *Complete cancellation of NRQCD factorization scale dependence* to two-loop accuracy, *no dependence on Λ in decay rates.* η_c result compatible with measurement.
- Large cancellation between corrections to wavefunction and two-loop corrections to short-distance coefficients, convergence is significantly improved: $\Gamma_{\eta_c \to \gamma\gamma} = \frac{16 N_c \pi \alpha^2 e_c^4}{m_\Upsilon^2} |\psi^{(0)}(0)|^2 \big(\underset{\text{LO}}{1} + \underset{O(\alpha_s)}{0.03} - \underset{O(v^2)}{0.10} + \underset{O(\alpha_s^2, \delta\psi/\psi^{(0)})}{0.14}\big)^2$

Summary and outlook

- In this work, we computed S-wave quarkonium wavefunctions at the origin from nonperturbative potential based on **first principles.**
- Wavefunctions at the origin are **renormalized in the $\overline{\rm MS}$ scheme,** *consistently with $\overline{\rm MS}$ calculations of short-distance coefficients*. *This has not been possible in existing nonperturbative methods.*
- **Exact cancellation** of **$\overline{\rm MS}$ scale dependence** occurs in decay constants and rates at two-loop level.
- Convergence of perturbative corrections is **significantly improved** by inclusion of corrections to wavefunctions.
- This calculation allows accurate and **model-independent predictions** of decay rates and decay constants based on **first principles**.
- This result can be extended to *higher orbital angular momentum* states (χ_c, χ_b, h_c, h_b), and to *higher orders in α_s (three loops and beyond).*

Numerical uncertainties in scheme conversion

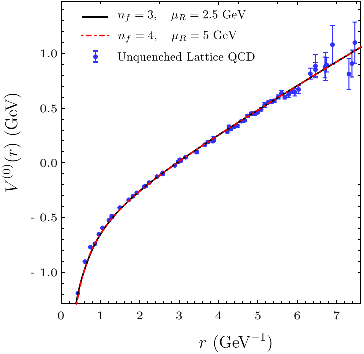
- To compute $\psi_n(0)|_{\overline{\rm MS}} = \psi_n(0)|_{\text{finite-}r} - \delta Z \psi_n^{(0)}(0)$, the strong r dependence in $\psi_n(0)|_{\text{finite-}r}$ must be cancelled by the $1/r$ and $\log r$ terms in δZ.
- This shows the mild residual r dependence in the non-Coulombic corrections to the wavefunctions at the origin.



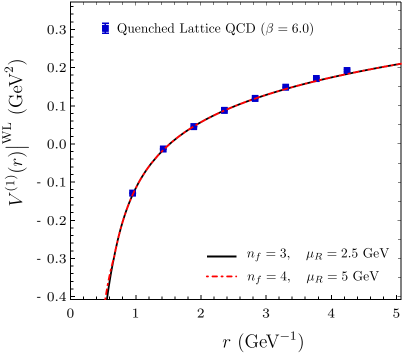
- The mild residual r dependences are included in uncertainties.

Static potential and $1/m$ potential at long distances

- Static potential and $1/m$ potential at long distances from lattice QCD



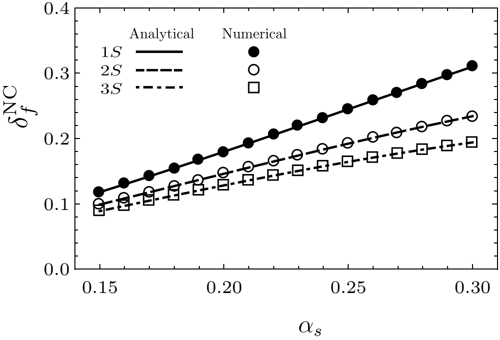
TXL, PRD62 (2000) 054503



Koma and Koma, PoS(LATTICE2012)140

Comparison with perturbative QCD results

- The numerical calculation in this work also reproduces perturbative QCD results for $\gamma^* \to Q\bar{Q}$ when nonperturbative effects are neglected.
- Numerical calculation of non-Coulombic corrections to vector decay constant f at various values of α_s compared to known analytical results:



Perturbative QCD results available from
Hoang and Teubner, PRD58 (1998) 114023
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Penin and Pivovarov, NPB549 (1999) 217
Yakovlev, PLB457 (1999) 170
Beneke, Signer, Smirnov, PLB454 (1999) 137
Nagano, Ota, Sumino, PRD60 (1999) 114014
Hoang and Teubner, PRD60 (1999) 114027
Penin and Pivovarov, Phys. Atom. Nucl. 64 (2001) 275
Hoang et al, EPJ direct 2 (2000) 3
Penin, Pineda, Smirnov, Steinhauser, NPB699 (2004) 183