The Research Council of Norway

University of Stavanger

Towards dissipative in-medium $Q\bar{Q}$ dynamics from the complex interquark potential

Alexander Rothkopf
Faculty of Science and Technology
Department of Mathematics and Physics
University of Stavanger

in collaboration with Y. Akamatsu, M. Asakawa, T. Miura (**Osaka**) &
J. Nordström, O. Ålund, F. Lauren (**Linköping**)

References:

O. Ålund, Y. Akamatsu, F. Laurén, T. Miura, J. Nordström, A.R. JCP 425 (2021) 109917
A.R. "Heavy Quarkonium in Extreme Conditions" Phys.Rept. 858 (2020)
T. Miura, Y. Akamatsu, M. Asakawa, A.R. PRD 101 (2020) 034011

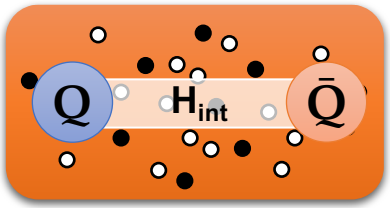
NPACT
Norwegian Particle, Astroparticle & Cosmology Theory network

The open quantum systems picture

University of Stavanger

- Need a general approach to describe quarkonium coupled to a thermal medium
  - Overall system is closed, hermitean Hamiltonian: von Neumann equation

$$H = H_{Q\bar{Q}} \otimes I_{med} + I_{Q\bar{Q}} \otimes H_{med} + H_{int} \qquad \frac{d\rho}{dt} = -i[H, \rho]$$

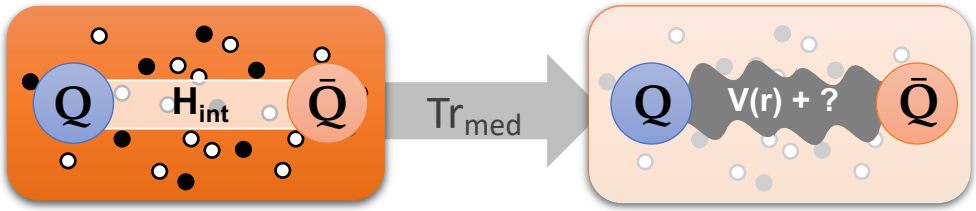


The open quantum systems picture

University of Stavanger

- Need a general approach to describe quarkonium coupled to a thermal medium
  - Overall system is closed, hermitean Hamiltonian: von Neumann equation

$$H = H_{Q\bar{Q}} \otimes I_{med} + I_{Q\bar{Q}} \otimes H_{med} + H_{int} \qquad \frac{d\rho}{dt} = -i[H, \rho]$$



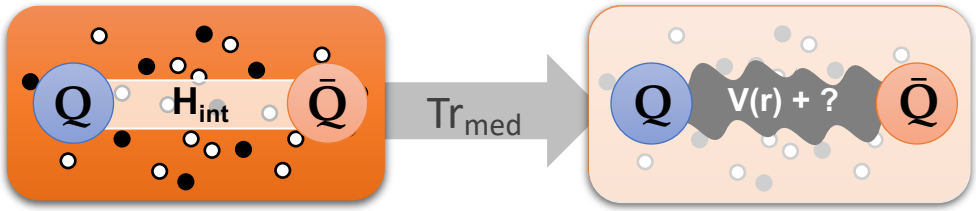
  - Dynamics of the reduced QQ system: $\rho_{Q\bar{Q}} = \mathrm{Tr}_{med}[\rho]$

The open quantum systems picture

University of Stavanger

- Need a general approach to describe quarkonium coupled to a thermal medium
  - Overall system is closed, hermitean Hamiltonian: von Neumann equation

$$H = H_{Q\bar{Q}} \otimes I_{med} + I_{Q\bar{Q}} \otimes H_{med} + H_{int} \qquad \frac{d\rho}{dt} = -i[H, \rho]$$



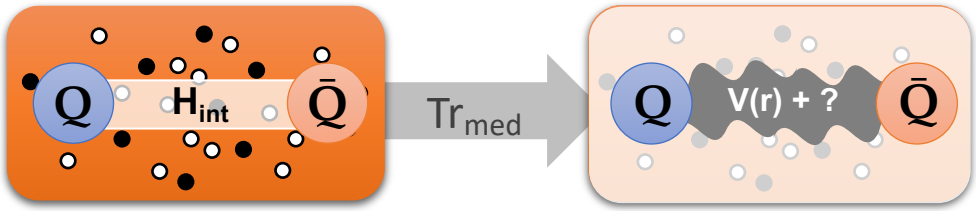
  - Dynamics of the reduced QQ system: $\rho_{Q\bar{Q}} = \mathrm{Tr}_{med}[\rho]$ $\qquad \frac{d}{dt}\rho_{Q\bar{Q}} = \mathcal{V}\rho_{Q\bar{Q}}$

The open quantum systems picture

University of Stavanger

- Need a general approach to describe quarkonium coupled to a thermal medium
  - Overall system is closed, hermitean Hamiltonian: von Neumann equation

$$H = H_{Q\bar{Q}} \otimes I_{med} + I_{Q\bar{Q}} \otimes H_{med} + H_{int} \qquad \frac{d\rho}{dt} = -i[H, \rho]$$



  - Dynamics of the reduced QQ system: $\rho_{Q\bar{Q}} = \mathrm{Tr}_{med}[\rho]$ $\quad \frac{d}{dt}\rho_{Q\bar{Q}} = \mathcal{V}\rho_{Q\bar{Q}}$

- **Separation of time-scales** determines the nature of the e.o.m. :

Environment relaxation scale τ_E :

$$\langle \Xi_m(t)\Xi_m(0)\rangle \sim e^{-t/\tau_E}$$

$Q\bar{Q}$ system scale τ_S :

$$\tau_S \sim 1/|\omega - \omega'|$$

$Q\bar{Q}$ relaxation scale τ_{rel} :

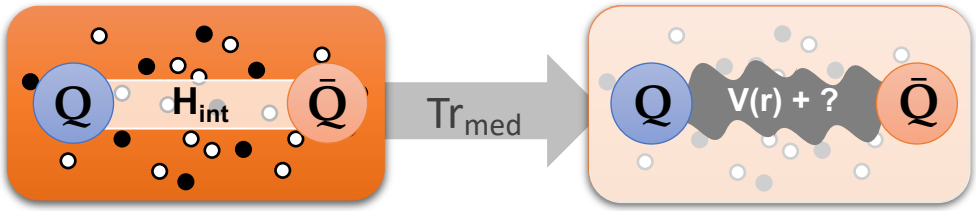
$$\langle p(t)\rangle \propto e^{-t/\tau_{\mathrm{rel}}}$$

The open quantum systems picture

University of Stavanger

- Need a general approach to describe quarkonium coupled to a thermal medium
  - Overall system is closed, hermitean Hamiltonian: von Neumann equation

$$H = H_{Q\bar{Q}} \otimes I_{med} + I_{Q\bar{Q}} \otimes H_{med} + H_{int} \qquad \frac{d\rho}{dt} = -i[H, \rho]$$



  - Dynamics of the reduced QQ system: $\rho_{Q\bar{Q}} = \mathrm{Tr}_{med}[\rho]$ $\quad \frac{d}{dt}\rho_{Q\bar{Q}} = \mathcal{V}\rho_{Q\bar{Q}}$

- **Separation of time-scales** determines the nature of the e.o.m. :

Environment relaxation scale τ_E : $\langle \Xi_m(t)\Xi_m(0)\rangle \sim e^{-t/\tau_E}$

$Q\bar{Q}$ system scale τ_S : $\tau_S \sim 1/|\omega - \omega'|$

$Q\bar{Q}$ relaxation scale τ_{rel} : $\langle p(t)\rangle \propto e^{-t/\tau_{\mathrm{rel}}}$

- In case of Markovian time evolution ($\tau_E \ll \tau_{\mathrm{rel}}$) leads to a **Lindblad equation**

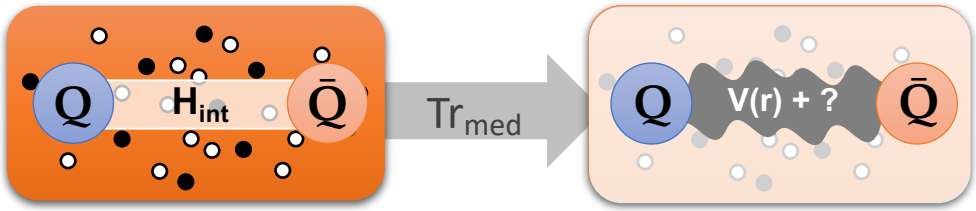
$$\frac{d}{dt}\rho_{Q\bar{Q}} = -i[\tilde{H}_{Q\bar{Q}}, \rho_{Q\bar{Q}}] + \sum_k \gamma_k \left(L_k \rho_{Q\bar{Q}} L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho_{Q\bar{Q}} - \frac{1}{2} \rho_{Q\bar{Q}} L_k^\dagger L_k \right)$$

The open quantum systems picture

University of Stavanger

- Need a general approach to describe quarkonium coupled to a thermal medium
  - Overall system is closed, hermitean Hamiltonian: von Neumann equation

$$H = H_{Q\bar{Q}} \otimes I_{med} + I_{Q\bar{Q}} \otimes H_{med} + H_{int} \qquad \frac{d\rho}{dt} = -i[H, \rho]$$



  - Dynamics of the reduced QQ system: $\rho_{Q\bar{Q}} = \mathrm{Tr}_{med}[\rho]$ $\quad \frac{d}{dt}\rho_{Q\bar{Q}} = \mathcal{V}\rho_{Q\bar{Q}}$

- **Separation of time-scales** determines the nature of the e.o.m. :

Environment relaxation scale τ_E : $\langle \Xi_m(t)\Xi_m(0)\rangle \sim e^{-t/\tau_E}$

$Q\bar{Q}$ system scale τ_S : $\tau_S \sim 1/|\omega - \omega'|$

$Q\bar{Q}$ relaxation scale τ_{rel} : $\langle p(t)\rangle \propto e^{-t/\tau_{\mathrm{rel}}}$

- In case of Markovian time evolution ($\tau_E \ll \tau_{\mathrm{rel}}$) leads to a **Lindblad equation**

$$\frac{d}{dt}\rho_{Q\bar{Q}} = -i[\tilde{H}_{Q\bar{Q}}, \rho_{Q\bar{Q}}] + \sum_k \gamma_k \left(L_k \rho_{Q\bar{Q}} L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} L_k^\dagger L_k \right)$$

$\langle n|\rho_{Q\bar{Q}}|n\rangle > 0, \forall n$

$\rho_{Q\bar{Q}}^\dagger = \rho_{Q\bar{Q}}, \quad \mathrm{Tr}[\rho_{Q\bar{Q}}] = 1$

From QCD towards models

University of Stavanger

QCD

From QCD towards models

University of Stavanger

QCD

Deterministic Schrödinger

$$i\partial_t\psi = \left[\frac{-\nabla^2}{2m_Q} + Re[V](R,T) - iIm[V](R,T)\right]\psi$$

From QCD towards models

University of Stavanger

QCD

non-relativistic quarks

NRQCD

Deterministic Schrödinger

$$i\partial_t \psi = \left[\frac{-\nabla^2}{2m_Q} + Re[V](R,T) - iIm[V](R,T)\right]\psi$$

From QCD towards models

University of Stavanger

QCD

non-relativistic quarks

NRQCD

weak coupling

Y. Akamatsu
PRD91 (2015) 056002

"quantum Brownian motion limit"

Lindblad equation

Deterministic Schrödinger

$$i\partial_t \psi = \left[\frac{-\nabla^2}{2m_Q} + Re[V](R,T) - iIm[V](R,T)\right]\psi$$

From QCD towards models

University of Stavanger

QCD

non-relativistic quarks

NRQCD

weak coupling
"quantum Brownian motion limit"

Y. Akamatsu
PRD91 (2015) 056002

Lindblad equation

color singlet projection
quantum state diffusion

T. Miura et.al.
PRD101 (2020) 034011

Non-linear stochastic Schrödinger

Deterministic Schrödinger

$$i\partial_t\psi = \left[\frac{-\nabla^2}{2m_Q} + Re[V](R,T) - iIm[V](R,T)\right]\psi$$

From QCD towards models

University of Stavanger

QCD

non-relativistic quarks

NRQCD

weak coupling
"quantum Brownian motion limit"

Y. Akamatsu
PRD91 (2015) 056002

Lindblad equation

color singlet projection
quantum state diffusion

T. Miura et.al.
PRD101 (2020) 034011

Non-linear stochastic Schrödinger

dissipationless limit

Y. Akamatsu, A.R.
PRD85 (2012) 105011

Stochastic Schrödinger

Deterministic Schrödinger

$$i\partial_t\psi = \left[\frac{-\nabla^2}{2m_Q} + Re[V](R,T) - iIm[V](R,T)\right]\psi$$

From QCD towards models

University of Stavanger

QCD

non-relativistic quarks

NRQCD

weak coupling
"quantum Brownian motion limit"

Y. Akamatsu
PRD91 (2015) 056002

Lindblad equation

color singlet projection
quantum state diffusion

T. Miura et.al.
PRD101 (2020) 034011

Non-linear stochastic Schrödinger

dissipationless limit

Y. Akamatsu, A.R.
PRD85 (2012) 105011

Stochastic Schrödinger

adiabatic approximation

Deterministic Schrödinger

$$i\partial_t\psi = \left[\frac{-\nabla^2}{2m_Q} + Re[V](R,T) - iIm[V](R,T)\right]\psi$$

From QCD towards models

University of Stavanger

QCD

non-relativistic quarks

NRQCD

weak coupling Y. Akamatsu PRD91 (2015) 056002

"quantum Brownian motion limit"

Lindblad equation

color singlet projection

quantum state diffusion T. Miura et.al. PRD101 (2020) 034011

Non-linear stochastic Schrödinger

dissipationless limit Y. Akamatsu, A.R. PRD85 (2012) 105011

Stochastic Schrödinger

adiabatic approximation

Deterministic Schrödinger

$$i\partial_t\psi = \left[\frac{-\nabla^2}{2m_Q} + Re[V](R,T) - iIm[V](R,T)\right]\psi$$

Quantum Brownian motion at high T

- Lindblad equation takes the standard form:

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

Quantum Brownian motion at high T

University of Stavanger

- Lindblad equation takes the standard form:

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- In-medium Hamiltonian exhibits a **screened and real-valued potential**

$$H_{Q\bar{Q}} = \frac{\mathbf{p}^2}{M_Q} + V_{Q\bar{Q}}(R)$$

$$V_{Q\bar{Q}}(R) = -\alpha_S \frac{e^{-m_D R}}{R} = Re[V_{\mathrm{EFT}}(R)]$$

Quantum Brownian motion at high T

University of Stavanger

- Lindblad equation takes the standard form:

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- In-medium Hamiltonian exhibits a **screened and real-valued potential**

$$H_{Q\bar{Q}} = \frac{\mathbf{p}^2}{M_Q} + V_{Q\bar{Q}}(R)$$

$$V_{Q\bar{Q}}(R) = -\alpha_S \frac{e^{-m_D R}}{R} = Re[V_{\mathrm{EFT}}(R)]$$

- Explicit form of the Lindblad operators:

$$L_{\mathbf{k},a} = \underbrace{\sqrt{\frac{D(\mathbf{k})}{2}}\left[1 - \frac{\mathbf{k}}{4m_Q T}\cdot\left(\frac{1}{2}\mathbf{P}_{\mathrm{CM}} + \mathbf{p}\right)\right] e^{i\mathbf{k}\cdot\mathbf{r}/2}\left(T^a \otimes 1\right)}_{\text{medium acting on the quark}}$$

$$\underbrace{-\sqrt{\frac{D(\mathbf{k})}{2}}\left[1 - \frac{\mathbf{k}}{4m_Q T}\cdot\left(\frac{1}{2}\mathbf{P}_{\mathrm{CM}} - \mathbf{p}\right)\right] e^{-i\mathbf{k}\cdot\mathbf{r}/2}\left(1 \otimes T^a\right)}_{\text{medium acting on the anti-quark}}$$

Quantum Brownian motion at high T

University of Stavanger

- Lindblad equation takes the standard form:

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- In-medium Hamiltonian exhibits a **screened and real-valued potential**

$$H_{Q\bar{Q}} = \frac{\mathbf{p}^2}{M_Q} + V_{Q\bar{Q}}(R) \qquad V_{Q\bar{Q}}(R) = -\alpha_S \frac{e^{-m_D R}}{R} = Re[V_{\mathrm{EFT}}(R)]$$

- Explicit form of the Lindblad operators:

fluctuations dissipation

$$L_{\mathbf{k},a} = \underbrace{\sqrt{\frac{D(\mathbf{k})}{2}}\left[1 - \frac{\mathbf{k}}{4m_Q T}\cdot\left(\frac{1}{2}\mathbf{P}_{\mathrm{CM}} + \mathbf{p}\right)\right] e^{i\mathbf{k}\cdot\mathbf{r}/2}\left(T^a \otimes 1\right)}_{\text{medium acting on the quark}}$$

fluctuations dissipation

$$-\underbrace{\sqrt{\frac{D(\mathbf{k})}{2}}\left[1 - \frac{\mathbf{k}}{4m_Q T}\cdot\left(\frac{1}{2}\mathbf{P}_{\mathrm{CM}} - \mathbf{p}\right)\right] e^{-i\mathbf{k}\cdot\mathbf{r}/2}\left(1 \otimes T^a\right)}_{\text{medium acting on the anti-quark}}$$

Quantum Brownian motion at high T

University of Stavanger

- Lindblad equation takes the standard form:

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- In-medium Hamiltonian exhibits a **screened and real-valued potential**

$$H_{Q\bar{Q}} = \frac{\mathbf{p}^2}{M_Q} + V_{Q\bar{Q}}(R)$$

$$V_{Q\bar{Q}}(R) = -\alpha_S \frac{e^{-m_D R}}{R} = Re[V_{\mathrm{EFT}}(R)]$$

- Explicit form of the Lindblad operators:

$$\tilde{D}(k) \sim Fourier(Im[V_{\mathrm{EFT}}(R)])$$

fluctuations dissipation

$$L_{\mathbf{k},a} = \underbrace{\sqrt{\frac{D(\mathbf{k})}{2}}\left[1 - \frac{\mathbf{k}}{4m_Q T}\cdot\left(\frac{1}{2}\mathbf{P}_{\mathrm{CM}} + \mathbf{p}\right)\right] e^{i\mathbf{k}\cdot\mathbf{r}/2}\left(T^a \otimes 1\right)}_{\text{medium acting on the quark}}$$

fluctuations dissipation

$$\underbrace{-\sqrt{\frac{D(\mathbf{k})}{2}}\left[1 - \frac{\mathbf{k}}{4m_Q T}\cdot\left(\frac{1}{2}\mathbf{P}_{\mathrm{CM}} - \mathbf{p}\right)\right] e^{-i\mathbf{k}\cdot\mathbf{r}/2}\left(1 \otimes T^a\right)}_{\text{medium acting on the anti-quark}}$$

Quantum Brownian motion at high T

University of Stavanger

- Lindblad equation takes the standard form:

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- In-medium Hamiltonian exhibits a **screened and real-valued potential**

$$H_{Q\bar{Q}} = \frac{\mathbf{p}^2}{M_Q} + V_{Q\bar{Q}}(R) \qquad V_{Q\bar{Q}}(R) = -\alpha_S \frac{e^{-m_D R}}{R} = Re[V_{\mathrm{EFT}}(R)]$$

- Explicit form of the Lindblad operators:

$$\tilde{D}(k) \sim Fourier(Im[V_{\mathrm{EFT}}(R)])$$

fluctuations dissipation

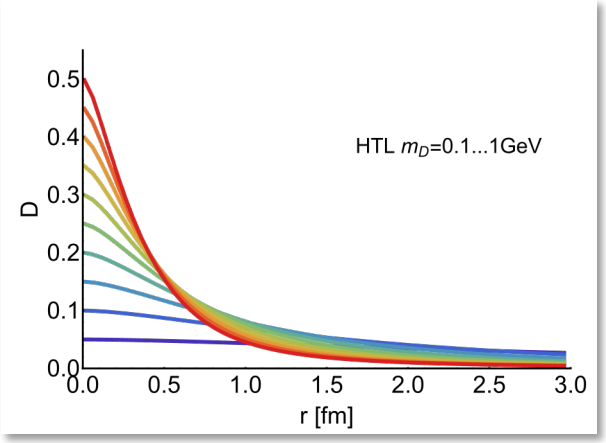
$$L_{\mathbf{k},a} = \sqrt{\frac{D(\mathbf{k})}{2}}\left[1 - \frac{\mathbf{k}}{4m_Q T}\cdot\left(\frac{1}{2}\mathbf{P}_{\mathrm{CM}} + \mathbf{p}\right)\right] e^{i\mathbf{k}\cdot\mathbf{r}/2}\left(T^a \otimes 1\right)$$

medium acting on the quark

fluctuations dissipation

$$-\sqrt{\frac{D(\mathbf{k})}{2}}\left[1 - \frac{\mathbf{k}}{4m_Q T}\cdot\left(\frac{1}{2}\mathbf{P}_{\mathrm{CM}} - \mathbf{p}\right)\right] e^{-i\mathbf{k}\cdot\mathbf{r}/2}\left(1 \otimes T^a\right)$$

medium acting on the anti-quark



Quantum Brownian motion at high T

University of Stavanger

- Lindblad equation takes the standard form:

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- In-medium Hamiltonian exhibits a **screened and real-valued potential**

$$H_{Q\bar{Q}} = \frac{\mathbf{p}^2}{M_Q} + V_{Q\bar{Q}}(R) \qquad V_{Q\bar{Q}}(R) = -\alpha_S \frac{e^{-m_D R}}{R} = Re[V_{\mathrm{EFT}}(R)]$$

- Explicit form of the Lindblad operators:

$$\tilde{D}(k) \sim Fourier(Im[V_{\mathrm{EFT}}(R)])$$

fluctuations dissipation

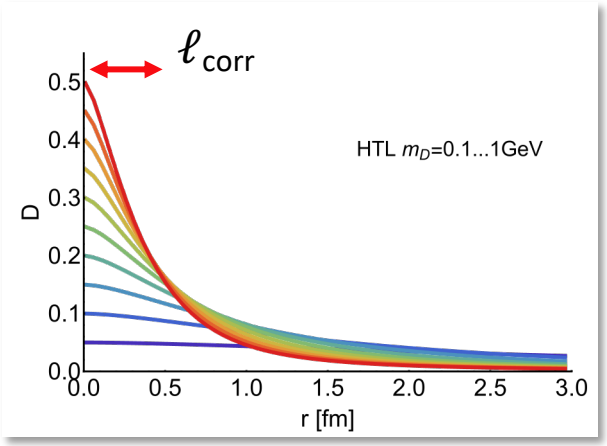
$$L_{\mathbf{k},a} = \sqrt{\frac{D(\mathbf{k})}{2}}\left[1 - \frac{\mathbf{k}}{4m_Q T}\cdot\left(\frac{1}{2}\mathbf{P}_{\mathrm{CM}} + \mathbf{p}\right)\right] e^{i\mathbf{k}\cdot\mathbf{r}/2}\left(T^a \otimes 1\right)$$

medium acting on the quark

fluctuations dissipation

$$-\sqrt{\frac{D(\mathbf{k})}{2}}\left[1 - \frac{\mathbf{k}}{4m_Q T}\cdot\left(\frac{1}{2}\mathbf{P}_{\mathrm{CM}} - \mathbf{p}\right)\right] e^{-i\mathbf{k}\cdot\mathbf{r}/2}\left(1 \otimes T^a\right)$$

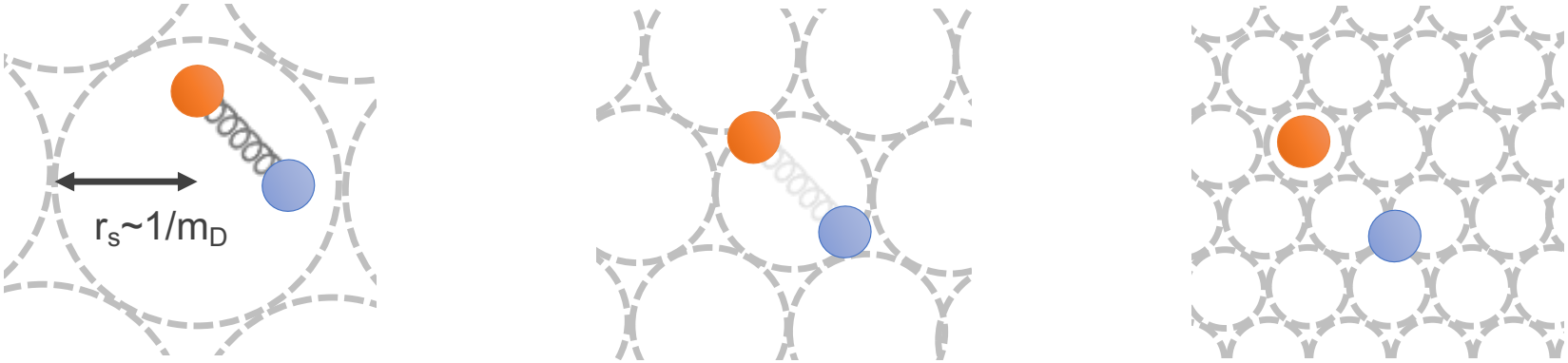
medium acting on the anti-quark



Screening and Decoherence

University of Stavanger

screening



binding force acts efficiently among Q and $\bar{Q}$

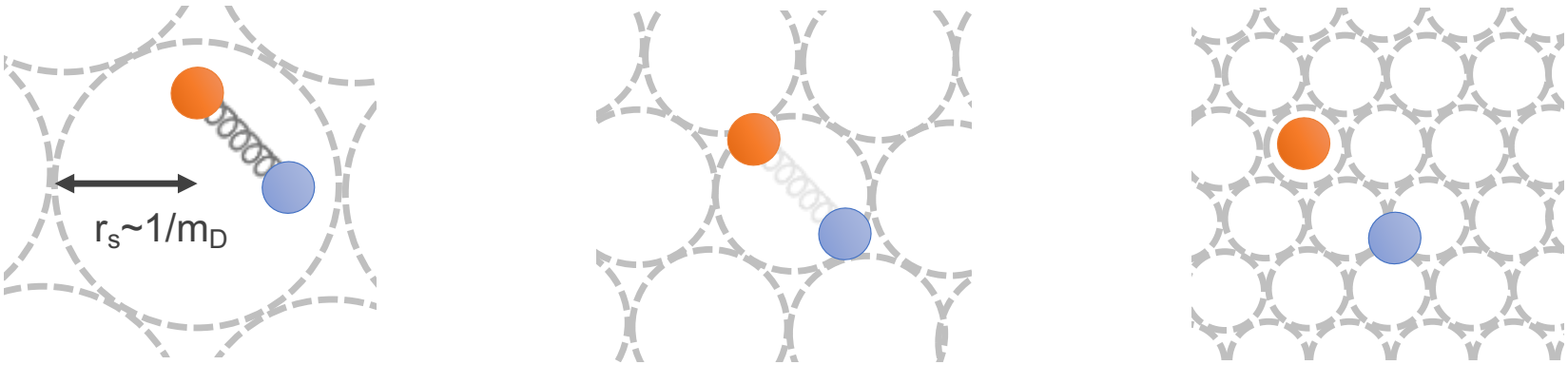
medium impedes gluons mediating force

increasing temperature

Screening and Decoherence

University of Stavanger

screening

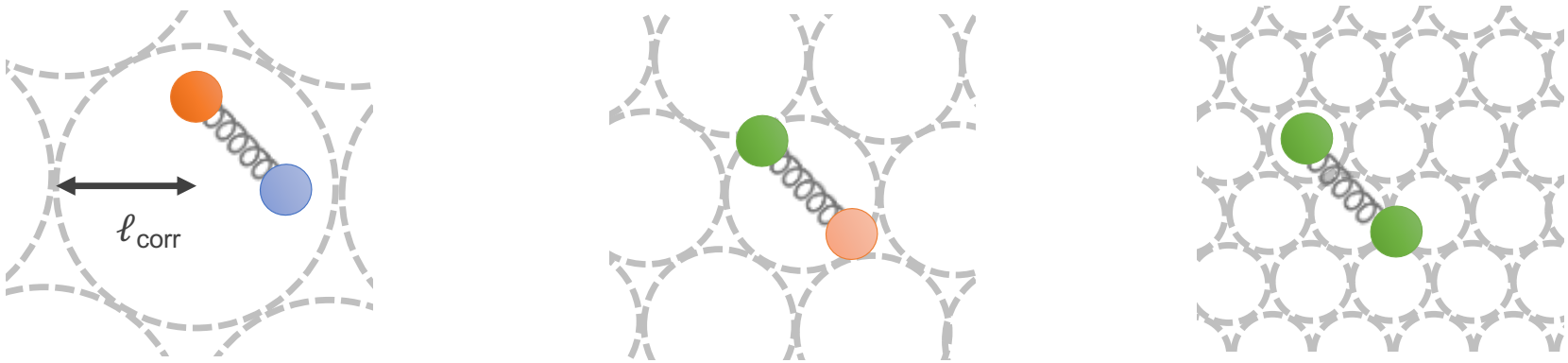


binding force acts efficiently among Q and $\bar{Q}$

medium impedes gluons mediating force

increasing temperature

decoherence



color rotation acts coherently on Q and $\bar{Q}$

color rotation acts individually on Q and Q

see discussion in S. Kajimoto, Y.Akamatsu, M. Asakawa, A.R., PRD97 (2018), 014003

Solving the 1d Lindblad equation I

University of Stavanger

- So far only one-dimensional, need to **establish numerical methods**

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

Solving the 1d Lindblad equation I

University of Stavanger

- So far only one-dimensional, need to **establish numerical methods**

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- Any Lindblad equation amenable to **Quantum State Diffusion** approach
  - Instead of 2d dim. density matrix ρ(**r**,**s**,t) evolve ensemble of d dim. wave functions via a **non-linear Schrödinger equation:**

$$|d\psi\rangle = -iH|\psi(t)\rangle dt + \sum_n \begin{pmatrix} 2\langle L_n^\dagger\rangle_\psi L_n - L_n^\dagger L_n \\ -\langle L_n^\dagger\rangle_\psi \langle L_n\rangle_\psi \end{pmatrix} |\psi(t)\rangle dt + \sum_n \left(L_n - \langle L_n\rangle_\psi\right) |\psi(t)\rangle d\xi_n,$$

Solving the 1d Lindblad equation I

University of Stavanger

- So far only one-dimensional, need to **establish numerical methods**

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- Any Lindblad equation amenable to **Quantum State Diffusion** approach
  - Instead of 2d dim. density matrix ρ(**r**,**s**,t) evolve ensemble of d dim. wave functions via a **non-linear Schrödinger equation:**

nonlinearity

$$|d\psi\rangle = -iH|\psi(t)\rangle dt + \sum_n \begin{pmatrix} 2\langle L_n^\dagger\rangle_\psi L_n - L_n^\dagger L_n \\ -\langle L_n^\dagger\rangle_\psi \langle L_n\rangle_\psi \end{pmatrix} |\psi(t)\rangle dt + \sum_n \left(L_n - \langle L_n\rangle_\psi\right)|\psi(t)\rangle d\xi_n,$$

Solving the 1d Lindblad equation I

University of Stavanger

- So far only one-dimensional, need to **establish numerical methods**

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- Any Lindblad equation amenable to **Quantum State Diffusion** approach
  - Instead of 2d dim. density matrix ρ(**r**,**s**,t) evolve ensemble of d dim. wave functions via a **non-linear Schrödinger equation:**

nonlinearity

ξ_n complex **noise terms**

$$|d\psi\rangle = -iH|\psi(t)\rangle dt + \sum_n \begin{pmatrix} 2\langle L_n^\dagger\rangle_\psi L_n - L_n^\dagger L_n \\ -\langle L_n^\dagger\rangle_\psi \langle L_n\rangle_\psi \end{pmatrix} |\psi(t)\rangle dt + \sum_n \left(L_n - \langle L_n\rangle_\psi\right) |\psi(t)\rangle d\xi_n,$$

Solving the 1d Lindblad equation I

University of Stavanger

- So far only one-dimensional, need to **establish numerical methods**

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- Any Lindblad equation amenable to **Quantum State Diffusion** approach
  - Instead of 2d dim. density matrix ρ(**r**,**s**,t) evolve ensemble of d dim. wave functions via a **non-linear Schrödinger equation:**

nonlinearity

ξ_n complex **noise terms**

$$|d\psi\rangle = -iH|\psi(t)\rangle dt + \sum_n \begin{pmatrix} 2\langle L_n^\dagger\rangle_\psi L_n - L_n^\dagger L_n \\ -\langle L_n^\dagger\rangle_\psi \langle L_n\rangle_\psi \end{pmatrix} |\psi(t)\rangle dt + \sum_n \left(L_n - \langle L_n\rangle_\psi\right)|\psi(t)\rangle d\xi_n,$$

  - Estimate density matrix from ensemble averages: $\rho_{Q\bar{Q}}(\mathbf{r},\mathbf{s}) = \langle \psi(\mathbf{r},t)\psi^*(\mathbf{s},t)\rangle_\xi$

Solving the 1d Lindblad equation I

University of Stavanger

- So far only one-dimensional, need to **establish numerical methods**

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- Any Lindblad equation amenable to **Quantum State Diffusion** approach
  - Instead of 2d dim. density matrix ρ(**r**,**s**,t) evolve ensemble of d dim. wave functions via a **non-linear Schrödinger equation:**

nonlinearity

ξ_n complex **noise terms**

$$|d\psi\rangle = -iH|\psi(t)\rangle dt + \sum_n \begin{pmatrix} 2\langle L_n^\dagger\rangle_\psi L_n - L_n^\dagger L_n \\ -\langle L_n^\dagger\rangle_\psi \langle L_n\rangle_\psi \end{pmatrix} |\psi(t)\rangle dt + \sum_n \left(L_n - \langle L_n\rangle_\psi\right)|\psi(t)\rangle d\xi_n,$$

  - Estimate density matrix from ensemble averages: $\rho_{Q\bar{Q}}(\mathbf{r},\mathbf{s}) = \langle \psi(\mathbf{r},t)\psi^*(\mathbf{s},t)\rangle_\xi$
  - Well suited for higher dimensions, results in 1d checked against direct solution method

Solving the 1d Lindblad equation I

University of Stavanger

- So far only one-dimensional, need to **establish numerical methods**

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- Any Lindblad equation amenable to **Quantum State Diffusion** approach
  - Instead of 2d dim. density matrix ρ(**r**,**s**,t) evolve ensemble of d dim. wave functions via a **non-linear Schrödinger equation:**

nonlinearity

ξ_n complex **noise terms**

$$|d\psi\rangle = -iH|\psi(t)\rangle dt + \sum_n \begin{pmatrix} 2\langle L_n^\dagger\rangle_\psi L_n - L_n^\dagger L_n \\ -\langle L_n^\dagger\rangle_\psi \langle L_n\rangle_\psi \end{pmatrix} |\psi(t)\rangle dt + \sum_n \left(L_n - \langle L_n\rangle_\psi\right)|\psi(t)\rangle d\xi_n,$$

  - Estimate density matrix from ensemble averages: $\rho_{Q\bar{Q}}(\mathbf{r},\mathbf{s}) = \langle \psi(\mathbf{r},t)\psi^*(\mathbf{s},t)\rangle_\xi$
  - Well suited for higher dimensions, results in 1d checked against direct solution method

- 1st QCD derivation of phenomenological models with non-linear Schrödinger eq.

c.f. e.g. R. Katz, P. Gossiaux Annals Phys. 368 (2016) 267

Solving the 1d Lindblad equation I

University of Stavanger

- So far only one-dimensional, need to **establish numerical methods**

$$\frac{d}{dt}\rho_{Q\bar{Q}}(t) = -i\left[H_{Q\bar{Q}}, \rho_{Q\bar{Q}}\right] + \sum_{i=1}^{N_{LB}} \gamma_i \left(\hat{L}_i \rho_{Q\bar{Q}} \hat{L}_i^\dagger - \frac{1}{2}\hat{L}_i \hat{L}_i^\dagger \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} \hat{L}_i \hat{L}_i^\dagger\right)$$

- Any Lindblad equation amenable to **Quantum State Diffusion** approach
  - Instead of 2d dim. density matrix ρ(**r**,**s**,t) evolve ensemble of d dim. wave functions via a **non-linear Schrödinger equation:**

nonlinearity

ξ_n complex **noise terms**

$$|d\psi\rangle = -iH|\psi(t)\rangle dt + \sum_n \begin{pmatrix} 2\langle L_n^\dagger\rangle_\psi L_n - L_n^\dagger L_n \\ -\langle L_n^\dagger\rangle_\psi \langle L_n\rangle_\psi \end{pmatrix} |\psi(t)\rangle dt + \sum_n \left(L_n - \langle L_n\rangle_\psi\right)|\psi(t)\rangle d\xi_n,$$

  - Estimate density matrix from ensemble averages: $\rho_{Q\bar{Q}}(\mathbf{r},\mathbf{s}) = \langle \psi(\mathbf{r},t)\psi^*(\mathbf{s},t)\rangle_\xi$
  - Well suited for higher dimensions, results in 1d checked against direct solution method

- 1st QCD derivation of phenomenological models with non-linear Schrödinger eq.

  c.f. e.g. R. Katz, P. Gossiaux Annals Phys. 368 (2016) 267

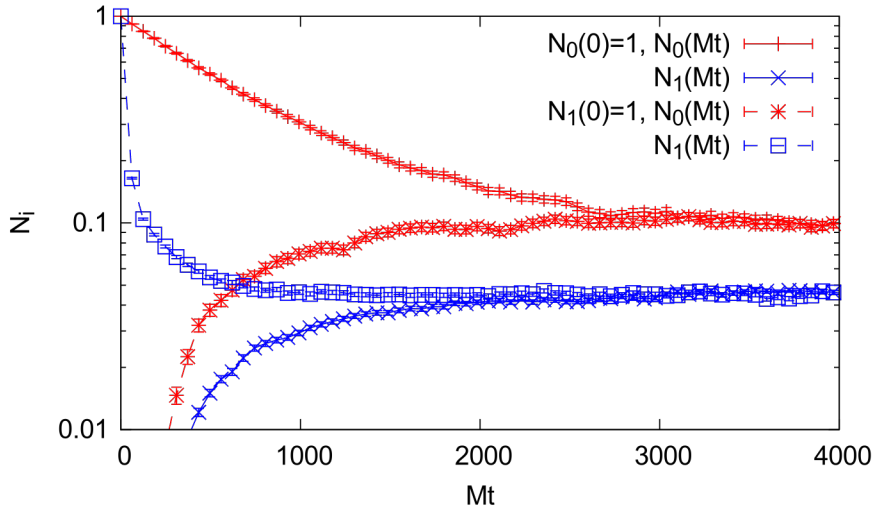
- 1st genuine potential based Linblad implementation: avoids unphysical behavior in ρ

  cf. prior work e.g. by D. De Boni, JHEP 1708 (2017) 064

Thermalization via dissipative dynamics

University of Stavanger

- Encouraging: it is possible to **thermalize** quarkonium in a **fully quantum** fashion

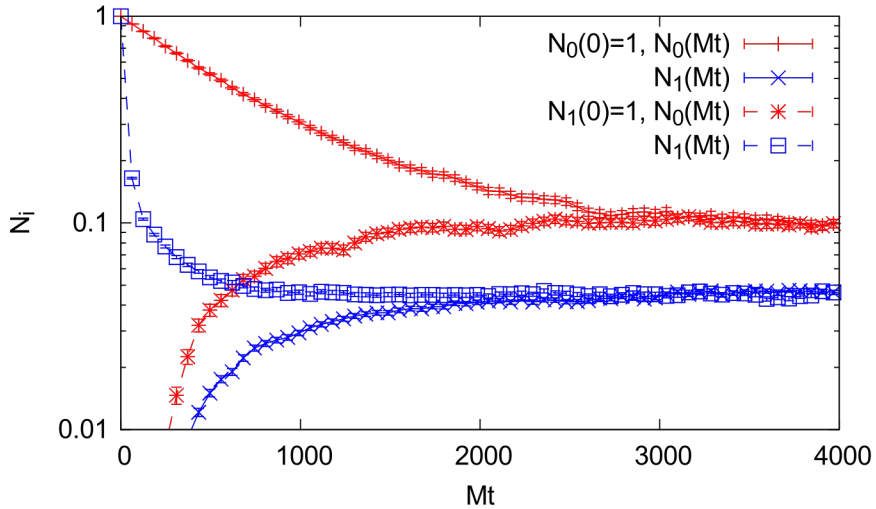


- Start either with single ground state or single excited state and monitor survival probability
- Late-time results independent of initial conditions: steady-state

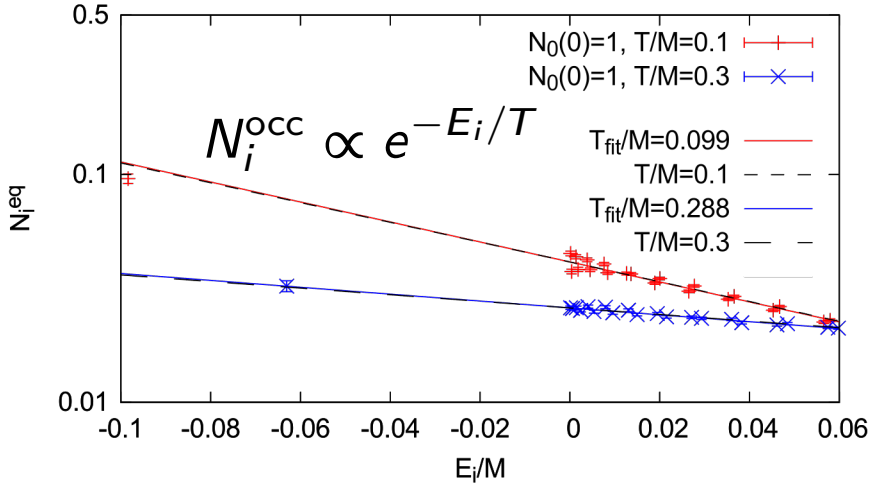
Thermalization via dissipative dynamics

University of Stavanger

- Encouraging: it is possible to **thermalize** quarkonium in a **fully quantum** fashion



- Start either with single ground state or single excited state and monitor survival probability
- Late-time results independent of initial conditions: steady-state

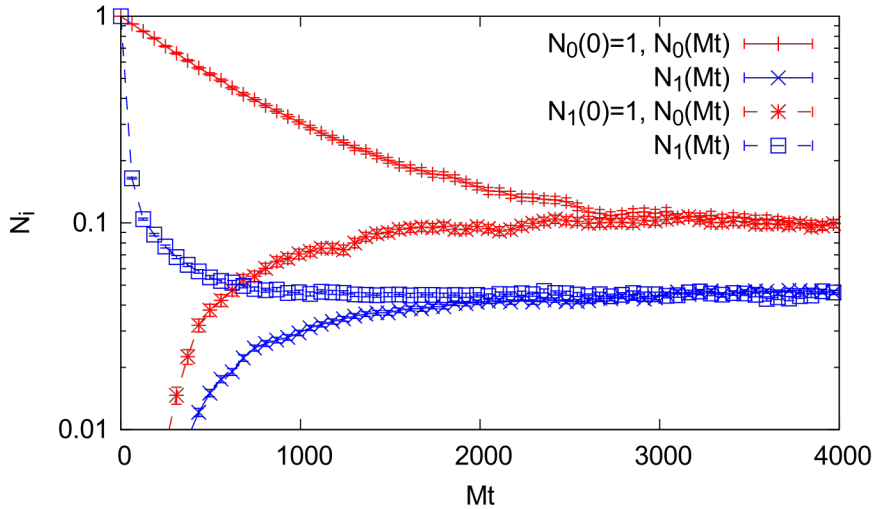


- Steady state characterized by Boltzmann distributed occupation numbers. Quarkonium temperature very close to medium temperature.

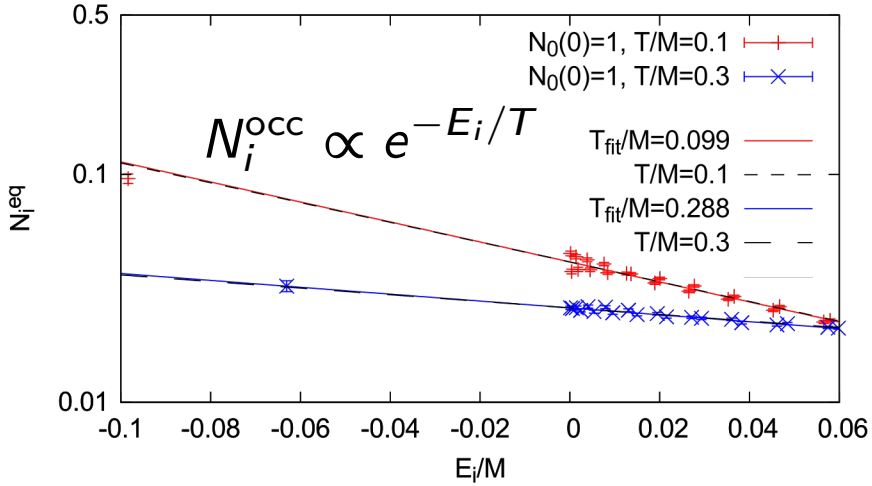
Thermalization via dissipative dynamics

University of Stavanger

- Encouraging: it is possible to **thermalize** quarkonium in a **fully quantum** fashion



- Start either with single ground state or single excited state and monitor survival probability
- Late-time results independent of initial conditions: steady-state

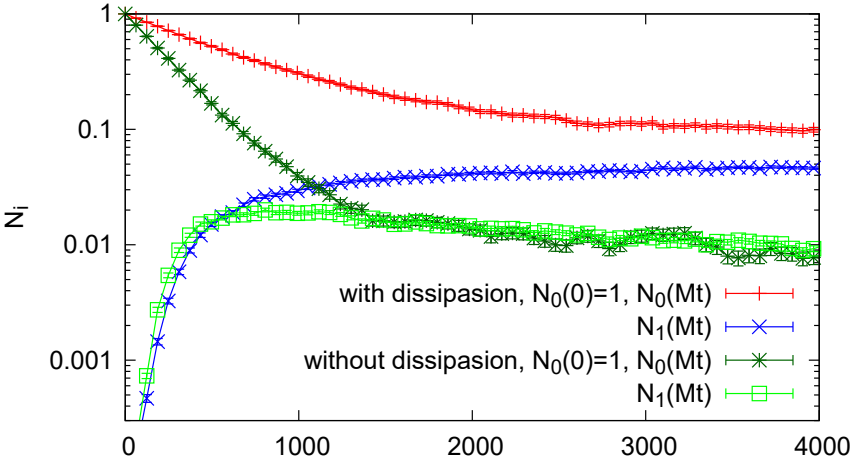


- Steady state characterized by Boltzmann distributed occupation numbers. Quarkonium temperature very close to medium temperature.
- First potential based real-time approach that can thermalize quarkonium states

Effects of dissipation

University of Stavanger

- Thermalization requires balance of fluctuations and dissipation.

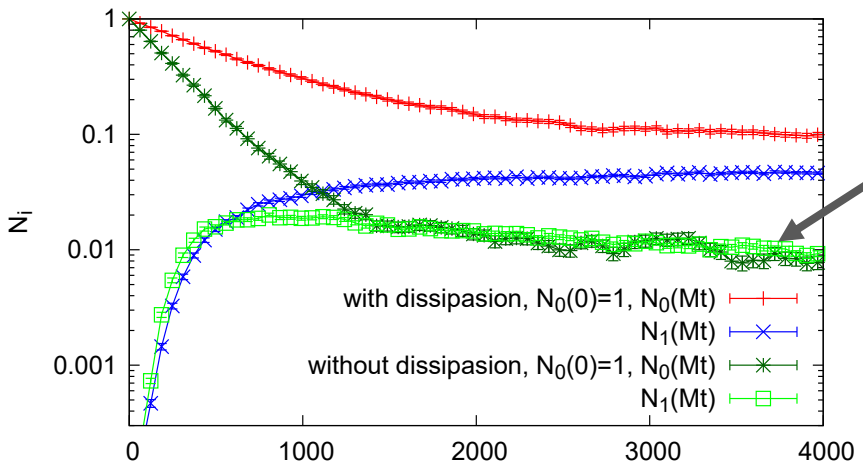


- In absence of dissipation, fluctuations heat up quarkonium system until fully randomized

Effects of dissipation

University of Stavanger

- Thermalization requires balance of fluctuations and dissipation.

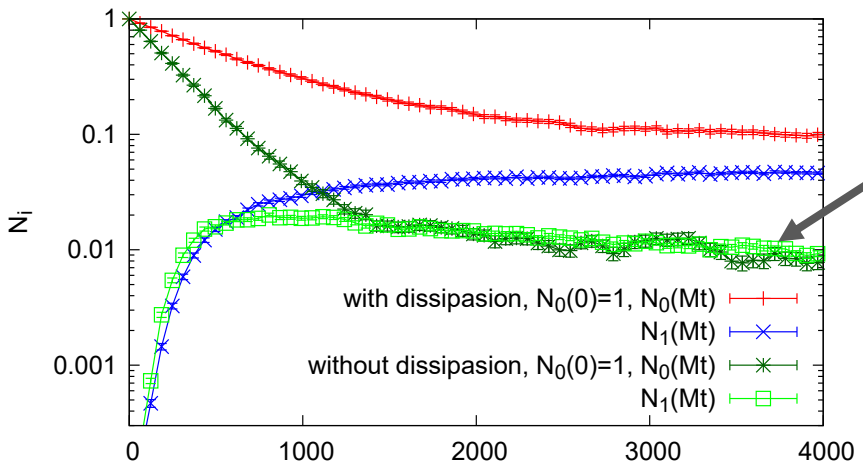


- In absence of dissipation, fluctuations heat up quarkonium system until fully randomized

Effects of dissipation

University of Stavanger

- Thermalization requires balance of fluctuations and dissipation.

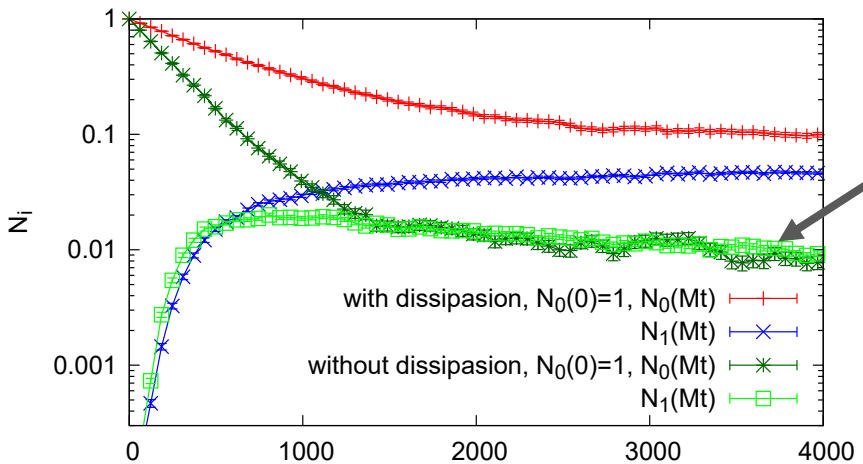


- In absence of dissipation, fluctuations heat up quarkonium system until fully randomized
- Note that dissipative effects are relevant for ground state from beginning of the evolution

Effects of dissipation

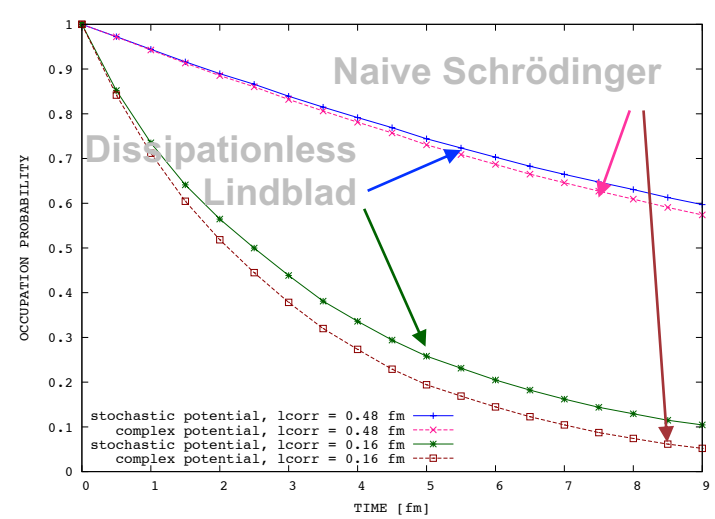
University of Stavanger

- Thermalization requires balance of fluctuations and dissipation.



- In absence of dissipation, fluctuations heat up quarkonium system until fully randomized
- Note that dissipative effects are relevant for ground state from beginning of the evolution

- With each additional approximation (dissipationless limit, adiabatic limit) ground state survival more and more underestimated.



S.Kajimoto, Y.Akamatsu, M. Asakawa, A.R., PRD97 (2018) 014003

Challenges for a direct solution

University of Stavanger

- Preservation of the continuum properties?

1d Lindblad equation in coord. space

$$\partial_t \rho(\mathbf{x}, \mathbf{y}, t) = \mathcal{L}[\mathbf{x}, \mathbf{y}]\rho(\mathbf{x}, \mathbf{y}, t)$$

Challenges for a direct solution

University of Stavanger

- Preservation of the continuum properties?

required for the probability interpretation

$$\text{Positivity}: \quad \forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C}, \tag{8}$$

$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

$$\text{Hermiticity}: \quad \rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$$

1d Lindblad equation in coord. space

$$\partial_t \rho(\mathbf{x}, \mathbf{y}, t) = \mathcal{L}[\mathbf{x}, \mathbf{y}] \rho(\mathbf{x}, \mathbf{y}, t)$$

Challenges for a direct solution

University of Stavanger

1d Lindblad equation in coord. space

$$\partial_t \rho(\mathbf{x}, \mathbf{y}, t) = \mathcal{L}[\mathbf{x}, \mathbf{y}]\rho(\mathbf{x}, \mathbf{y}, t)$$

- Preservation of the continuum properties?

required for the probability interpretation

$$\text{Positivity}: \quad \forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C}, \tag{8}$$

$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

$$\text{Hermiticity}: \quad \rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$$

- What about the trace conservation? More involved!

required for the physics interpretation

$$\text{Unit trace}:$$

$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 \delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{y}_1) \ldots \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) = 1$$

Challenges for a direct solution

University of Stavanger

- Preservation of the continuum properties?

required for the probability interpretation

$$\text{Positivity}: \quad \forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C}, \tag{8}$$

$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

$$\text{Hermiticity}: \quad \rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$$

- What about the trace conservation? More involved!

required for the physics interpretation

$$\text{Unit trace}:$$

$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 \delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{y}_1) \ldots \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) = 1$$

1d Lindblad equation in coord. space

$$\partial_t \rho(\mathbf{x}, \mathbf{y}, t) = \mathcal{L}[\mathbf{x}, \mathbf{y}] \rho(\mathbf{x}, \mathbf{y}, t)$$

proof of trace conservation needs continuum like change between x,y and z,z':

$$z = x - y, \quad z' = x + y \qquad \frac{\partial}{\partial z'} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

$$\int dx \int dy\, \delta(x - y)\, F_3\left(\frac{x + y}{2}\right) \frac{\partial}{\partial x} \frac{\partial}{\partial y} \rho(x, y, t)$$

$$= \int dz \int dz'\, \delta(z)\, F_3\left(\frac{z'}{2}\right) \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z'^2}\right) \rho(z, z', t)$$

Challenges for a direct solution

University of Stavanger

- Preservation of the continuum properties?

required for the probability interpretation

$$\text{Positivity}: \quad \forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C}, \tag{8}$$
$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

$$\text{Hermiticity}: \quad \rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$$

- What about the trace conservation? More involved!

required for the physics interpretation

$$\text{Unit trace}:$$
$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 \delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{y}_1) \ldots \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) = 1$$

1d Lindblad equation in coord. space

$$\partial_t \rho(\mathbf{x}, \mathbf{y}, t) = \mathcal{L}[\mathbf{x}, \mathbf{y}] \rho(\mathbf{x}, \mathbf{y}, t)$$

proof of trace conservation needs continuum like change between x,y and z,z':

$$z = x - y, \quad z' = x + y \qquad \frac{\partial}{\partial z'} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

$$\int dx \int dy\, \delta(x-y)\, F_3\left(\frac{x+y}{2}\right) \frac{\partial}{\partial x}\frac{\partial}{\partial y} \rho(x, y, t)$$
$$= \int dz \int dz'\, \delta(z)\, F_3\left(\frac{z'}{2}\right) \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z'^2}\right) \rho(z, z', t)$$

- Solution: a reparameterization invariant SBP finite difference operator

O. Ålund, Y. Akamatsu, F. Laurén, T.Miura, J. Nordström, A.R. JCP 425 (2021) 109917

Challenges for a direct solution

University of Stavanger

- Preservation of the continuum properties?

required for the probability interpretation

$$\text{Positivity}: \quad \forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C}, \tag{8}$$
$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

$$\text{Hermiticity}: \quad \rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$$

- What about the trace conservation? More involved!

required for the physics interpretation

$$\text{Unit trace}:$$
$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 \delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{y}_1) \ldots \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) = 1$$

1d Lindblad equation in coord. space

$$\partial_t \rho(\mathbf{x}, \mathbf{y}, t) = \mathcal{L}[\mathbf{x}, \mathbf{y}] \rho(\mathbf{x}, \mathbf{y}, t)$$

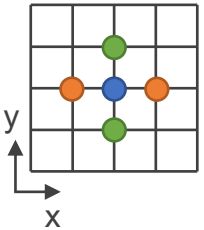
proof of trace conservation needs continuum like change between x,y and z,z':

$$z = x - y, \qquad z' = x + y \qquad \frac{\partial}{\partial z'} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

$$\int dx \int dy\, \delta(x-y)\, F_3\left(\frac{x+y}{2}\right) \frac{\partial}{\partial x}\frac{\partial}{\partial y} \rho(x, y, t)$$
$$= \int dz \int dz'\, \delta(z)\, F_3\left(\frac{z'}{2}\right) \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z'^2}\right) \rho(z, z', t)$$

- Solution: a reparameterization invariant SBP finite difference operator

O. Ålund, Y. Akamatsu, F. Laurén, T.Miura, J. Nordström, A.R. JCP 425 (2021) 109917



$$\mathbb{D}_x u = \frac{u(x_{i+1}, y_j) - u(x_{i-1}, y_j)}{2\Delta}$$

$$\mathbb{D}_y u = \frac{u(x_i, y_{j+1}) - u(x_i, y_{j-1})}{2\Delta}$$

Challenges for a direct solution

University of Stavanger

- Preservation of the continuum properties?

required for the probability interpretation

$$\text{Positivity}: \quad \forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C}, \tag{8}$$

$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

$$\text{Hermiticity}: \quad \rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$$

- What about the trace conservation? More involved!

required for the physics interpretation

$$\text{Unit trace}: \quad \int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 \delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{y}_1) \ldots \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) = 1$$

1d Lindblad equation in coord. space

$$\partial_t \rho(\mathbf{x}, \mathbf{y}, t) = \mathcal{L}[\mathbf{x}, \mathbf{y}] \rho(\mathbf{x}, \mathbf{y}, t)$$

proof of trace conservation needs continuum like change between x,y and z,z':

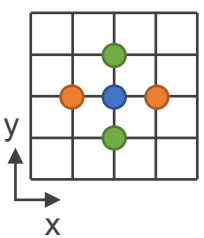
$$z = x - y, \quad z' = x + y \qquad \frac{\partial}{\partial z'} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

$$\int dx \int dy\, \delta(x-y)\, F_3\left(\frac{x+y}{2}\right) \frac{\partial}{\partial x}\frac{\partial}{\partial y} \rho(x, y, t)$$

$$= \int dz \int dz'\, \delta(z)\, F_3\left(\frac{z'}{2}\right) \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z'^2}\right) \rho(z, z', t)$$

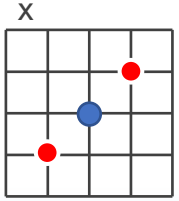
- Solution: a reparameterization invariant SBP finite difference operator

O. Ålund, Y. Akamatsu, F. Laurén, T.Miura, J. Nordström, A.R. JCP 425 (2021) 109917



$$\mathbb{D}_x u = \frac{u(x_{i+1}, y_j) - u(x_{i-1}, y_j)}{2\Delta}$$

$$\mathbb{D}_y u = \frac{u(x_i, y_{j+1}) - u(x_i, y_{j-1})}{2\Delta}$$



$$\frac{1}{2}\left(\mathbb{D}_x + \mathbb{D}_y\right) u \neq \mathbb{D}_{z'} u = \frac{u(x_{i+1}, y_{j+1}) - u(x_{i-1}, y_{j-1})}{2\Delta}$$

Challenges for a direct solution

University of Stavanger

- Preservation of the continuum properties?

required for the probability interpretation

$$\text{Positivity}: \quad \forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C}, \tag{8}$$

$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

$$\text{Hermiticity}: \quad \rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$$

- What about the trace conservation? More involved!

required for the physics interpretation

$$\text{Unit trace}: \quad \int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 \delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{y}_1) \ldots \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) = 1$$

1d Lindblad equation in coord. space

$$\partial_t \rho(\mathbf{x}, \mathbf{y}, t) = \mathcal{L}[\mathbf{x}, \mathbf{y}] \rho(\mathbf{x}, \mathbf{y}, t)$$

proof of trace conservation needs continuum like change between x,y and z,z':

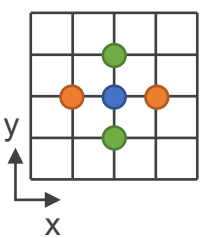
$$z = x - y, \quad z' = x + y \qquad \frac{\partial}{\partial z'} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

$$\int dx \int dy\, \delta(x-y)\, F_3\left(\frac{x+y}{2}\right) \frac{\partial}{\partial x}\frac{\partial}{\partial y} \rho(x, y, t)$$

$$= \int dz \int dz'\, \delta(z)\, F_3\left(\frac{z'}{2}\right) \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z'^2}\right) \rho(z, z', t)$$

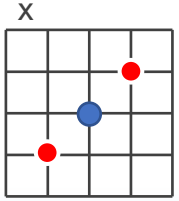
- Solution: a reparameterization invariant SBP finite difference operator

O. Ålund, Y. Akamatsu, F. Laurén, T.Miura, J. Nordström, A.R. JCP 425 (2021) 109917



$$\mathbb{D}_x u = \frac{u(x_{i+1}, y_j) - u(x_{i-1}, y_j)}{2\Delta}$$

$$\mathbb{D}_y u = \frac{u(x_i, y_{j+1}) - u(x_i, y_{j-1})}{2\Delta}$$

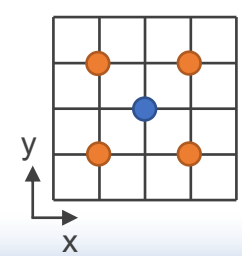


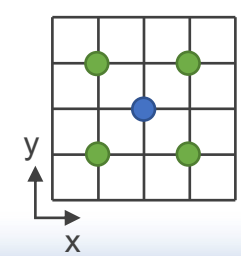
$$\frac{1}{2}\left(\mathbb{D}_x + \mathbb{D}_y\right) u \neq \mathbb{D}_{z'} u = \frac{u(x_{i+1}, y_{j+1}) - u(x_{i-1}, y_{j-1})}{2\Delta}$$

$$\mathbb{D}_x^{\mathrm{RN}} u = \frac{u(x_{i+1}, y_{j+1}) - u(x_{i-1}, y_{j+1}) + u(x_{i+1}, y_{j-1}) - u(x_{i-1}, y_{j-1})}{4\Delta}$$

$$\mathbb{D}_y^{\mathrm{RN}} u = \frac{u(x_{i+1}, y_{j+1}) - u(x_{i+1}, y_{j-1}) + u(x_{i-1}, y_{j+1}) - u(x_{i-1}, y_{j-1})}{4\Delta}$$

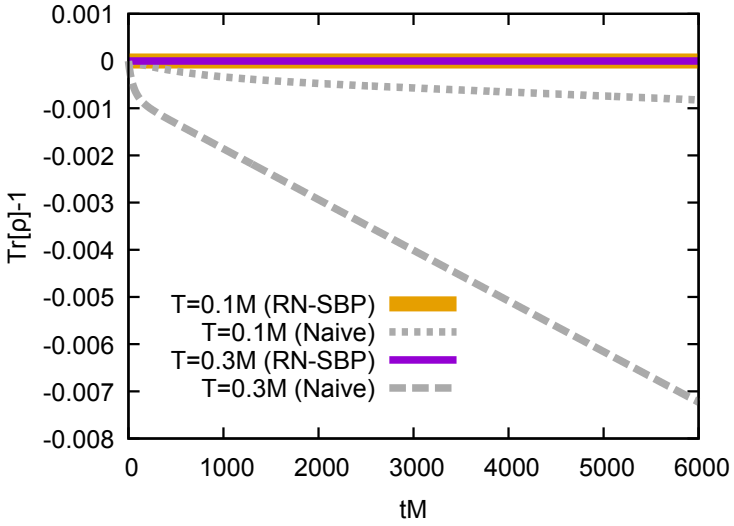
$$\frac{1}{2}\left(\mathbb{D}_x^{\mathrm{RN}} + \mathbb{D}_y^{\mathrm{RN}}\right) u = \mathbb{D}_{z'} u$$



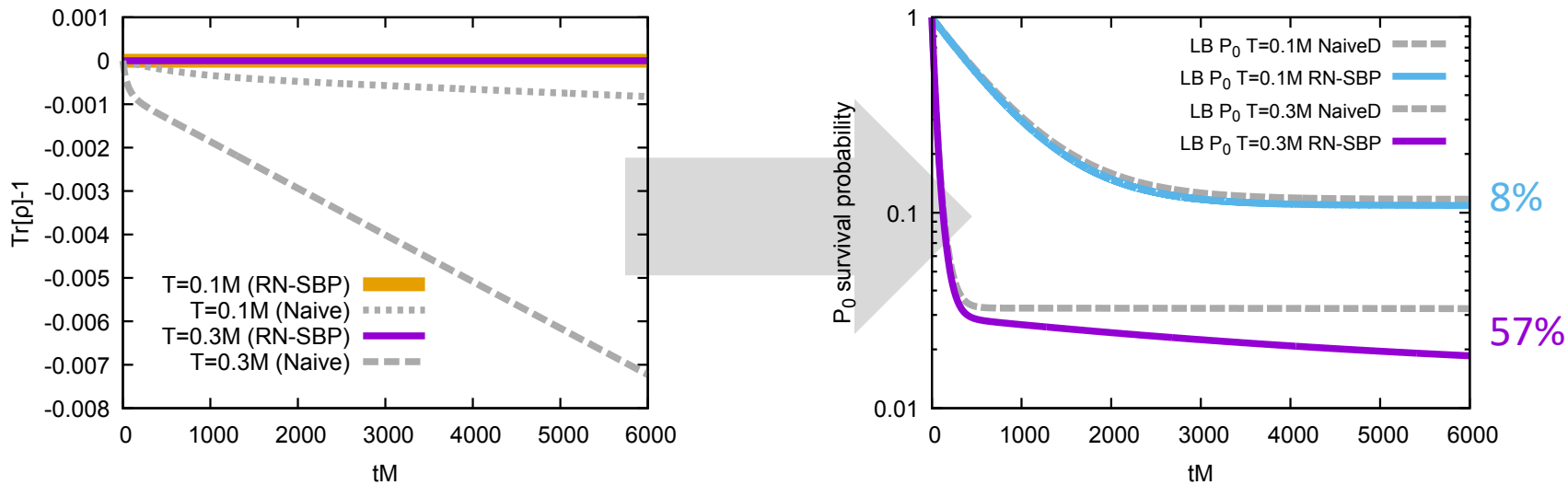


Master equation dynamics

University of Stavanger



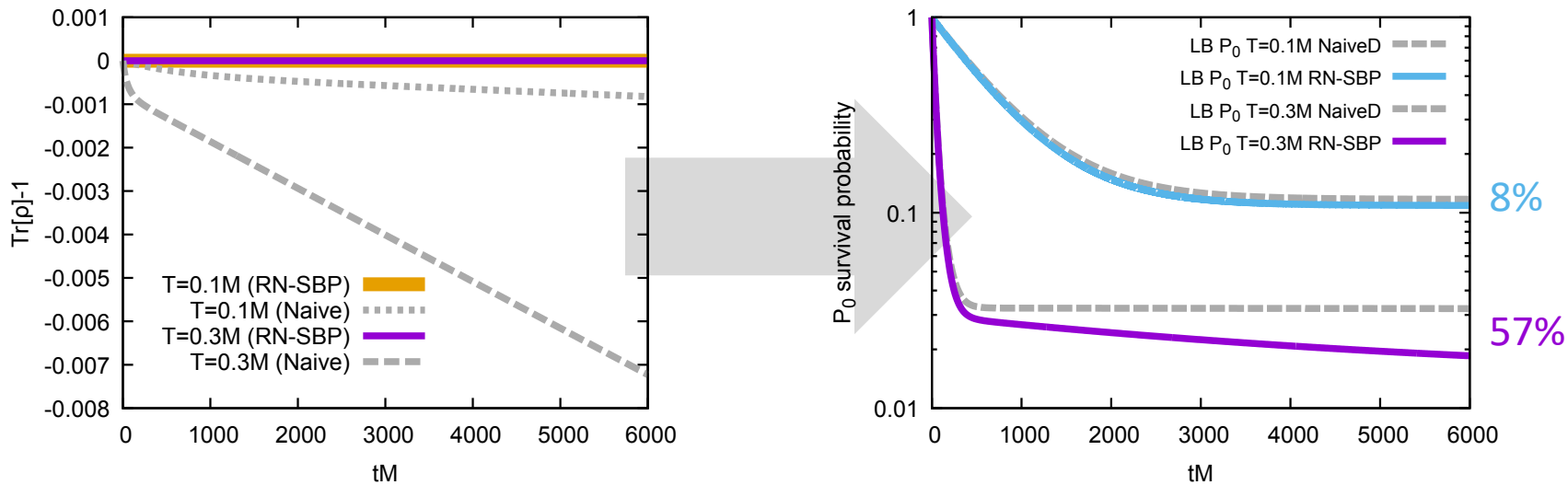
Master equation dynamics



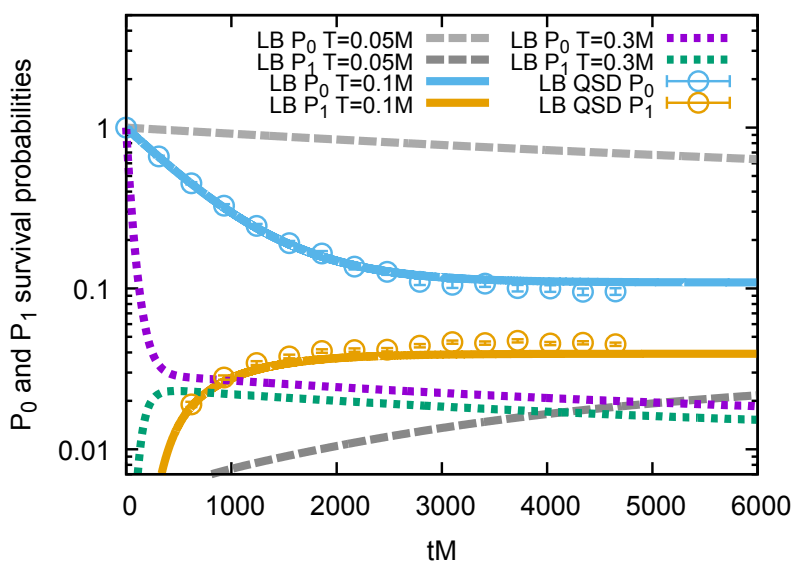
- Tiny deviations from overall trace conservation can translate into significant survival mismatch

Master equation dynamics

University of Stavanger



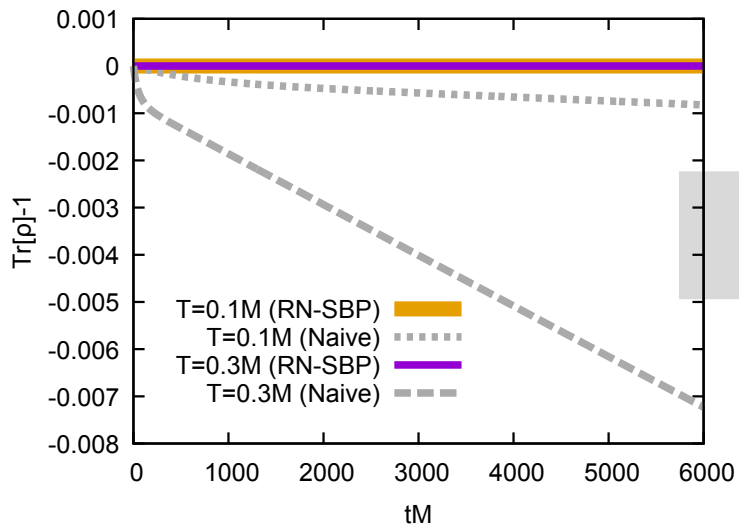
- Tiny deviations from overall trace conservation can translate into significant survival mismatch

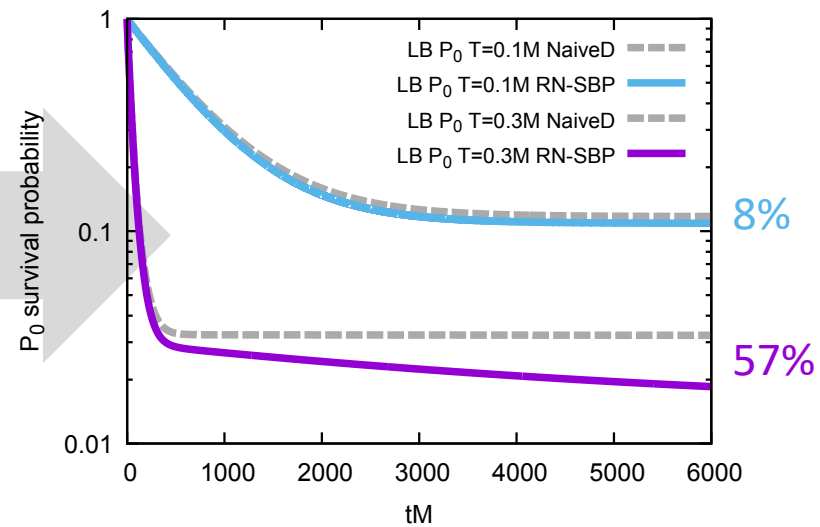


- Direct solution of master equation does not require additional approximations such as employed in stochastic unravelling (QSD)

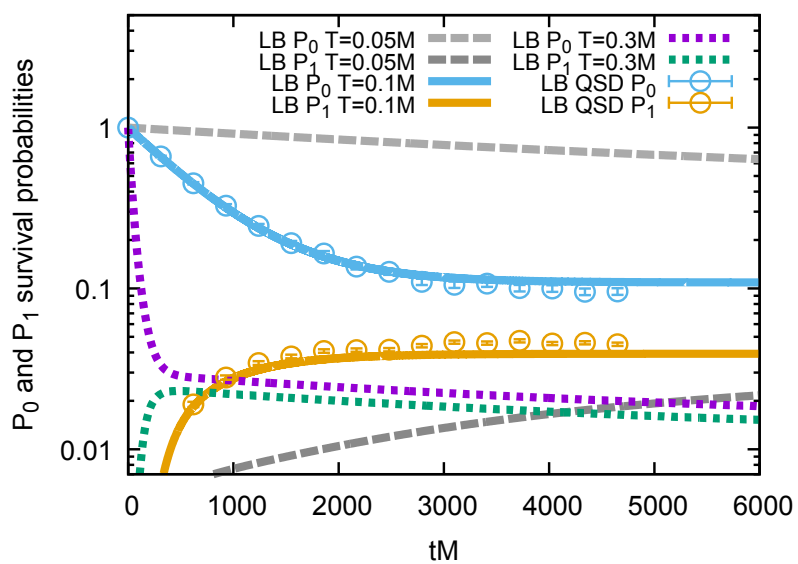
Master equation dynamics

University of Stavanger





- Tiny deviations from overall trace conservation can translate into significant survival mismatch



- Direct solution of master equation does not require additional approximations such as employed in stochastic unravelling (QSD)
- Curse of dimensionality comes back to haunt us in 3d: density matrix is a 6d object.

Conclusion

University of Stavanger

- Exciting progress in the understanding of **in-medium heavy-quarkonium dynamics** from QCD within the **Open-Quantum-Systems** framework
- Achieved **derivation** of several phenomenological models for **weakly coupled QCD** using a systematic chain of approximations
- The **Stavanger – Osaka approach** is based on a Lindblad equation in the **Quantum Brownian Motion** regime derived from QCD at high temperature
- Implementation in 1d via **quantum state diffusion** approach, crosschecked via direct simulation of master equation with full **trace preservation**.
- Inclusion of genuine **dissipative effects**: thermalization from quantum dynamics

Conclusion

University of Stavanger

- Exciting progress in the understanding of **in-medium heavy-quarkonium dynamics** from QCD within the **Open-Quantum-Systems** framework
- Achieved **derivation** of several phenomenological models for **weakly coupled QCD** using a systematic chain of approximations
- The **Stavanger – Osaka approach** is based on a Lindblad equation in the **Quantum Brownian Motion** regime derived from QCD at high temperature
- Implementation in 1d via **quantum state diffusion** approach, crosschecked via direct simulation of master equation with full **trace preservation**.
- Inclusion of genuine **dissipative effects**: thermalization from quantum dynamics

Thank you for your attention