C, P, T symmetries and Lorentz transformations in the theory of superalgebraic spinors

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Mathematical theories of spinors

- 1. **Covariant spinors** (matrix columns or rows) - Élie Cartan.
- 2. **Algebraic spinors -** approach is based on theory of Clifford algebras. Matrix representation in a 2m dimensional complex space in the form of square matrices $2^m \cdot 2^m$.
- 3. **Superalgebraic spinors** – extension of the theory of algebraic spinors and of axiomatic algebraic QFT. Theory of C*-algebras. Grassmann variables and derivatives with respect to them. CAR-algebra of second quantization of fermions (CAR – Canonical Anticommutation Relations)

Theory of superalgebraic spinors

1. M. Pavšič. A theory of quantized fields based on orthogonal and symplectic Clifford algebras. *Advances in Applied Clifford Algebras*, 2012, v.22, p.449-481.
2. V. Monakhov. Superalgebraic representation of Dirac matrices. *Theoretical and Mathematical Physics*. 2016. v. 186. p.70–82.
3. V. Monakhov. Dirac matrices as elements of superalgebraic matrix algebra. *Bulletin of the Russian Academy of Sciences: Physics*, 2016, v.80, p. 985–988.
4. V. Monakhov. Superalgebraic structure of Lorentz transformations. *J. of Physics: Conf. Series*, 2018, v.1051, 012023.
5. V.Monakhov. Generalization of Dirac conjugation in the superalgebraic theory of spino rs *Theoretical and Mathematical Physics*, 2019, v.200, p.1026-1042.
6. V. Monakhov. Vacuum and spacetime signature in the theory of superalgebraic spinors. *Universe*, 2019,v.5 (7), 162.
7. V. Monakhov. Spacetime and inner space of spinors in the theory of superalgebraic spinors. *Journal of Physics: Conference Series*, 2020, v.1557(1), 12031.
8. V. Monakhov. Generation of Electroweak Interaction by Analogs of Dirac Gamma Matrices Constructed from Operators of the Creation and Annihilation of Spinors. *Bulletin of the Russian Academy of Sciences: Physics*, 2020, Vol. 84, No. 10, pp. 1216–1220.

4-component superalgebraic spinors*

$$\Psi = \int d^3 p \left(\psi^{\alpha}(p) \frac{\partial}{\partial \theta^{\alpha}(p)} + \psi^{\tau}(p) \theta^{\tau}(p) \right)$$

$$\theta^{a}(p)^{+} = \frac{\partial}{\partial \theta^{a}(p)}; \ \{ \frac{\partial}{\partial \theta^{k}(p)}, \theta^{l}(p')\} = \delta_{k}^{l}\, \delta(p - p')$$

$$\frac{\partial}{\partial \theta^{1}(p)} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \frac{\partial}{\partial \theta^{2}(p)} \cong \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \theta^{3}(p) \cong \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \theta^{4}(p) \cong \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\theta^{1}(p) \cong \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad \theta^{2}(p) \cong \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial \theta^{3}(p)} \cong \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \frac{\partial}{\partial \theta^{4}(p)} \cong \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

Gamma-operators (analogs of matrices): two additional compared to Dirac's theory!

$$\hat{A} = [A, \bullet] \Rightarrow \hat{A}\Psi = [A, \Psi] = A\Psi - \Psi A$$

$$\hat{\gamma}^0 = \int \mathrm{d}^3 p \, [\frac{\partial}{\partial \theta^1(p)}\theta^1(p) + \frac{\partial}{\partial \theta^2(p)}\theta^2(p) + \frac{\partial}{\partial \theta^3(p)}\theta^3(p) + \frac{\partial}{\partial \theta^4(p)}\theta^4(p), \bullet]$$

$$\hat{\gamma}^1 = \int \mathrm{d}^3 p \, [\frac{\partial}{\partial \theta^1(p)}\frac{\partial}{\partial \theta^4(p)} - \theta^4(p)\theta^1(p) + \frac{\partial}{\partial \theta^2(p)}\frac{\partial}{\partial \theta^3(p)} - \theta^3(p)\theta^2(p), \bullet]$$

$$\hat{\gamma}^2 = i\int \mathrm{d}^3 p \, [-\frac{\partial}{\partial \theta^1(p)}\frac{\partial}{\partial \theta^4(p)} - \theta^4(p)\theta^1(p) + \frac{\partial}{\partial \theta^2(p)}\frac{\partial}{\partial \theta^3(p)} + \theta^3(p)\theta^2(p), \bullet]$$

$$\hat{\gamma}^3 = \int \mathrm{d}^3 p \, [\frac{\partial}{\partial \theta^1(p)}\frac{\partial}{\partial \theta^3(p)} - \theta^3(p)\theta^1(p) - \frac{\partial}{\partial \theta^2(p)}\frac{\partial}{\partial \theta^4(p)} + \theta^4(p)\theta^2(p), \bullet]$$

$$\hat{\gamma}^4 = i\hat{\gamma}^5 = i\int \mathrm{d}^3 p \, [\frac{\partial}{\partial \theta^1(p)}\frac{\partial}{\partial \theta^3(p)} + \theta^3(p)\theta^1(p) + \frac{\partial}{\partial \theta^2(p)}\frac{\partial}{\partial \theta^4(p)} + \theta^4(p)\theta^2(p), \bullet]$$

$$\hat{\gamma}^6 = i\int \mathrm{d}^3 p \, [\frac{\partial}{\partial \theta^1(p)}\frac{\partial}{\partial \theta^2(p)} + \theta^2(p)\theta^1(p) - \frac{\partial}{\partial \theta^3(p)}\frac{\partial}{\partial \theta^4(p)} - \theta^4(p)\theta^3(p), \bullet]$$

$$\hat{\gamma}^7 = \int \mathrm{d}^3 p \, [\frac{\partial}{\partial \theta^1(p)}\frac{\partial}{\partial \theta^2(p)} - \theta^2(p)\theta^1(p) + \frac{\partial}{\partial \theta^3(p)}\frac{\partial}{\partial \theta^4(p)} - \theta^4(p)\theta^3(p), \bullet]$$

Operators of annihilation and creation of spinor. Operator of generalized Dirac conjugation

$$b_{\alpha}(p_i) = \exp(\hat{\gamma}^{0k}\varphi_k)\frac{\partial}{\partial\theta^{\alpha}(0)}\Big|_{p=0\to p=p_i}$$

$$\overline{b_{\alpha}}(p_i) = \exp(\hat{\gamma}^{0k}\varphi_k)\theta^{l}(0)\Big|_{p=0\to p=p_i}$$

$$\overline{\Psi} = (M\Psi)^{+}$$

$$\text{Signature} = (+---) \Rightarrow M = \hat{\gamma}^{0}$$

$$\overline{b_{\alpha}}(p) = (\hat{\gamma}^{0}b_{\alpha}(p))^{+}$$

$$\overline{\Psi} = (\hat{\gamma}^{0}\Psi)^{+} = (\bullet)^{+}\hat{\gamma}^{0}\Psi$$

Discretization of momentum space. Spinor vacuum

$$\{\frac{\partial}{\partial\theta^k(p_i)},\theta^l(p_j)\}=\delta_k^l\frac{1}{\Delta^3 p_i}\delta_j^i;\quad \delta(p_i-p_j)=\frac{1}{\Delta^3 p_i}\delta_j^i$$

$$\{\frac{\partial}{\partial\theta^k(p_i)},\frac{\partial}{\partial\theta^l(p_j)}\}=\{\theta^k(p_i),\theta^l(p_j)\}=0$$

$$\Psi_V=\prod_i \Psi_V(p_i)$$

$$\Psi_V(0)=(\Delta^3 p\,|_{p=0})^4\frac{\partial}{\partial\theta^1(0)}\theta^1(0)\frac{\partial}{\partial\theta^2(0)}\theta^2(0)\frac{\partial}{\partial\theta^3(0)}\theta^3(0)\frac{\partial}{\partial\theta^4(0)}\theta^4(0)$$

$$\Psi_V(p_i)=(\Delta^3 p_i)^4\, b_1(p_i)\overline{b}_1(p_i)b_2(p_i)\overline{b}_2(p_i)b_3(p_i)\overline{b}_3(p_i)b_4(p_i)\overline{b}_4(p_i)$$

Properties of the spinor vacuum

$$\Psi_V(p_i)^+ = \Psi_V(-p_i) \;\; => \Psi_V{}^+ = \Psi_V$$

$$(\Psi_V)^2 = \Psi_V$$

$b_1(p_i)\Psi_V = 0$, annihilation operator

$\bar{b}_1(p_i)\Psi_V \neq 0$, creation operator

Ψ_V is primitive Hermitian idempotent.

Clifford algebra: operators of reflection

Operator A transforms Clifford vector X as

$$X' = AXA^{-1} = (\lambda A)X(\lambda A)^{-1},$$

i.e. A is defined up to numerical factor λ

Operator A transforms spinor Ψ as

$$\Psi' = A\Psi,$$

$$(\Psi', \Psi') = (\Psi, \Psi) \Rightarrow \lambda = \mathrm{e}^{\mathrm{i}\varphi}$$

$A = \mathrm{i}\hat{\gamma}^0 \Rightarrow \hat{\gamma}^0{}' = \hat{\gamma}^0,\ \hat{\gamma}^k{}' = -\hat{\gamma}^k, k = 1,2,3,6,7,5$ – reflects $\hat{\gamma}^k$

$A = \hat{\gamma}^{ab} \Rightarrow \hat{\gamma}^a{}' = -\hat{\gamma}^a,\ \hat{\gamma}^b{}' = -\hat{\gamma}^b,\ \hat{\gamma}^c{}' = \hat{\gamma}^c, a \neq c \neq b$

– reflects $\hat{\gamma}^a$ and $\hat{\gamma}^b$

CAR algebra: operators of reflection

Operator A transforms Clifford vector X as

$$X' = AXA^{-1} = (\lambda A) X (\lambda A)^{-1}, \text{ numerical factor } \lambda$$

Operator A transforms spinor Ψ as

$$\Psi' = A\Psi, \quad \text{numerical factor } \lambda = \mathrm{e}^{i\varphi}.$$

New : Operator A transforms antispinor $\overline{\Psi}$ as

$$\Psi' = A\overline{\Psi}.$$

$$\text{New : CAR algebra } \{\lambda \frac{\partial}{\partial \theta^k(p)}, \lambda \theta^l(p')\} = \delta_k^l \, \delta(p - p')$$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

Reflection operator of all axes except t

$$1+d\hat{G}=1+[dG,\bullet],$$

$$(1+d\hat{G})\Psi_1\ldots\Psi_k=1+[dG,\Psi_1]\ldots\Psi_k+\ldots+\Psi_1[dG,\Psi_k]=$$

$$=(e^{d\hat{G}}\Psi_1)(e^{d\hat{G}}\Psi_2)\ldots(e^{d\hat{G}}\Psi_k)$$

$$e^{\hat{G}}\Psi_1\Psi_2\ldots\Psi_k=(e^{\hat{G}}\Psi_1)(e^{\hat{G}}\Psi_2)\ldots(e^{\hat{G}}\Psi_k)$$

$$d\hat{G}=i\hat{\gamma}^0 d\omega_0, \qquad e^{i\hat{\gamma}^0\omega_0}=\cos\omega_0+i\hat{\gamma}^0\sin\omega_0$$

$$\omega_0=\frac{\pi}{2} \Rightarrow P_{i\hat{\gamma}^0}=e^{i\hat{\gamma}^0\frac{\pi}{2}}=i\hat{\gamma}^0$$

$$P_{i\hat{\gamma}^0}\Psi=i[\gamma^0,\Psi],$$

$$P_{i\hat{\gamma}^0}\Psi_1\Psi_2\ldots\Psi_k=(P_{i\hat{\gamma}^0}\Psi_1)(P_{i\hat{\gamma}^0}\Psi_2)\ldots(P_{i\hat{\gamma}^0}\Psi_k)$$

Operators Q and P

$$\hat{Q} = \mathrm{i}\hat{\gamma}^6\hat{\gamma}^7 =$$

$$\int d^3p\,[\frac{\partial}{\partial\theta^1(p)}\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\theta^2(p) - \frac{\partial}{\partial\theta^3(p)}\theta^3(p) - \frac{\partial}{\partial\theta^4(p)}\theta^4(p),\bullet]$$

– generator of rotations in the plane $\hat{\gamma}^6$, $\hat{\gamma}^7$.

Operator of charge in the theory of second quantization.

$$\hat{Q}\Psi = \Psi, \qquad \hat{Q}\overline{\Psi} = -\overline{\Psi},$$

$$e^{\mathrm{i}\hat{Q}\varphi}\Psi = e^{\mathrm{i}\varphi}\Psi, \quad e^{\mathrm{i}\hat{Q}\varphi}\overline{\Psi} = e^{-\mathrm{i}\varphi}\overline{\Psi}.$$

$P = P_{\hat{\gamma}^0\hat{Q}}$ – spatial reflection

Operator T1 of time reflection

$$\Psi' = U\Psi = (1 + \hat{\gamma}^{05} d\omega_{05})\Psi$$

$$\overline{\Psi'} = (\hat{\gamma}^{0}(1 + \hat{\gamma}^{05} d\omega_{05})\Psi)^{+} = (1 - \hat{\gamma}^{05} d\omega_{05})(\hat{\gamma}^{0}\Psi)^{+} = U^{-1}\overline{\Psi}.$$

After integration $\Psi' = \hat{\gamma}^{05}\Psi, \qquad \overline{\Psi'} = -\hat{\gamma}^{05}\overline{\Psi}.$

- improper transformation of conjugated spinors.

For both cases proper transformation of vectors:

$$\hat{\gamma}^{0}{}' = -\hat{\gamma}^{0},$$

$$\hat{\gamma}^{5}{}' = -\hat{\gamma}^{5},$$

$$\hat{\gamma}^{k}{}' = \hat{\gamma}^{k}, k = 1,2,3,6,7.$$

Operator T1 converts vacuum to alternative vacuum

$$\Psi_{\mathrm{V}} = \prod_i \Psi_{\mathrm{V}}(p_i),$$

$$P_{\hat{\gamma}^{05}} \Psi_{\mathrm{V}}(0) = \Psi_{\mathrm{altV}}(0),$$

$$\Psi_{\mathrm{V}}(0) = (\Delta^3 p\,|_{p=0})^4 \frac{\partial}{\partial\theta^1(0)}\theta^1(0)\frac{\partial}{\partial\theta^2(0)}\theta^2(0)\frac{\partial}{\partial\theta^3(0)}\theta^3(0)\frac{\partial}{\partial\theta^4(0)}\theta^4(0)$$

$$\Psi_{\mathrm{altV}}(0) = (\Delta^3 p\,|_{p=0})^4 \theta^1(0)\frac{\partial}{\partial\theta^1(0)}\theta^2(0)\frac{\partial}{\partial\theta^2(0)}\theta^3(0)\frac{\partial}{\partial\theta^3(0)}\theta^4(0)\frac{\partial}{\partial\theta^4(0)}$$

Creation operators became annihilation operators and vice versa

Transposition and time reflection T

Transposition

$$(\bullet)^T \hat{\gamma}^a \Psi = -\hat{\gamma}^a \Psi^T, \; a = 0,2,4,6,$$

$$(\bullet)^T \hat{\gamma}^b \Psi = \hat{\gamma}^b \Psi^T, \; b = 1,3,7.$$

Time reflection

$$T = -(\bullet)^T \hat{\gamma}^{26} = -\hat{\gamma}^{26} (\bullet)^T \; - one\ particle$$

$$T = (\bullet)^T P_{-\hat{\gamma}^{26}} = P_{-\hat{\gamma}^{26}} (\bullet)^T - common\ case$$

$$T \Psi_V = \Psi_V$$

Charge conjugation C

Complex conjugation in the Clifford algebra differs from the usual complex conjugation. It leaves basis Clifford vectors unchanged, even if in the considered representation they are complex.

$$\hat{\gamma}^{2}* = -\hat{\gamma}^{2}, \hat{\gamma}^{6}* = -\hat{\gamma}^{6} \Rightarrow$$

$$C = (\bullet)^{*} P_{\hat{\gamma}^{26}} = P_{\hat{\gamma}^{26}} (\bullet)^{*}$$

$$\hat{Q} C\Psi = -C\Psi, \quad \hat{Q} C\overline{\Psi} = C\overline{\Psi}$$

Conclusion 1

$$P = P_{\hat{\gamma}^0 \hat{Q}}$$

$$T = (\bullet)^T P_{-\hat{\gamma}^{26}} = P_{-\hat{\gamma}^{26}} (\bullet)^T$$

$$C = (\bullet)^* P_{\hat{\gamma}^{26}} = P_{\hat{\gamma}^{26}} (\bullet)^*$$

$$\overline{\Psi} = (\hat{\gamma}^0 \hat{Q} \Psi)^+ = TCP\Psi$$

$$\Psi = TCP\overline{\Psi}$$

Conclusion 2

- Operators T and C are not consistent with generalized Dirac conjugation.
- They can only be approximate symmetry operators.
- Symmetry will be observed for phenomena described by tensor quantities and involving only spinors or only conjugated spinors.
- P, TC, TCP can be symmetry operators of spinors.