

New experimental methods and observables for anisotropic flow analyses in high-energy physics

Phys. Rev. C 102, 024907 (2020), Phys. Rev. C 102, 024910 (2020)

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“**NUCLEUS 2020**”, 12/10/2020



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Outline



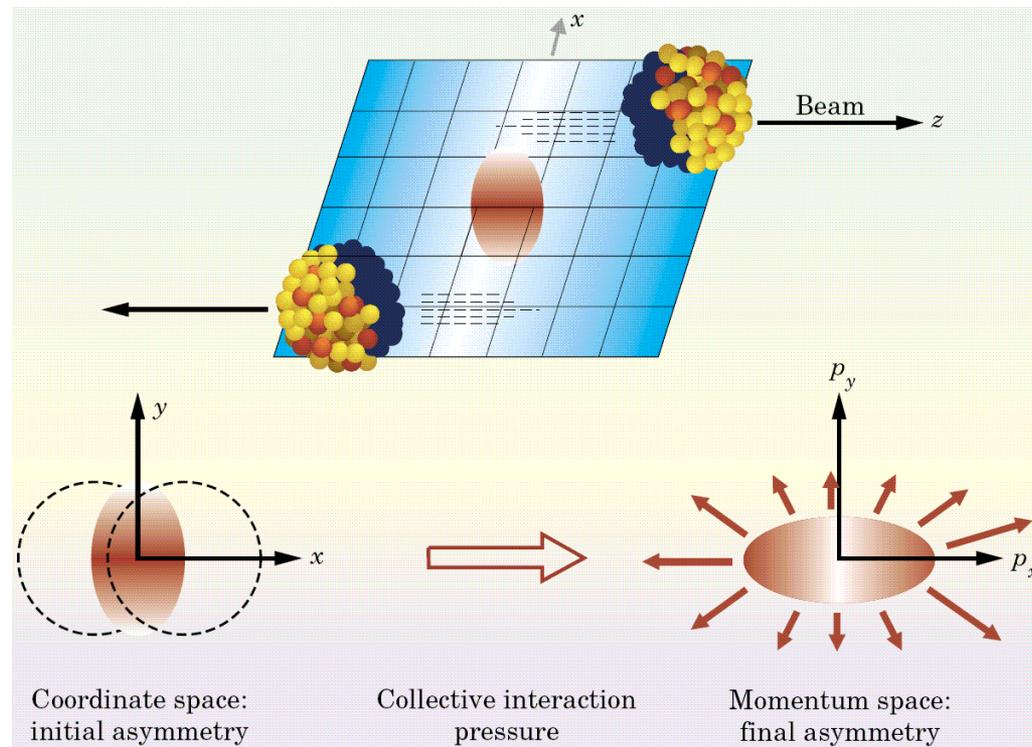
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- Introduction
 - Anisotropic flow
 - Multiparticle azimuthal correlations
- Higher order Symmetric Cumulants: $SC(k,l,m)$

C. Mordasini, AB, D. Karakoç, F. Taghavi, Phys. Rev. C **102**, 024907 (2020)
- New estimator for symmetry plane correlations
AB, M. Lesch, F. Taghavi, Phys. Rev. C **102**, 024910 (2020)

Anisotropic flow phenomenon

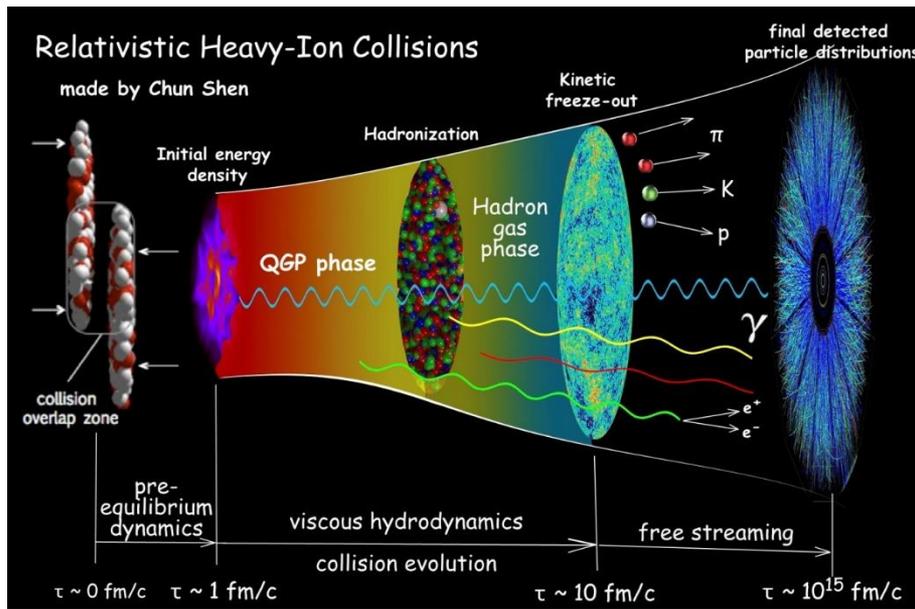
- Transfer of anisotropy from the initial coordinate space into the final momentum space via the thermalized medium:



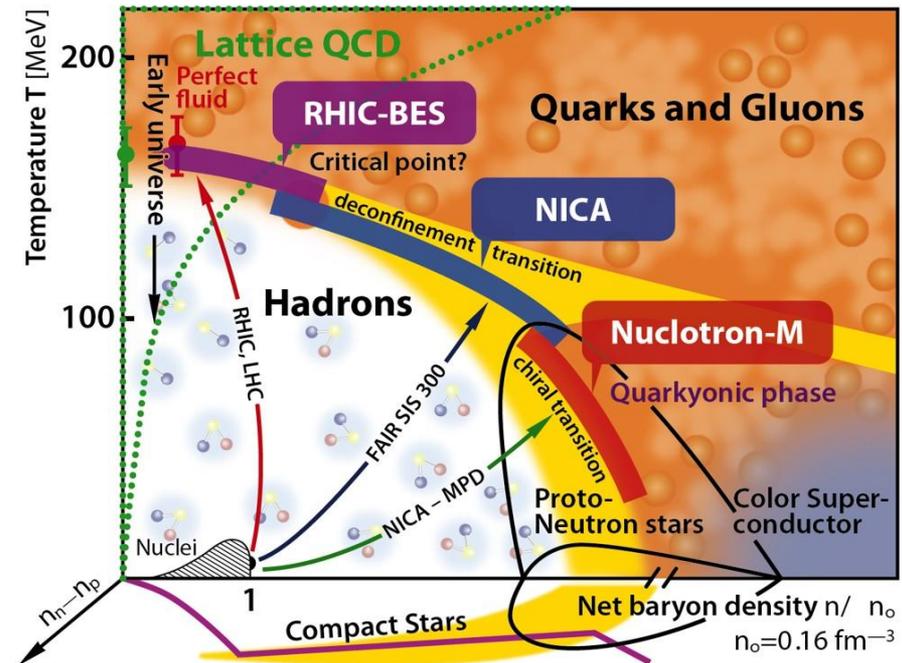
- J.Y. Ollitrault, Phys. Rev. D **46** (1992) 229

Quark-Gluon Plasma (QGP)

Heavy-ion evolution



Phase diagram of QCD



- In the QGP stage quarks are deconfined
- Anisotropic flow is a sensitive probe of QGP properties (e.g. of its shear viscosity)

Fourier series

- We use Fourier series to describe anisotropic emission of particles in the plane transverse to the beam direction after each heavy-ion collision:

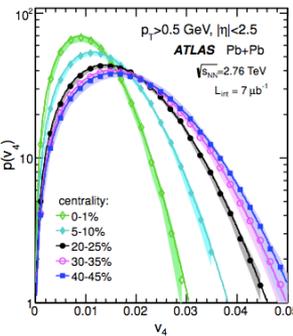
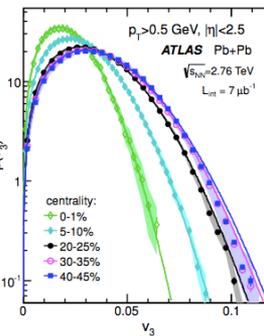
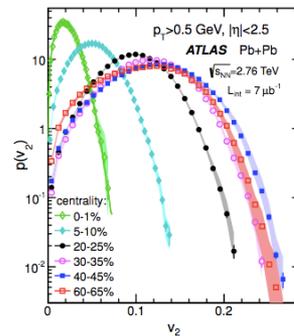
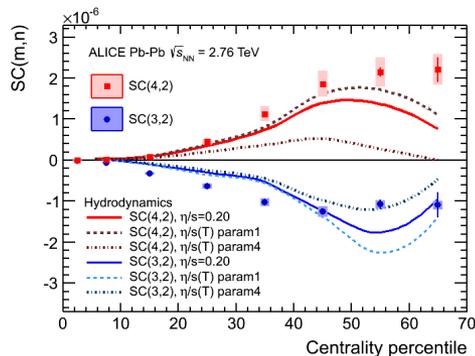
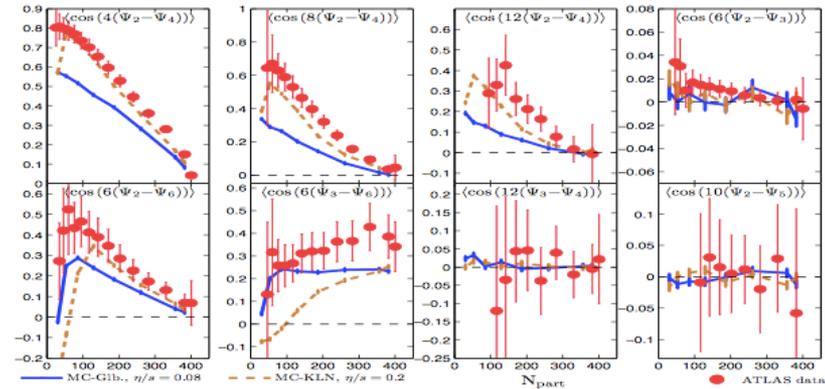
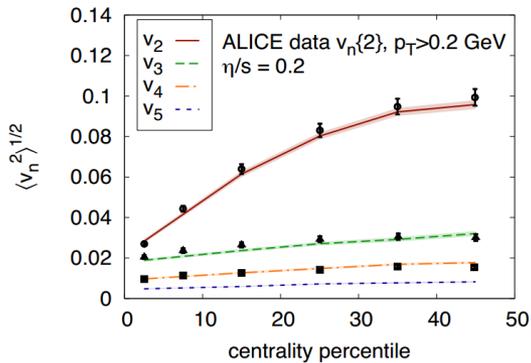
$$f(\varphi) = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} v_n \cos[n(\varphi - \Psi_n)] \right]$$

- v_n : flow amplitudes
- Ψ_n : symmetry planes
- Anisotropic flow is quantified with v_n and Ψ_n
 - v_1 is directed flow
 - v_2 is elliptic flow
 - v_3 is triangular flow
 - v_4 is quadrangular flow, etc.

Flow observables

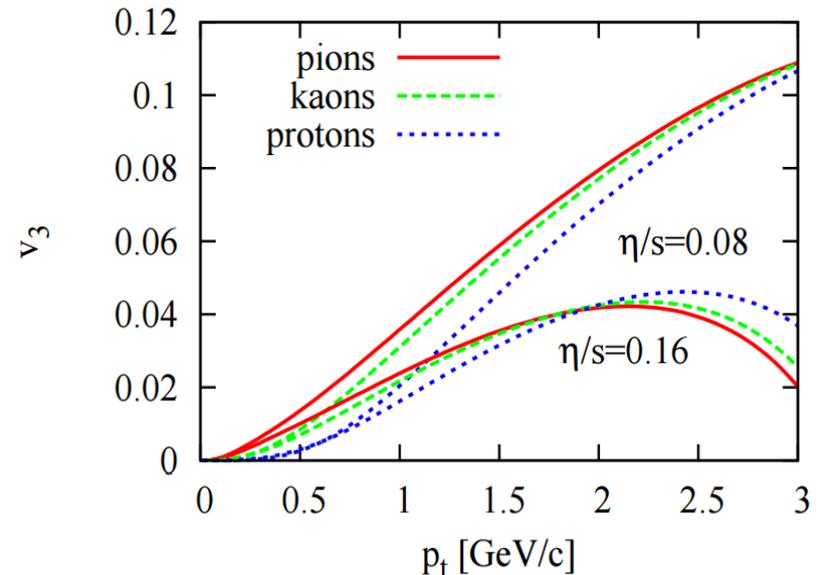
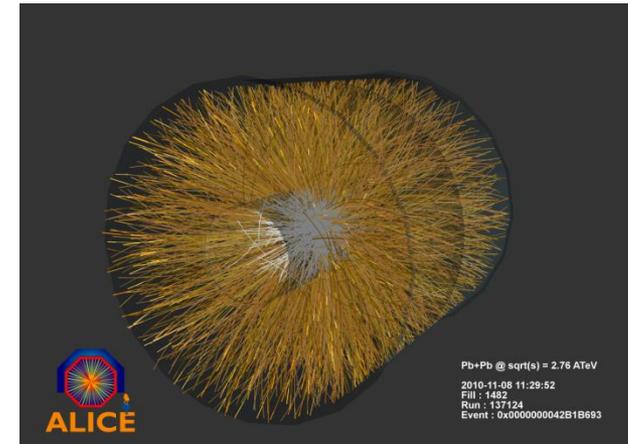
- Individual flow harmonics: $v_1, v_2, v_3, v_4, \dots$
- Correlations between harmonics: $\langle v_m^2 v_n^2 \rangle - \langle v_m^2 \rangle \langle v_n^2 \rangle$
- Symmetry plane correlations: $\langle \cos[mn(\Psi_m - \Psi_n)] \rangle$
- Probability density function: $\mathbf{P}(v_n)$

•
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Why so many flow observables?

- Heavy-ion collision is a very complex system and we cannot describe everything only with a few parameters
 - In head-on collisions, more than 10K particles are produced
- Different observables exhibit different sensitivities to QGP properties
 - Example from theory: Transverse momentum dependence of triangular flow is different for different values of QGP's shear viscosity



Multiparticle azimuthal correlations

Multiparticle azimuthal correlations

- The most general result, which relates multiparticle azimuthal correlators and flow degrees of freedom:

$$\langle \cos[n_1 \varphi_1 + \dots + n_k \varphi_k] \rangle = v_{n_1} \dots v_{n_k} \cos[n_1 \Psi_{n_1} + \dots + n_k \Psi_{n_k}]$$

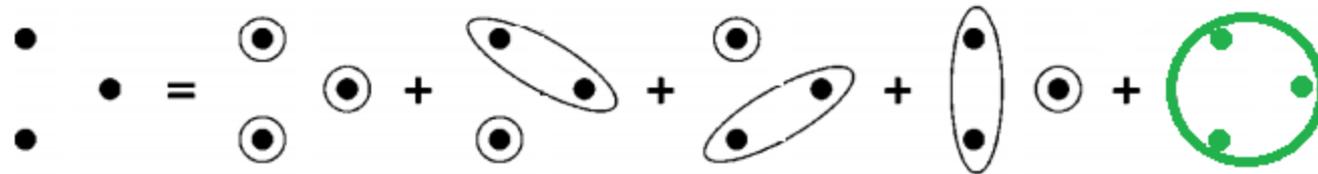
R. S. Bhalerao, M. Luzum and J.-Y. Ollitrault, Phys. Rev. C **84** 034910 (2011)

- A lot of non-trivial and independent flow observables for different choices of harmonics n_i
 - Examples: 2- and 4-particle azimuthal correlations

$$\begin{aligned} \langle \cos[n(\varphi_1 - \varphi_2)] \rangle &= v_n^2 \\ \langle \cos[n(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)] \rangle &= v_n^4 \end{aligned}$$

Multiparticle cumulants

- Consider the following diagram representation of the most general decomposition of 3-particle correlation:



- The very last term, which cannot be decomposed further, is by definition 3-particle cumulant
 - Cumulant term exists for any number of particles, it is always unique, and it isolates the genuine collective contribution
- Introduced in flow analyses by Ollitrault *et al*

N. Borghini, P. M. Dinh, and J.-Y. Ollitrault, Phys. Rev. C **63**, 054906 (2001)

N. Borghini, P. M. Dinh, and J.-Y. Ollitrault, Phys. Rev. C **64**, 054901 (2001)

3-particle cumulants in general

- Working recursively from higher to lower orders, we eventually have 3-particle cumulant expressed in terms of measured 3-, 2-, and 1-particle averages

$$\begin{aligned}\langle X_1 X_2 X_3 \rangle_c &= \langle X_1 X_2 X_3 \rangle \\ &- \langle X_1 X_2 \rangle \langle X_3 \rangle - \langle X_1 X_3 \rangle \langle X_2 \rangle - \langle X_2 X_3 \rangle \langle X_1 \rangle \\ &+ 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle\end{aligned}$$

- General result, true for any choice of stochastic observables
- In the same way, cumulants can be expressed in terms of measurable averages for any number of observables
 - The number of terms grows rapidly

Cumulants in flow analyses

- Traditionally, azimuthal angles are chosen as random observables in the cumulant expansion
- Based on this approach, one can derive $v_n\{4\}$ observable
 - It gives an estimate for flow harmonic v_n by using 4-particle azimuthal cumulant (not 4-p azimuthal correlator!)
 - For large multiplicities, $v_n\{4\}$ suppresses well nonflow effects
- But this traditional approach yields to very weird and inconsistent results when applied to the correlations of different flow amplitudes

Why the traditional cumulant approach with azimuthal angles which worked so well in the past fails when applied in the studies of correlations of different flow amplitudes?

Higher order Symmetric Cumulants

C. Mordasini, AB, D. Karakoç, F. Taghavi, Phys. Rev. C **102**, 024907 (2020)

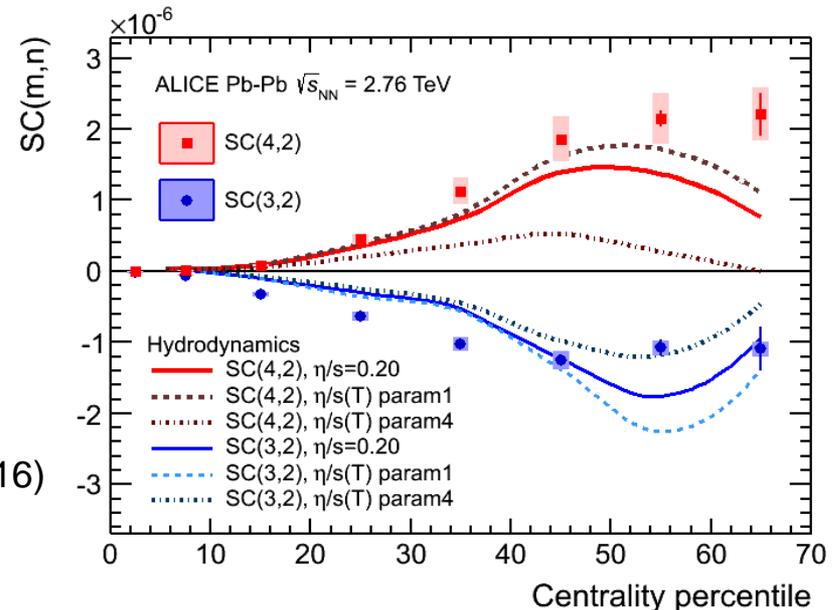
Reminder: Symmetric Cumulants $SC(m,n)$

- How to quantify experimentally the correlation between two different flow amplitudes?
 - Symmetric Cumulants (Section IVC in Phys. Rev. C **89** (2014) no.6, 064904)

$$\begin{aligned}
 \langle \langle \cos(m\varphi_1 + n\varphi_2 - m\varphi_3 - n\varphi_4) \rangle \rangle_c &= \langle \langle \cos(m\varphi_1 + n\varphi_2 - m\varphi_3 - n\varphi_4) \rangle \rangle \\
 &\quad - \langle \langle \cos[m(\varphi_1 - \varphi_2)] \rangle \rangle \langle \langle \cos[n(\varphi_1 - \varphi_2)] \rangle \rangle \\
 &= \langle v_m^2 v_n^2 \rangle - \langle v_m^2 \rangle \langle v_n^2 \rangle
 \end{aligned}$$

- SC observables are sensitive to differential $\eta/s(T)$ parametrizations
- Individual flow amplitudes are dominated by averages $\langle \eta/s(T) \rangle$
- Independent constraints both on initial conditions and QGP properties

ALICE Collaboration, Phys. Rev. Lett. 117, 182301 (2016)



New paradigm in flow analyses

- Example: General 2-particle cumulant

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

- Traditional approach: fundamental observable is an angle

$$X_1 \rightarrow e^{in\varphi_1}, \quad X_2 \rightarrow e^{-in\varphi_2}$$

- New approach: fundamental observable is a flow amplitude

$$X_1 \rightarrow v_n^2, \quad X_2 \rightarrow v_m^2$$

- Two approaches yield **accidentally the same results** for lower order Symmetric Cumulants $SC(k,l)$, but differ for higher orders $SC(k,l,m)$, $SC(k,l,m,n)$, ...
 - Which one is correct?

Generalization: Multi-harmonic SC

- New paradigm:

1/ Cumulant expansion directly on flow amplitudes:

$$SC(k, l, m) \equiv \langle v_k^2 v_l^2 v_m^2 \rangle - \langle v_k^2 v_l^2 \rangle \langle v_m^2 \rangle - \langle v_k^2 v_m^2 \rangle \langle v_l^2 \rangle - \langle v_l^2 v_m^2 \rangle \langle v_k^2 \rangle + 2 \langle v_k^2 \rangle \langle v_l^2 \rangle \langle v_m^2 \rangle$$

2/ Azimuthal angles are used merely to build an estimator for the above observable:

$$\begin{aligned} SC(k, l, m) = & \langle \langle \cos[k\varphi_1 + l\varphi_2 + m\varphi_3 - k\varphi_4 - l\varphi_5 - m\varphi_6] \rangle \rangle \\ & - \langle \langle \cos[k\varphi_1 + l\varphi_2 - k\varphi_3 - l\varphi_4] \rangle \rangle \langle \langle \cos[m(\varphi_5 - \varphi_6)] \rangle \rangle \\ & - \langle \langle \cos[k\varphi_1 + m\varphi_2 - k\varphi_5 - m\varphi_6] \rangle \rangle \langle \langle \cos[l(\varphi_3 - \varphi_4)] \rangle \rangle \\ & - \langle \langle \cos[l\varphi_3 + m\varphi_4 - l\varphi_5 - m\varphi_6] \rangle \rangle \langle \langle \cos[k(\varphi_1 - \varphi_2)] \rangle \rangle \\ & + 2 \langle \langle \cos[k(\varphi_1 - \varphi_2)] \rangle \rangle \langle \langle \cos[l(\varphi_3 - \varphi_4)] \rangle \rangle \langle \langle \cos[m(\varphi_5 - \varphi_6)] \rangle \rangle \end{aligned}$$

C. Mordasini, AB, D. Karakoç, F. Taghavi: *'Higher order Symmetric Cumulants'*, Phys. Rev. C **102**, 024907 (2020)

Generalization: Multi-harmonic SC



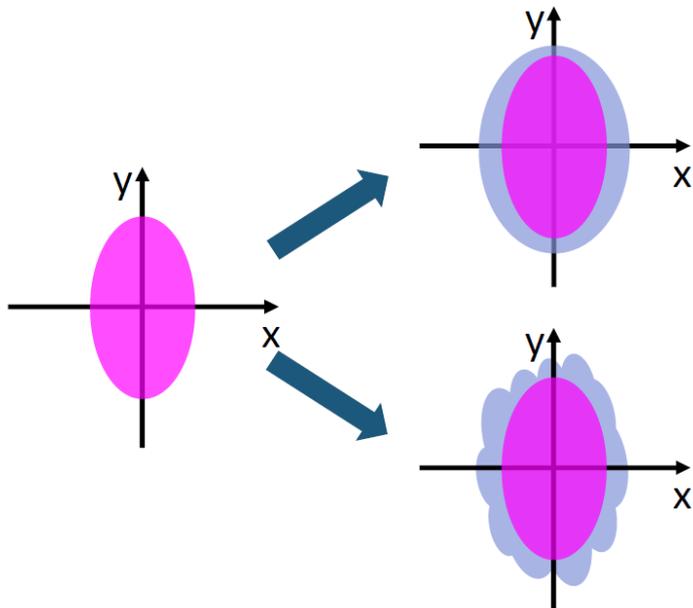
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- **The main conclusion:** One cannot perform cumulant expansion in one set of stochastic observables, then in the resulting expression perform the transformation to some new set of observables, and then claim that the cumulant properties are preserved in the new set of observables
 - After such transformation, the fundamental properties of cumulants are lost in general
- Flow amplitudes v_k, v_l, \dots are stochastic observables in a sense that there exists underlying multivariate p.d.f. $f(v_k, v_l, \dots)$ from which they are sampled event-by-event
 - The formalism of cumulants can be applied directly on flow amplitudes

C. Mordasini, AB, D. Karakoç, F. Taghavi: *'Higher order Symmetric Cumulants'*, Phys. Rev. C **102**, 024907 (2020)

What can we measure?

- Can we attempt for the first time to separate with these new observables effects of shear (η) and bulk (ξ) viscosities?

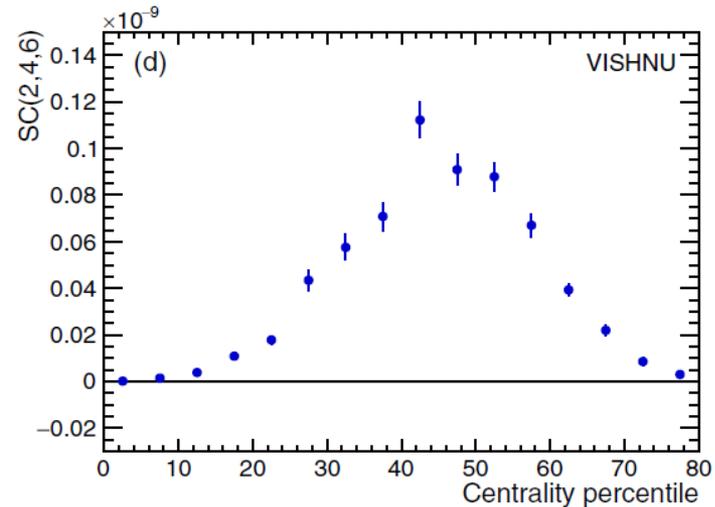
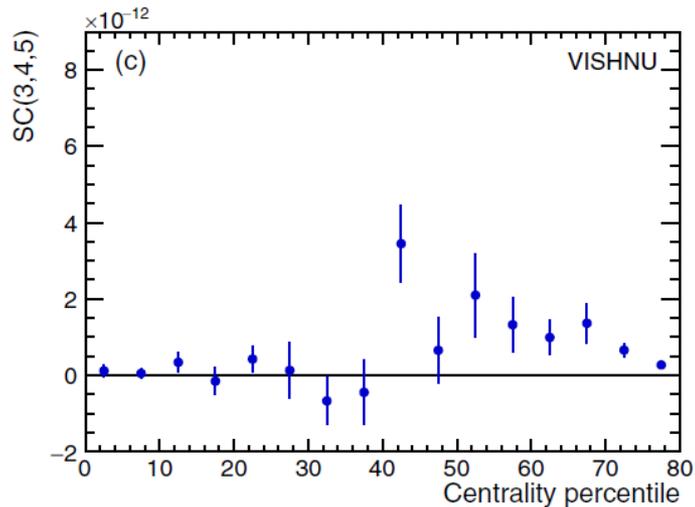
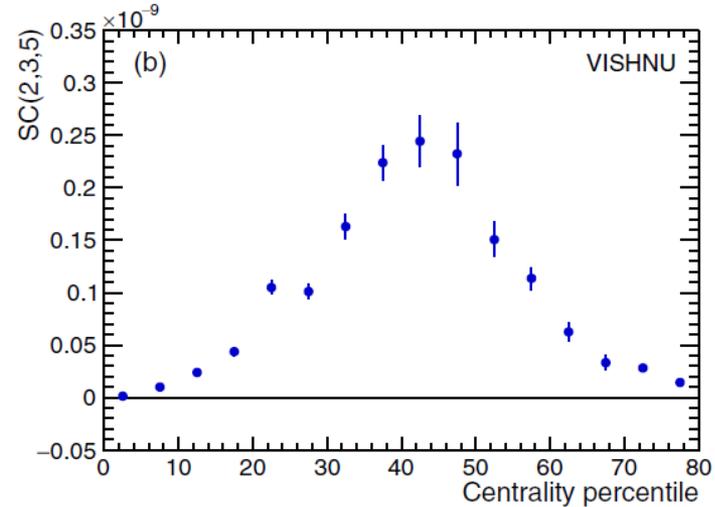
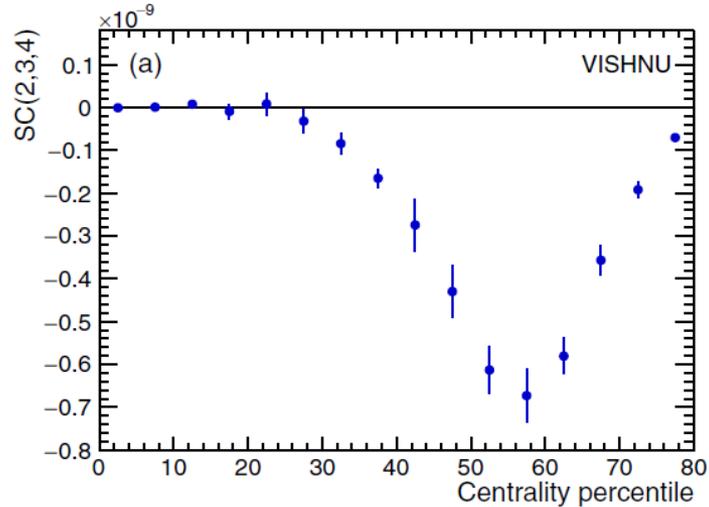


Credits: C. Mordasini

- Isotropic fluctuations
 - Neighbouring layers move at equal velocities
 - Generally preserves the ellipse shape
 - Main sensitivity to ξ/s
- Shape fluctuations
 - Neighbouring layers move at different velocities
 - Sensitivity to η/s

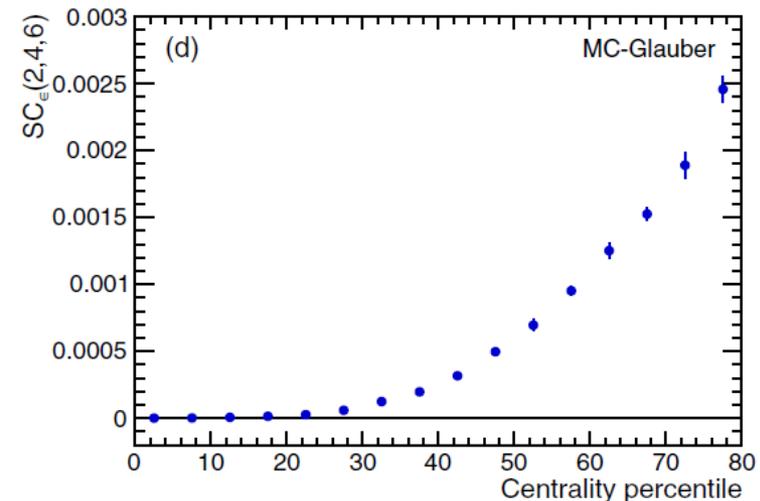
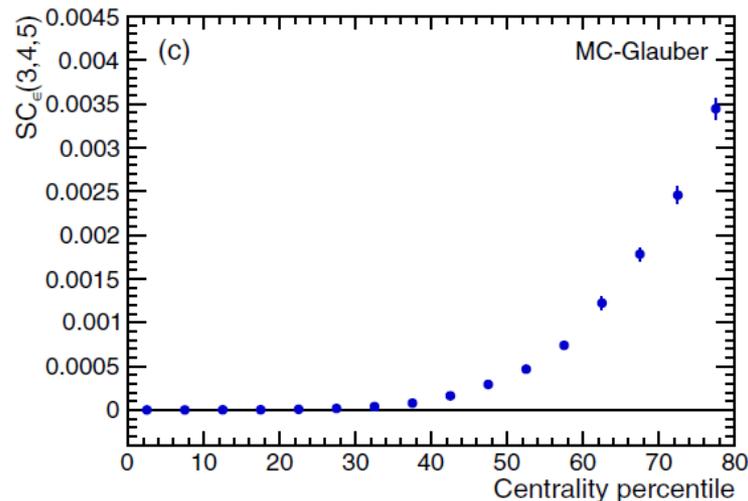
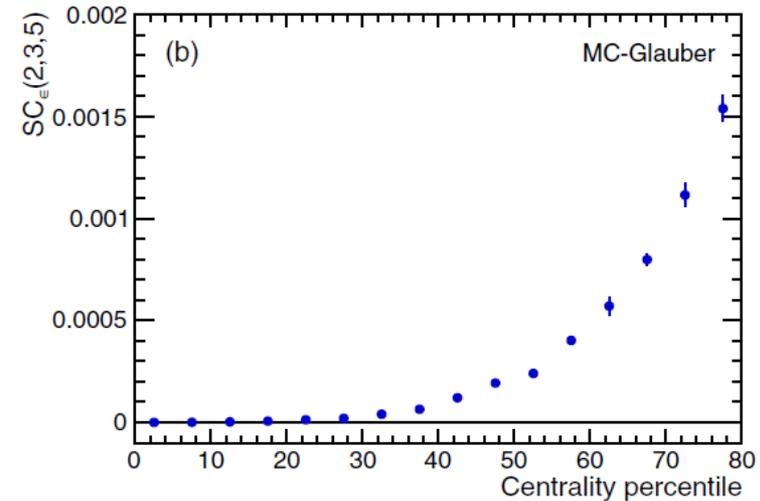
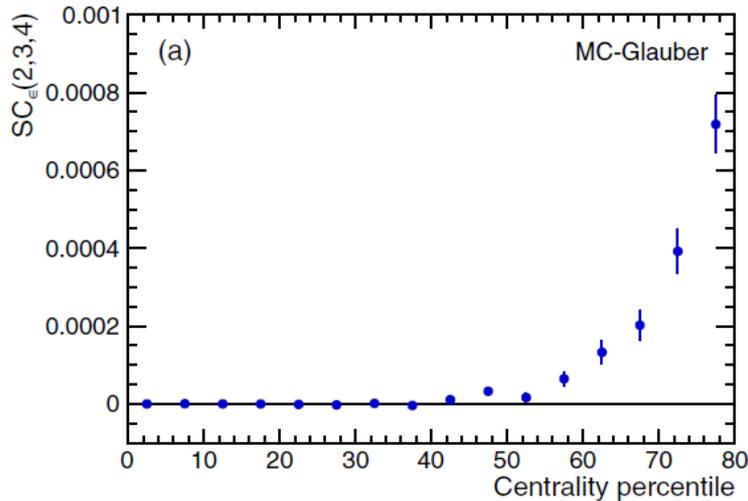
Selected Monte Carlo studies

- Predictions from VISHNU for the final momentum space:



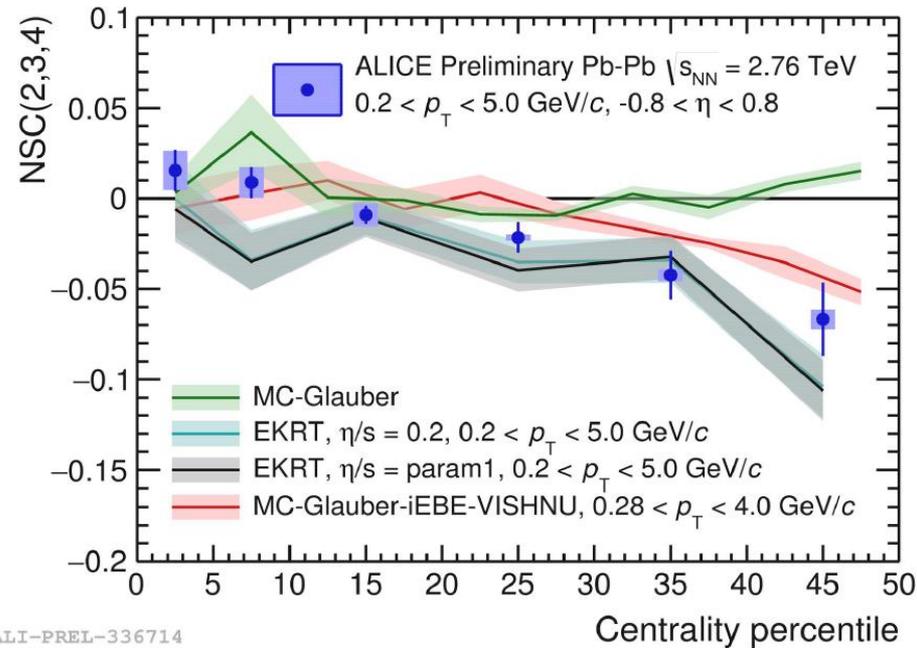
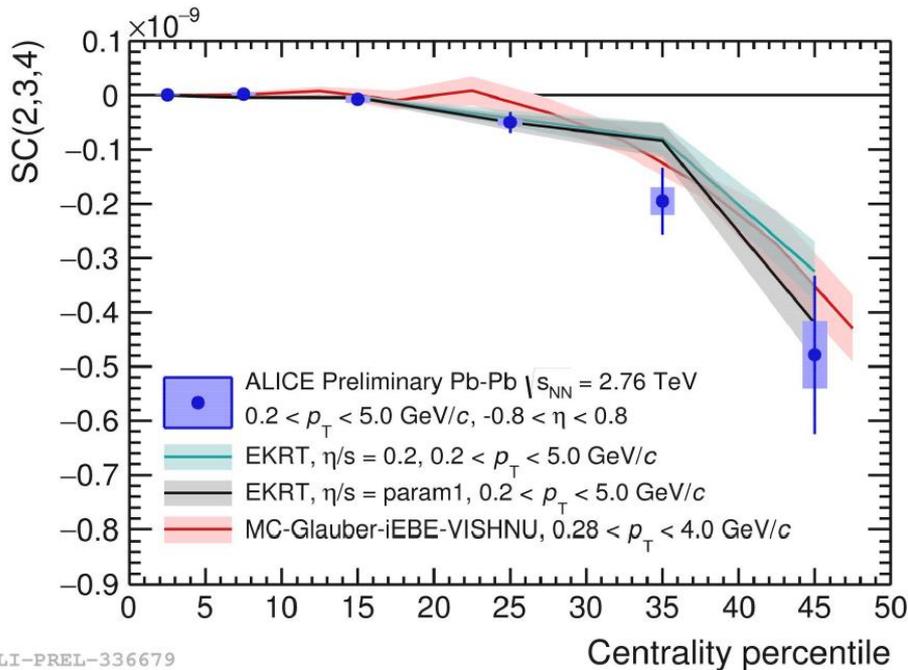
Selected Monte Carlo studies

- Predictions from MC-Glauber for the initial coordinate space:



First experimental results

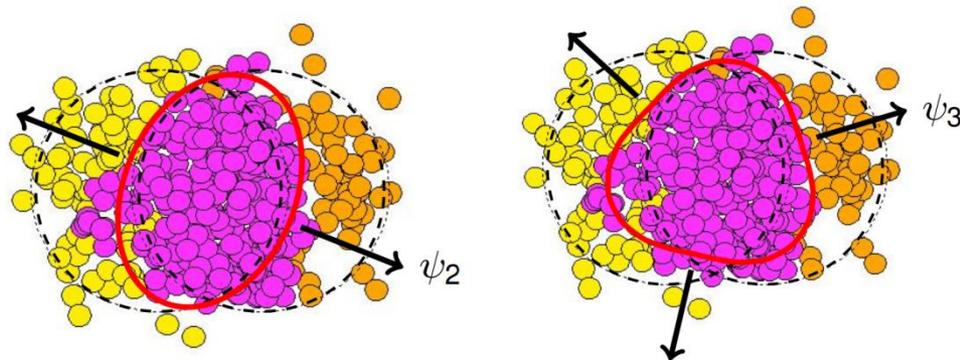
- $SC(k,l,m)$ in Pb+Pb collisions at LHC:



Cindy Mordasini, 'Multi-harmonic correlations in ALICE', poster #275 at QM 2019
(<https://indico.cern.ch/event/792436/contributions/3533796/>)

New estimator for symmetry plane correlations

AB, M. Lesch, F. Taghavi, Phys. Rev. C **102**, 024910 (2020)



Fourier series, recap.

- We use Fourier series to describe anisotropic flow:

$$f(\varphi) = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} v_n \cos[n(\varphi - \Psi_n)] \right]$$

- v_n : flow amplitudes
- Ψ_n : symmetry planes
 - Due to event-by-event flow fluctuations, v_n and Ψ_n are independent and equally important degrees of freedom to quantify anisotropic flow
 - But they are also fundamentally different (Ψ_n depend on the choice of coordinate system, not uniquely defined (periodicity))
- Observables: Symmetry plane correlations
 - Event Plane method
 - Scalar Product method

Symmetry plane correlations (SPC)

- Definition of SPC:

$$\langle e^{i(a_1 n_1 \Psi_{n_1} + \dots + a_k n_k \Psi_{n_k})} \rangle, \quad \sum_i a_i n_i = 0$$

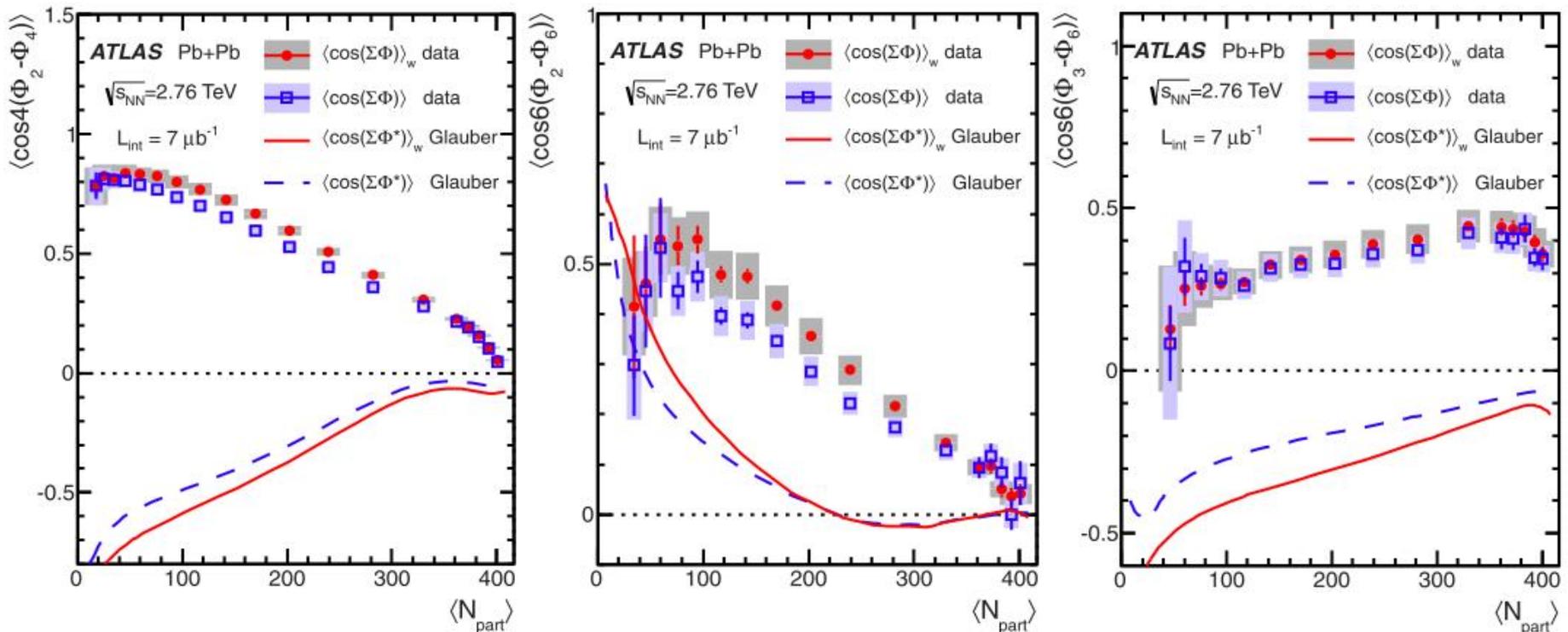
- n_i are flow harmonics
- a_i are positive integers: the number of appearances of harmonic n_i associated with different azimuthal angles on the RHS in the analytic expression below:

$$v_{n_1}^{a_1} \dots v_{n_k}^{a_k} e^{i(a_1 n_1 \Psi_{n_1} + \dots + a_k n_k \Psi_{n_k})} = \langle e^{i(a_1 n_1 \varphi_1 + \dots + a_k n_k \varphi_k)} \rangle$$

R. S. Bhalerao, M. Luzum and J.-Y. Ollitrault, PRC 84 034910 (2011)

Symmetry plane correlations (SPC)

- ATLAS: Phys. Rev. C **90**, 024905



- Correlations of symmetry planes in coordinate space are not equal to correlations of symmetry planes in momentum space

However...

- In all current SPC measurements, there is a very bold assumption that flow amplitudes \mathbf{v}_n are independent
 - Can really the denominator be fully factorised in the Scalar Product method?

$$\begin{aligned} & \langle \cos (a_1 n_1 \Psi_{n_1} + \dots + a_k n_k \Psi_{n_k}) \rangle_{SP} \\ &= \frac{\langle v_{n_1}^{a_1} \dots v_{n_k}^{a_k} \cos (a_1 n_1 \Psi_{n_1} + \dots + a_k n_k \Psi_{n_k}) \rangle}{\sqrt{\langle v_{n_1}^{2a_1} \rangle \dots \langle v_{n_k}^{2a_k} \rangle}} \end{aligned}$$

- Experimentally, we know that is not true, i.e. different flow amplitudes are definitely strongly correlated!
 - Can we improve the measurements of symmetry plane correlations by taking this correlation into account?

New estimator for symmetry plane correlations

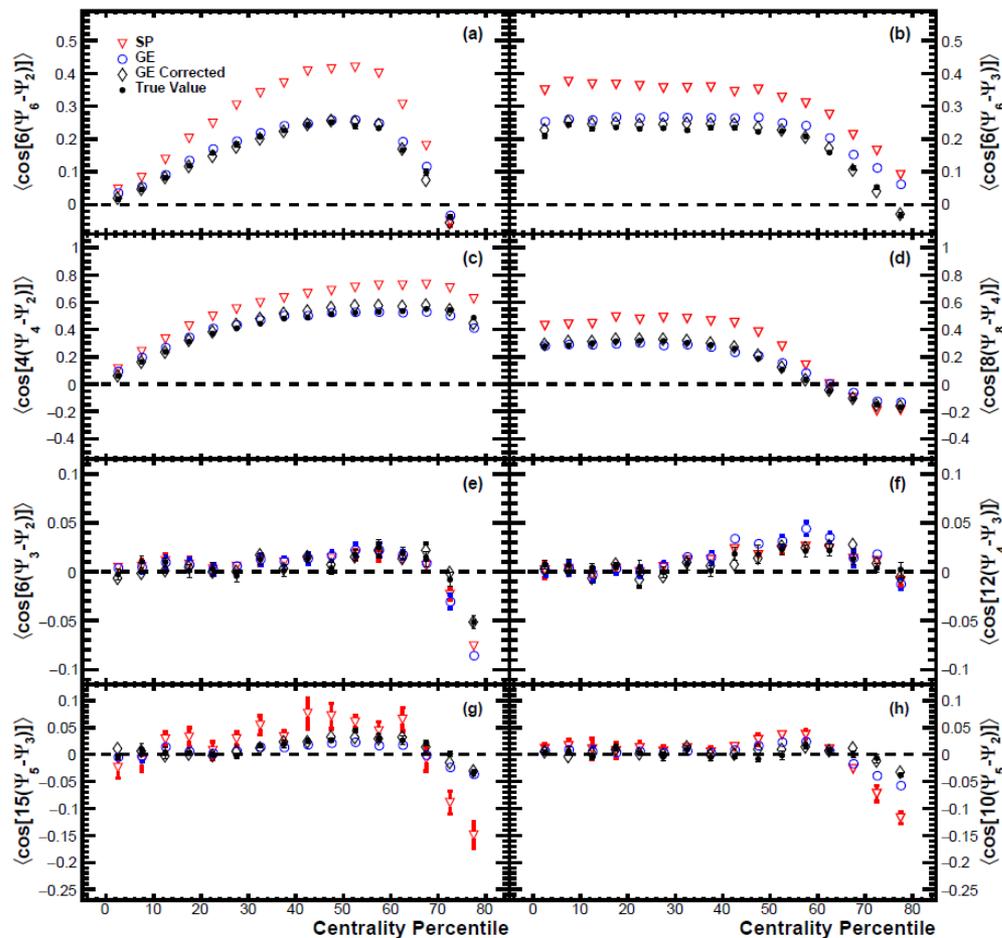
- The new estimator, which correctly accounts for the biases from correlations of flow amplitudes:

$$\begin{aligned} & \langle \cos (a_1 n_1 \Psi_{n_1} + \dots + a_k n_k \Psi_{n_k}) \rangle_{\text{GE}} \\ & \simeq \sqrt{\frac{\pi}{4}} \frac{\langle v_{n_1}^{a_1} \dots v_{n_k}^{a_k} \cos (a_1 n_1 \Psi_{n_1} + \dots + a_k n_k \Psi_{n_k}) \rangle}{\sqrt{\langle v_{n_1}^{2a_1} \dots v_{n_k}^{2a_k} \rangle}} \end{aligned}$$

- The new denominator can be estimated with suitable chosen multiparticle azimuthal correlators
 - How important is the difference?

AB, M. Lesch, F. Taghavi: 'New estimator for symmetry plane correlations in anisotropic flow analyses', Phys. Rev. C **102**, 024910 (2020)

New estimator for symmetry plane correlations



- Clear improvement over other existing estimators (e.g. the one based on traditional Scalar Product (SP) method)
- For centralities in which SP estimator (red markers) fails to reproduce the true values (black markers), our new estimator is still doing a great job!
- Works also if different flow amplitudes are correlated

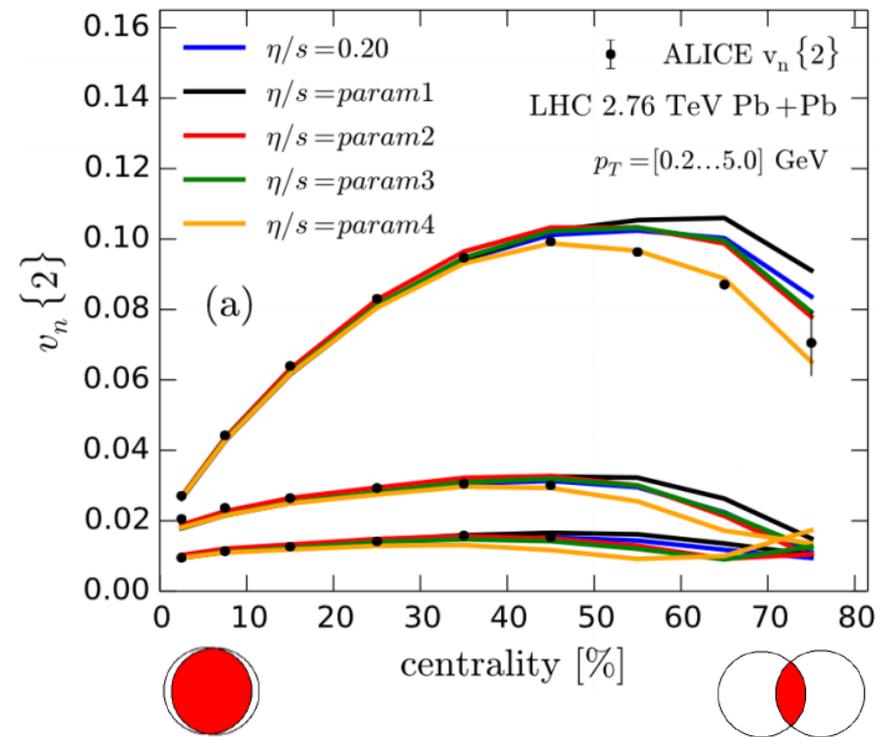
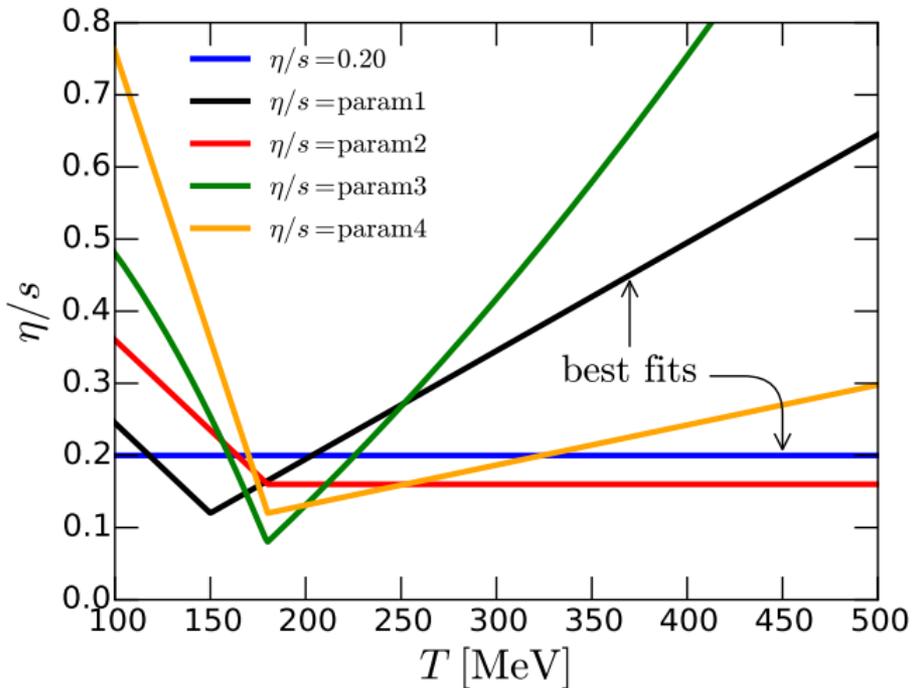
AB, M. Lesch, F. Taghavi: 'New estimator for symmetry plane correlations in anisotropic flow analyses', Phys. Rev. C **102**, 024910 (2020)

Thanks!

Backup slides

'Classical' flow observables

- Insensitivity to temperature dependence of η/s



H. Niemi, K. J. Eskola, R. Paatelainen, Phys. Rev. C 93, 024907 (2016)

The essence of the idea

- Estimating flow harmonics with 2-particle correlation:

$$\begin{aligned}
 \begin{array}{l} \text{event} \\ \text{average} \end{array} & \rightarrow \langle\langle e^{in(\varphi_1 - \varphi_2)} \rangle\rangle = \langle\langle e^{in(\varphi_1 - \Psi_n - (\varphi_2 - \Psi_n))} \rangle\rangle \\
 \begin{array}{l} \text{particle} \\ \text{average} \end{array} & \rightarrow \langle\langle e^{in(\varphi_1 - \Psi_n)} \rangle\rangle \langle\langle e^{-in(\varphi_2 - \Psi_n)} \rangle\rangle \\
 & = v_n^2
 \end{aligned}$$

- The ‘trick’ works for any number of particles in the correlator
 - k -particle correlations estimate v_n^k
- But in the real world, there are subtleties...
 - Trivial self-correlations
 - Other sources of physical correlations (‘nonflow’)
 - Detector artifacts

Q -vectors

- Q -vectors (or flow vectors) are among the most important fundamental objects in flow analyses nowadays
- Three definitions:
 - M -particle Q -vector

$$Q_n \equiv \sum_{i=1}^M e^{in\varphi_i}$$

- Unit Q -vector

$$u_n \equiv e^{in\varphi}$$

- Reduced Q -vector

$$q_n \equiv \frac{Q_n}{\sqrt{M}}$$

Q -vectors

- What Q -vectors have to do with multi-particle correlation techniques?
- Remarkably, we can **analytically express any multi-particle azimuthal correlator in terms of Q -vectors** in such a way that all self-correlations are exactly removed
 - First realized by S. Voloshin ~ 15 years ago
 - This realization is the most important breakthrough in the field of correlation techniques of late

• Example:

$$\begin{aligned}
 \langle 2 \rangle &\equiv \langle \cos(n(\varphi_1 - \varphi_2)) \rangle \\
 &\equiv \frac{1}{\binom{M}{2} 2!} \sum_{\substack{i,j=1 \\ (i \neq j)}}^M e^{in(\varphi_i - \varphi_j)} \\
 &= \frac{1}{\binom{M}{2} 2!} \times [|Q_n|^2 - M]
 \end{aligned}$$

Q-vectors

- Example: Analytic result for 4-p correlation

$$\begin{aligned} \langle 4 \rangle &\equiv \langle \cos(n(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)) \rangle \\ &= \frac{1}{\binom{M}{4} 4!} \sum_{\substack{i,j,k,l=1 \\ (i \neq j \neq k \neq l)}}^M e^{in(\varphi_i + \varphi_j - \varphi_k - \varphi_l)} \end{aligned}$$

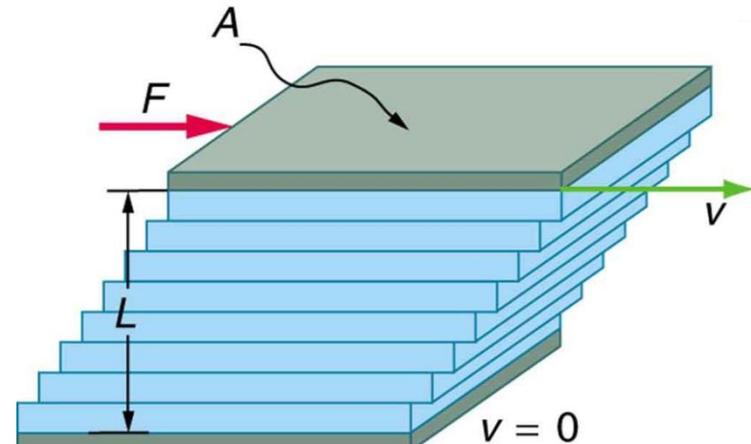
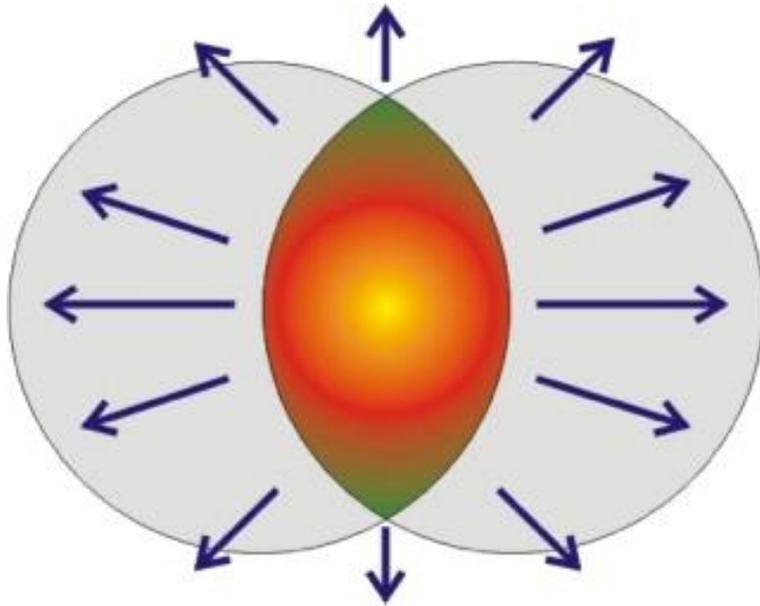
$$\begin{aligned} Q_n &= \sum_{i=1}^M e^{in\varphi_i} \\ Q_{2n} &= \sum_{i=1}^M e^{i2n\varphi_i} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\binom{M}{4} 4!} \times [|Q_n|^4 + |Q_{2n}|^2 - 2 \cdot \Re [Q_{2n} Q_n^* Q_n^*] - 4(M-2) |Q_n|^2 \\ &\quad + 2M(M-3)] \end{aligned}$$

- The key point: The RHS can be obtained in the single loop over all azimuthal angles of particles
 - Both exact and fast formalism

Hydro flow in-plane

- Non-trivial effect which is sensitive to transport coefficients of QGP (e.g. its shear viscosity)

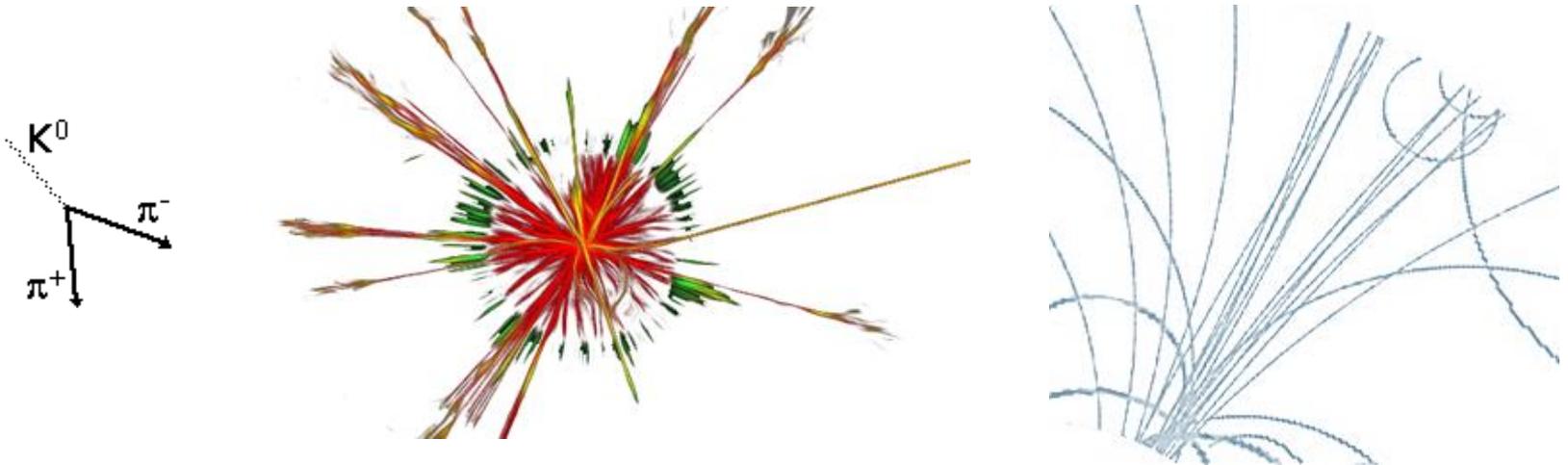


If anisotropic flow has developed, neighboring layers are moving at different relative velocities, parallel displacement is opposed by shear viscosity

large anisotropic flow \Leftrightarrow small shear viscosity

Nonflow examples

- **Physical:** Resonance decays, jets, etc.
- **Detector artifacts:** Track splitting in reconstruction, etc.
- **Computational:** Autocorrelations



$$\langle e^{in(\varphi_1 - \varphi_2)} \rangle, \quad \varphi_1 \neq \varphi_2$$

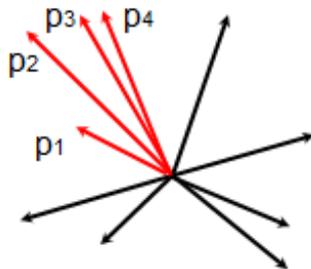
$$\langle e^{in(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)} \rangle, \quad \varphi_1 \neq \varphi_2 \neq \varphi_3 \neq \varphi_4$$

Nonflow scaling

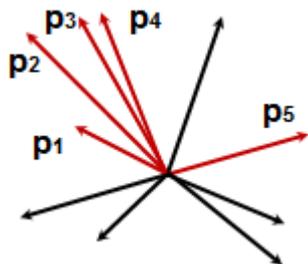
- Since nonflow is generally correlation among few particles, we can establish the following probabilistic argument:
 - For k -particle azimuthal correlator, nonflow contributes as:

$$\delta_k \sim \frac{1}{M^{k-1}}$$

- Example: 4-particle correlation $\langle e^{in(\varphi_1+\varphi_2-\varphi_3-\varphi_4)} \rangle$



$$\delta_4 \sim \frac{3}{M-1} \frac{2}{M-2} \frac{1}{M-3} \approx \frac{1}{M^3}$$



$$\delta_4 \sim \frac{4}{M-1} \frac{3}{M-2} \frac{2}{M-3} \approx \frac{1}{M^3}$$

Nonflow scaling

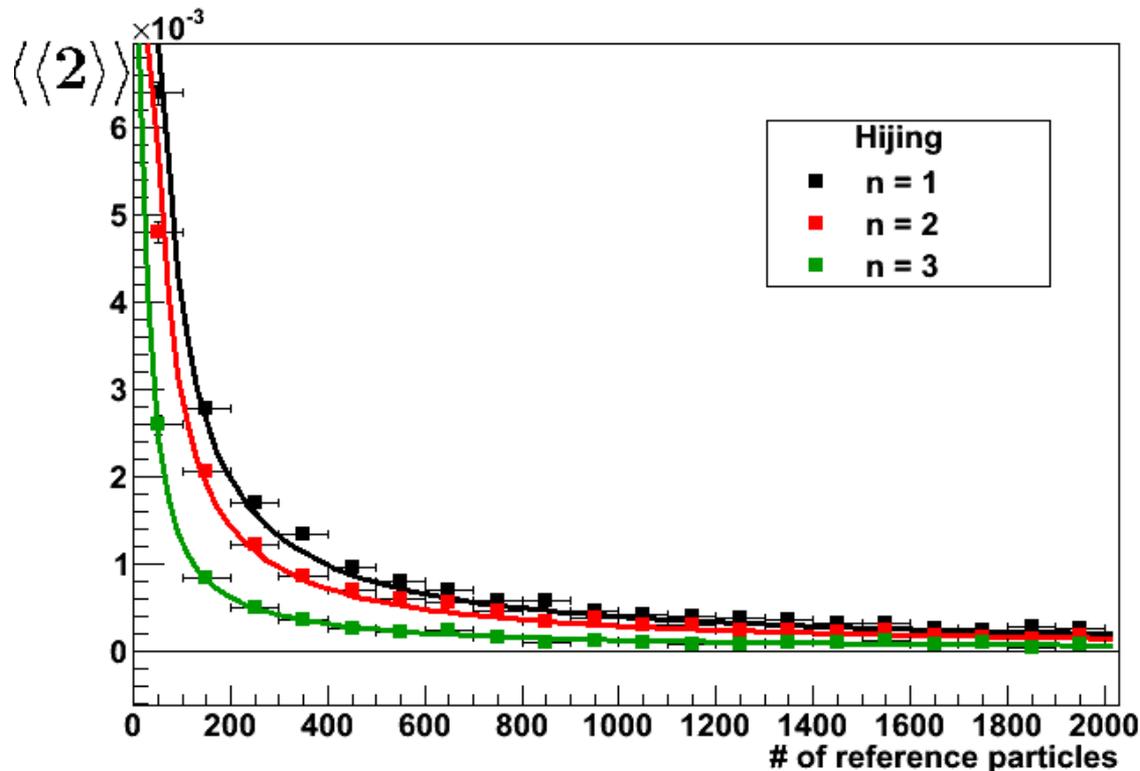
- Since flow is a collective effect, such a scaling *vs.* multiplicity never holds:

$$\delta_4(\text{flow}) \sim \frac{M-1}{M-1} \frac{M-2}{M-2} \frac{M-3}{M-3} \sim 1$$

- If k -particle azimuthal correlator is dominated by flow, $1/M^{k-1}$ scaling must be broken
 - Very robust flow signature
- Next observations:
 - It's less probable for resonance to decay in 3 than in 2 particles
 - It's less probable for resonance to decay in 4 than in 3 particles
 - ...
- Higher order correlations are biased less by nonflow, but flow is collective effect => its contribution doesn't diminish!

Nonflow scaling: Proof

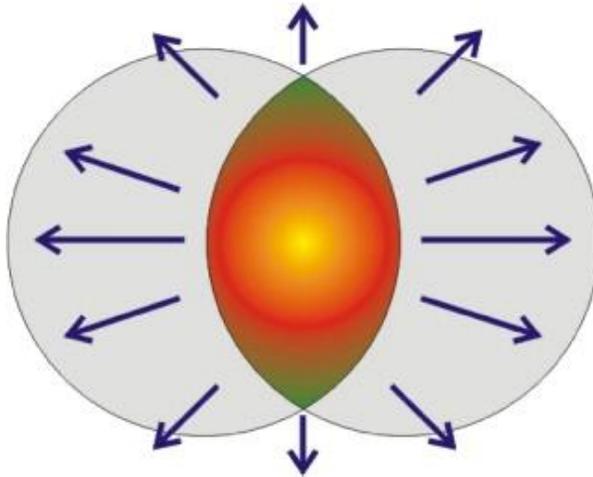
- Hijing is a heavy-ion generator which implements all sources of physical correlations, except flow
 - 2-particle azimuthal correlation in Hijing measures only nonflow



Perfect $\sim 1/M$ scaling of 2-particle correlation, for three different harmonics

The 'flow principle'

- Correlations among all produced particles are induced solely by correlation of each single particle to the collision geometry



- Analogy with the falling bodies in gravitational field (rhs)
- Whether or not particles are emitted simultaneously, or one by one, trajectories are the same

○ These are **statistically independent** trajectories

Statistical independence, back to flow

- If anisotropic flow is the only source of correlations between produced particles, their joint n -variate p.d.f.

$$f(\varphi_1, \dots, \varphi_n)$$

factorizes into product of n single-particle marginal p.d.f.'s:

$$f(\varphi_1, \dots, \varphi_n) = f_{\varphi_1}(\varphi_1) \cdots f_{\varphi_n}(\varphi_n)$$

- From 'flow principle': All marginal p.d.f.'s are the same, and therefore parameterized by the same Fourier series:

$$f(\varphi_1, \dots, \varphi_n) = f(\varphi_1) \cdots f(\varphi_n)$$

$$f(\varphi) = \frac{1}{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} v_n \cos[n(\varphi - \Psi_n)] \right]$$

Scaling of stat. and sys. errors

- Scaling of statistical uncertainty (N is number of events, M is multiplicity, v is flow strength, k is order of correlator):

$$\sigma_v \sim \frac{1}{\sqrt{N}} \frac{1}{M^{k/2}} \frac{1}{v^{k-1}}$$

- Nonflow scaling:

$$\delta_k \sim \frac{1}{M^{k-1}}$$

- For both reasons, multiparticle correlations is a precision technique only for: a) large multiplicities, b) large flow

Example 1

- 3-particle cumulant for 3 independent observables:

$$\begin{aligned}\langle X_1 X_2 X_3 \rangle_c &= \langle X_1 X_2 X_3 \rangle \\ &\quad - \langle X_1 X_2 \rangle \langle X_3 \rangle - \langle X_1 X_3 \rangle \langle X_2 \rangle - \langle X_2 X_3 \rangle \langle X_1 \rangle \\ &\quad + 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle \\ &= \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle \\ &\quad - \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle - \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle - \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle \\ &\quad + 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle \\ &= 0\end{aligned}$$

Example 2

- 3-particle cumulant for 2 independent observables (let's say X_1 and X_2 are both independent of X_3 , but X_1 and X_2 can be correlated):

$$\begin{aligned}\langle X_1 X_2 X_3 \rangle_c &= \langle X_1 X_2 X_3 \rangle \\ &\quad - \langle X_1 X_2 \rangle \langle X_3 \rangle - \langle X_1 X_3 \rangle \langle X_2 \rangle - \langle X_2 X_3 \rangle \langle X_1 \rangle \\ &\quad + 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle \\ &= \langle X_1 X_2 \rangle \langle X_3 \rangle \\ &\quad - \langle X_1 X_2 \rangle \langle X_3 \rangle - \langle X_1 \rangle \langle X_3 \rangle \langle X_2 \rangle - \langle X_2 \rangle \langle X_3 \rangle \langle X_1 \rangle \\ &\quad + 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle \\ &= 0\end{aligned}$$